AN INTRODUCTION TO THE FOURIER TRANSFORM

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I. INTRODUCTION

We are going to be looking at how to describe and analyze a two-dimensional wave $f(x,t)$—i.e., a function of one spatial variable x and time t. Since the spatial and temporal dependences of such a wave are related, we don't really need to be considering a function of two variables. We could, for example, sit at one spot, say $x=0$, and examine the wave's temporal dependence, given by $f(x=0,t)$, as it passes that spot. On the other hand, we could study how the wave looks in space at a particular time, say $t=0$; i.e., we could look at the function $f(x,t=0)$. In this document we are going to make the latter choice and study the function

$$f(x)\equiv f(x,t=0)\;,$$

but you will want to keep in mind that everything we do can also be applied to a function of time.

II. SUPERPOSITIONS OF MONOCHROMATIC WAVES

The simplest kind of wave is a monochromatic wave

$$f(x,t)=A\cos(kx-\omega t+\delta)\;,$$

for which our function $f(x)$ is

$$f(x)=A\cos(kx+\delta)\;.$$

Though a monochromatic wave has the virtue of having a precise angular frequency ω and a precise wave number k, it is unphysical because it extends over all of space. A physically realistic wave, called a *wave packet*, extends only over a finite region of space, with a well defined start and finish. The purpose of this document is to show how a wave packet can be constructed as a superposition of monochromatic waves. Thus monochromatic waves are important not just because they are simple, but more importantly because they provide the building blocks for all other kinds of waves.

We begin our investigation by considering the superposition of just two monochromatic waves, which have the same amplitude and phase and nearly the same wave number:

$$f(x)=A\cos(k_1x+\delta)+A\cos(k_2x+\delta)\;. \tag{1}$$

To see what this function looks like, it is useful to introduce phasors and to write $f(x)$ as

$$f(x)=\mathrm{Re}\Big(Ae^{i(k_1x+\delta)}+Ae^{i(k_2x+\delta)}\Big)\;.$$

Now we go through some mathematical manipulations,

$$\begin{aligned}f(x)&=\mathrm{Re}\bigg(Ae^{i[(k_1+k_2)x/2+\delta]}\Big(e^{i(k_1-k_2)x/2}+e^{-i(k_1-k_2)x/2}\Big)\bigg)\\&=\mathrm{Re}\bigg(Ae^{i[(k_1+k_2)x/2+\delta]}2\cos\Big(\frac{k_1-k_2}{2}x\Big)\bigg)\;,\end{aligned}$$

which put $f(x)$ in the form

$$f(x)=2A\cos\Big(\frac{k_1-k_2}{2}x\Big)\cos\Big(\frac{k_1+k_2}{2}x+\delta\Big)\;. \tag{2}$$

To say that the two wave numbers are nearly the same means that the wave-number difference $\Delta k \equiv k_1 - k_2$ is (in magnitude) much smaller than the average wave number $\bar{k} \equiv (k_1 + k_2)/2$, i.e.,

$$|\Delta k| \ll \frac{k_1 + k_2}{2} \ .$$

Thus we can think of $f(x)$ in Eq. (2) as a wave $\cos(\bar{k}x + \delta)$ that has the average wave number $\bar{k}$ and the phase δ, but whose amplitude $2A\cos(\Delta k\, x/2)$ varies slowly and sinusoidally at half the wave-number difference. When this amplitude has its largest absolute value of $2A$, i.e., when $\Delta k\, x = n\pi$ where n is an *even* integer, the two waves superposed in Eq. (1) are said to interfere *constructively*; in contrast, when the amplitude is zero, i.e., when $\Delta k\, x = n\pi$ where n is an *odd* integer, the waves in Eq. (1) are said to interfere *destructively.* The functions $\pm 2A\cos(\Delta k\, x/2)$ form an *envelope* for the rapid oscillations of $f(x)$; the rapid oscillations with wave number $\bar{k}$ are bounded by this envelope.

The lesson here is that in a superposition of two monochromatic waves, destructive interference makes the wave go to zero at certain places. Perhaps if we use more than two monochromatic waves in the superposition, we can arrange the destructive interference so that the function $f(x)$ is nonzero only in an isolated region—*voilà*, a wave packet.

To see how this can be achieved, we will use what you already know about Fourier series—that any periodic function can be written as a superposition of cosines and sines. Specifically, suppose $f_L(x)$ is periodic over a length L. Then we can write $f_L(x)$ as the *Fourier series*

$$f_L(x) = \sum_{n=0}^{\infty} a_n \cos k_n x + b_n \sin k_n x \ , \tag{3}$$

where the wave numbers

$$k_n = 2n\pi/L \ , \qquad n = 0, 1, 2, \dots \ ,$$

are chosen so that we include in the series all the sinusoidal functions that are periodic over the length L. We can replace the trigonometric functions $\cos k_n x$ and $\sin k_n x$ by complex exponentials $e^{\pm i k_n x}$ and rewrite the Fourier series (3) as

$$f_L(x) = \sum_{n=-\infty}^{\infty} c_n e^{i k_n x} \ . \tag{4}$$

It is sensible to let this sum run over both positive and negative integers, thereby including, for any nonnegative n, both $e^{ik_n x}$ and $e^{-ik_n x} = e^{ik_{-n} x}$. The price we pay for this is that we have to remember that k_n can be either positive or negative. By using Euler's relation for the complex exponential,

$$e^{ik_n x} = \cos k_n x + i \sin k_n x \ ,$$

we can relate the two kinds of Fourier coefficients:

$$\begin{aligned} a_0 &= c_0 \ , & b_0 &= 0 \ , \\ a_n &= c_n + c_{-n} \ , & b_n &= i(c_n - c_{-n}) \ , \quad n = 1, 2, \dots \ . \end{aligned}$$

The separate monochromatic terms in Eq. (4) are called *Fourier components* of the function $f_L(x)$. We can find the *Fourier coefficient* c_n for the nth Fourier component—i.e., we can invert the Fourier series—by doing the integral

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} dx\, f_L(x) e^{-ik_n x} \ . \tag{5}$$

Now we're ready to see what kind of Fourier series is required to represent a function that is nonzero only over small part of each period. For specificity, let's consider a function defined on the interval $-L/2 \le x \le L/2$ by

$$f_L(x) = \begin{cases} 1 \ , & \text{for } |x| \le a/2, \\ 0 \ , & \text{for } a/2 \le |x| \le L/2. \end{cases}$$

This function is zero except that it has a bump of height 1 and width a centered at the origin. Of course, since $f_L(x)$ is periodic, the same bump is repeated at each integral multiple of L. For this function the Fourier coefficients (5) become

$$c_n = \frac{1}{L}\int_{-a/2}^{a/2} dx\, e^{-ik_n x} = \frac{\sin(k_n a/2)}{k_n L/2} = \frac{\sin(n\pi a/L)}{n\pi} \ . \tag{6}$$

We should record a couple of observations about these Fourier coefficients. First, for small values of $|n|$, for which the period (or wavelength, if you wish) $2\pi/|k_n|$ of the Fourier component is much bigger than a, the Fourier coefficients all have approximately the same value,

$$c_n \simeq \frac{a}{L} \ , \qquad \text{for } |k_n|a \ll 2\pi.$$

The reason is that the Fourier function $e^{ik_n x}$ doesn't have room even to begin to oscillate within the integral (6) and thus it can be replaced by its value at the origin, which is 1. Second, for large values of $|n|$, for which the period $2\pi/|k_n|$ is much bigger than a, the magnitude of the Fourier coefficients falls off as $1/|n|$. The reason is that in this situation the integral (6) averages over many periods of the Fourier function. The lesson is that to represent a function with periodic bumps of width a, separated by distance L, we need to have roughly equal contributions from the Fourier components with $|k_n|a \lesssim 2\pi$, with decreasing contributions from shorter-period Fourier components. The Fourier components interfere constructively within the bumps at each integral multiple of L and interfere destructively otherwise.

III. THE FOURIER TRANSFORM

To eliminate the periodic structure, we need to include even more Fourier components; for example, it should be clear that we have to include Fourier functions whose period is longer than L. We can do this by considering the function $f_L(x)$ to be defined on the central interval $-L/2 \le x \le L/2$ and taking the limit $L \to \infty$ while keeping a fixed. The limit pushes all the bumps except the central one out beyond infinity, leaving a function $f(x)$ with a single bump of width a centered at the origin:

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1\ , & \text{for } |x| \le a/2, \\ 0\ , & \text{for } |x| > a/2. \end{cases}$$

We now have to take the required limit in the Fourier series (4) and in the Fourier coefficient (6). Dealing with the Fourier series first, we write Eq. (4) as

$$f_L(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi/L} c_n e^{ik_n x} = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L c_n e^{ik_n x} \ . \tag{7}$$

Here $\Delta k = 2\pi/L$ is the difference between successive values of k_n. We now define a function $\tilde{f}(k)$ of wave number k by

$$\tilde{f}(k) \equiv \lim_{L\to\infty} L c_n = \lim_{L\to\infty} L c_{kL/2\pi} \ .$$

In taking this limit, we write the Fourier coefficient as $c_n = c_{kL/2\pi}$ and think of it as a function of k instead of n. Taking the limit in Eq. (6) gives

$$\tilde{f}(k) = \frac{\sin(ka/2)}{k/2} \ ,$$

and the limit of Eq. (7) is

$$f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L c_n e^{ik_n x} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k) e^{ikx} \ . \tag{8}$$

The function $\tilde{f}(k)$ is called the *Fourier transform* of $f(x)$. Equation (8) gives the single-bump function $f(x)$ as a continuous superposition of Fourier components, the Fourier coefficient for each component now being given by the continuous function $\tilde{f}(k)$.

The lesson here is nearly the same as above. To represent a *single* bump of width a requires a continuous superposition of monochromatic waves. The Fourier transform $\tilde{f}(k)$ tells how much and with what phase each monochromatic wave contributes to the superposition. The Fourier transform is roughly constant for small wave numbers satisfying $|k|a \lesssim 2\pi$ and then falls off as $1/|k|$ for larger wave numbers. You should notice that the relation between the width of the function and the effective width of the Fourier transform is an expression of the uncertainty principle.

It takes only a little work now to find the general relation between a function and its Fourier transform. We start with a periodic function $f_L(x)$, whose Fourier series is given by Eq. (4) and whose Fourier coefficients coefficients are given by Eq. (5). We then take the limit $L \to \infty$ so that the central interval $-L/2 \le x \le L/2$ occupies the entire real line. The resulting function $f(x)$ is related to the Fourier transform as in Eq. (8),

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k) e^{ikx} \;, \tag{9}$$

and the Fourier transform $\tilde{f}(k) \equiv \lim_{L\to\infty} Lc_n = \lim_{L\to\infty} Lc_{kL/2\pi}$ is now given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \;, \tag{10}$$

where we take the limit in Eq. (5). Equations (9) and (10) are called a *Fourier transform pair. The Fourier transform is the most important integral transform in physics. You should make yourself thoroughly familiar with it.*

You will often see the Fourier transform in slightly different guises. For example, the $1/2\pi$ in the k integral can be put in the x integral instead, or it can be distributed symmetrically as a $1/\sqrt{2\pi}$ in both the k and x integrals. I like the convention used here, because *all you have to remember is that the integration measure for wave number is always $dk/2\pi$.* As another example, when dealing with time and angular frequency instead of space and wave number, physicists almost universally use the opposite sign convention in the complex exponentials, i.e.,

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{f}(\omega) e^{-i\omega t} \qquad \text{and} \qquad \tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t}$$

(can you figure out why?). Engineers, on the other hand, tend to stick with the original sign convention when dealing with time and frequency.

Following is a list of properties of the Fourier transform.

1. Linearity: $h(x) = \alpha f(x) + \beta g(x) \iff \tilde{h}(k) = \alpha \tilde{f}(k) + \beta \tilde{g}(k)$,
2. Conjugation: $h(x) = f^*(x) \iff \tilde{h}(-k) = \tilde{f}^*(k)$,
3. Reality: $f(x) = f^*(x) \iff \tilde{f}(-k) = \tilde{f}^*(k)$,
4. Product: $h(x) = f(x)g(x) \iff \tilde{h}(k) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{f}(k')\tilde{g}(k-k')$,
5. Convolution: $h(x) = \int_{-\infty}^{\infty} dx'\, f(x')g(x-x') \iff \tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$,
6. Derivative: $h(x) = \frac{d^n f(x)}{dx^n} \iff \tilde{h}(k) = (ik)^n \tilde{f}(k)$.
7. Completeness: $f(x) = \delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')} \iff \tilde{f}(k) = e^{-ikx'}$.

You should be able to verify all of these properties.

Perhaps the most useful of the properties is the completeness property, so called because if the Fourier functions are a complete set of expansion functions, then we must have

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k) e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \int_{-\infty}^{\infty} dx'\, f(x') e^{-ikx'} = \int_{-\infty}^{\infty} dx'\, f(x') \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}$$

for all functions $f(x)$. The only way this can be true for all functions is to have

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')} = \delta(x-x')\ ,$$

which is the completeness property. The physical way to state the completeness property is as the inverse of the statement above, i.e., that the Fourier transform of a plane wave is a δ function:

$$f(x) = e^{ikx} \quad \Longleftrightarrow \quad \tilde{f}(k') = \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\delta(k-k')\ .$$

The only properties that might present some difficulty are the product and convolution properties, but even these are fairly easy to check. The product and convolution properties are inverses of one another. The product property says that the Fourier transform of a product of two functions is the *convolution* of the Fourier transforms of those two functions, whereas the the convolution property says that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the two functions. These two properties are important in the theory of dispersion and in the theory of signal processing. An important special case of the product property occurs when $g(x) = f^*(x)$, so that $h(x) = |f(x)|^2$:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 e^{ikx} = \tilde{h}(k) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{f}(k')\tilde{f}^*(k'-k)\ .$$

This special case is called *Parseval's relation.* When evaluated at $k=0$, Parseval's relation reduces to

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, |\tilde{f}(k)|^2\ .$$

Following are some examples of Fourier transform pairs.

$$f(x) = \begin{cases} e^{i\kappa x}\ , & \text{for } |x| \le a/2, \\ 0\ , & \text{for } |x| > a/2. \end{cases} \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{\sin(k-\kappa)a/2}{(k-\kappa)/2}\ ,$$

$$f(x) = \begin{cases} \cos\kappa x\ , & \text{for } |x| \le a/2, \\ 0\ , & \text{for } |x| > a/2. \end{cases} \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{1}{2}\left(\frac{\sin(k-\kappa)a/2}{(k-\kappa)/2} + \frac{\sin(k+\kappa)a/2}{(k+\kappa)/2} \right)\ ,$$

$$f(x) = \begin{cases} \sin\kappa x\ , & \text{for } |x| \le a/2, \\ 0\ , & \text{for } |x| > a/2. \end{cases} \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{1}{2i}\left(\frac{\sin(k-\kappa)a/2}{(k-\kappa)/2} - \frac{\sin(k+\kappa)a/2}{(k+\kappa)/2} \right)\ ,$$

$$f(x) = e^{-\gamma x} e^{i\kappa x} \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{2\gamma}{(k-\kappa)^2+\gamma^2}\ ,$$

$$f(x) = e^{-\gamma x}\cos\kappa x \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{1}{2}\left(\frac{2\gamma}{(k-\kappa)^2+\gamma^2} + \frac{2\gamma}{(k+\kappa)^2+\gamma^2} \right)\ ,$$

$$f(x) = e^{-\gamma x}\sin\kappa x \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{1}{2i}\left(\frac{2\gamma}{(k-\kappa)^2+\gamma^2} - \frac{2\gamma}{(k+\kappa)^2+\gamma^2} \right)\ ,$$

$$f(x) = e^{-\gamma x^2} e^{i\kappa x} \quad \Longleftrightarrow \quad \tilde{f}(k) = \sqrt{\frac{\pi}{\gamma}} e^{-(k-\kappa)^2/4\gamma}\ ,$$

$$f(x) = e^{-\gamma x^2}\cos\kappa x \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{1}{2}\sqrt{\frac{\pi}{\gamma}} \left(e^{-(k-\kappa)^2/4\gamma} + e^{-(k+\kappa)^2/4\gamma} \right)\ ,$$

$$f(x) = e^{-\gamma x^2}\sin\kappa x \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{1}{2i}\sqrt{\frac{\pi}{\gamma}} \left(e^{-(k-\kappa)^2/4\gamma} - e^{-(k+\kappa)^2/4\gamma} \right)\ ,$$

You should think about verifying these examples. Moreover, you should notice that in each example the function $f(x)$ can be thought of as a wave packet; the relation between the width of the wave packet and the width of its Fourier transform is an example of the uncertainty principle.

IV. SOLVING EQUATIONS USING THE FOURIER TRANSFORM

One of the most important applications of Fourier transforms in physics is to the problem of finding the solution of linear ordinary or partial differential equations. The derivative property of Fourier transforms allows one to transform a linear ordinary or partial differential equation into an algebraic equation or to transform a partial differential equation into an ordinary differential equation. Let's take a look at the latter technique by solving the two-dimensional scalar wave equation,

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \ . \tag{11}$$

The scalar function $f(x,t)$ could be the magnitude of the electric field for a linearly polarized plane electromagnetic wave propagating along the x axis. Now let's introduce the Fourier transform of $f(x,t)$ with respect to x only:

$$\tilde{f}(k,t) = \int_{-\infty}^{\infty} dx\, f(x,t)e^{-ikx} \ .$$

If we Fourier transform the wave equation (11) using the derivative property, we obtain an ordinary differential equation for $\tilde{f}(k,t)$,

$$\frac{d^2\tilde{f}(k,t)}{dt^2} + (vk)^2\tilde{f}(k,t) = 0 \ , \ . \tag{12}$$

This is an ordinary differential equation because the wave number k is just a parameter as far as the equation is concerned. The solution of Eq. (12) should be familiar:

$$\tilde{f}(k,t) = \tilde{f}_R(k)e^{-ikvt} + \tilde{f}_L(k)e^{ikvt} \ .$$

Here $\tilde{f}_R(k)$ and $\tilde{f}_L(k)$ are arbitrary functions of the wave number k. We get the solution of the wave equation by doing the inverse Fourier transform to find

$$f(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}(k,t)e^{ikx} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}_R(k)e^{ik(x-vt)} + \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}_L(k)e^{ik(x+vt)} \ .$$

We see that the general solution is a superposition of a wave traveling to the right and a wave traveling to the left:

$$f(x,t) = f_R(x-vt) + f_L(x+vt) \ .$$

Moreover, we see that the waves traveling to the right and to the left are completely arbitrary and that they are built up from arbitrary superpositions of monochromatic waves. The tool for handling such superpositions is the Fourier transform.

V. GROUP VELOCITY

Suppose now that one has a wave packet traveling to the right,

$$f(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}_R(k)e^{ik[x-v(k)t]} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{f}_R(k)e^{i(kx-\omega t)} \ . \tag{13}$$

The difference from the above discussion is that we now allow $\omega/k \equiv v(k)$, the *phase velocity* of the wave, to be a function of wave number k (or of angular frequency ω), a phenomenon called *dispersion*. Such a dispersive wave packet, though *not* a solution of the wave equation, does correspond to many physical situations. One example is the propagation of matter waves in nonrelativistic quantum mechanics, where $\hbar\omega = p^2/2m = \hbar^2k^2/2m$, which implies that $\omega = \hbar k^2/2m$ and $v(k) = \omega/k = \hbar k/2m = p/2m$. Another example is the propagation of electromagnetic waves in dispersive materials.

We consider the situation where the Fourier amplitude $\tilde{f}_R(k)$ has substantial support only over a narrow interval of wave numbers centered at wave number k_0. We assume that the width Δk of the narrow interval is so small that $v(k)$ doesn't vary much across the interval. This allows us to expand $\omega(k)$ in a Taylor series about the center of the interval:

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k=k_0} (k - k_0) + \frac{1}{2} \left.\frac{d^2\omega}{dk^2}\right|_{k=k_0} (k - k_0)^2 + \text{(higher-order terms)} \ . \tag{14}$$

We shall see that the linear term describes a wave packet that moves to the right with the *group velocity*

$$v_g \equiv \left.\frac{d\omega}{dk}\right|_{k=k_0} \ . \tag{15}$$

The quadratic and higher-order terms describe spreading, or dispersion, of the wave packet. In this discussion we shall neglect the dispersive terms, retaining only the linear term in the Taylor expansion.

Notice that no matter how fast $v(k)$ varies with k, we can always consider a wave packet that is narrow enough in k that we can justify keeping only the linear term in the Taylor expansion. This means, however, in accordance with our discussion in Sec. III, that the wave packet has an extent $a \gtrsim 2\pi/\Delta k$ in position. If we are forced to make Δk very small to justify keeping only the linear term, the wave packet will become very extended in position. Certainly not all wave packets are of this sort, and thus not all wave packets can be treated in the group-velocity approximation.

Returning now to the Taylor expansion (14), let's retain only the linear term and plug the expansion into the expression (13) for the wave packet:

$$\begin{aligned} f(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_R(k) e^{i\left(kx - [\omega(k_0) + v_g(k-k_0)]t\right)} \\ &= e^{-i[\omega(k_0) - v_g k_0]t} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_R(k) e^{ik(x - v_g t)\big)} \\ &= e^{-i[\omega(k_0) - v_g k_0]t} f(x - v_g t, 0) \ . \end{aligned}$$

Aside from the time-dependent phase, this is, as promised, a wave packet that propagates to the right with a speed given by the group velocity. Each Fourier component moves through the wave packet at its phase velocity $\omega/k = v(k)$, but the various Fourier components interfere so that the whole wave packet moves at the group velocity. Notice that for a nonrelativistic matter wave, the group velocity $v_g = d\omega/dk = \hbar k/m = p/m$ is in accord with classical notions of the velocity of a particle.