Chapter 4

Delay: Approximation and Optimization

4.1 Introduction

While studying magnitude approximations in Chapter 3, real rational functions were selected in order to achieve desirable attenuation in the stop band and pass band. Not much attention was paid on the unintentional occurrence of phase shift or delay. It has been observed that this delay is neither the same for different approximations nor is it constant. In fact, filter delay is highly dependent on the type of approximation, order of the filter and its parameters. As an example, normalized delay for a fourth-order maximally flat, Chebyshev, inverse Chebyshev and Cauer or elliptic filter, with equal order of pass band attenuation in each case, is shown in Figure 4.1. It can be observed that the delay performance in the maximally flat case is the closest to the ideal requirement of constant delay with frequency. Performance is the worst for the elliptic filter and intermediate for the Chebyshev and the inverse Chebyshev cases. It is important to note that though Figure 4.1 is only a representative sketch, the nature of the functions remain the same: maxima of the delay occur near the pass band edge and relative maxima are dependent on filter specifications in the pass and stop bands.

It is important to have the group delay as constant or the phase response of the filter as linear with frequency. This requirement for filters is necessary in order to process the magnitude and because in many applications, the amount of delay needs to be controlled at the same time. For example, in a telephonic conversation through satellite, generally, the communication channel is active one way and the listener speaks only when the speaker from the other side stops. However, if the delay is too large, two-way conversation will become difficult and sometimes overlapping.

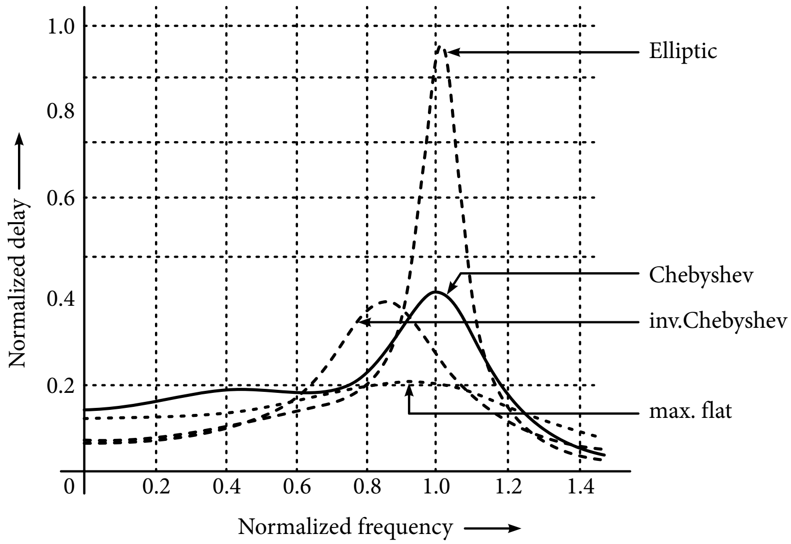

Figure 4.1 Comparison between delay performances of fourth-order elliptic, Chebyshev, inverse Chebyshev, and maximally flat responses.

From this brief discussion, it is clear that in order to avoid distortion due to different phase/group delay in a filter, an ideal transfer function should have its delay constant with frequency.

In Section 4.2, delay is defined. Its relation with the phase change in a filter section and the normalized transfer function of a filter realizing delay are also discussed in detail. As Bessel polynomials are used in realizing such transfer function, these polynomials are reviewed and the Bessel filter design shown in the next section. One of the main functions of Bessel filters is *delay equalization*, that is, trying to make as much uniform delay as possible in the pass band. Such an optimization is done through first-order, second-order, or still higher-order filters; the utilization of these filters is shown in Sections 4.5 to 4.7.

4.2 Delay

If a sine wave with $V_{in} = V_m \sin(\omega t + \varphi)$ is applied to a network (say a filter section) which contains frequency-dependent reactance elements like capacitors or inductors, its output magnitude and phase changes depending on the characteristics of the filter. However, if the magnitude of the output voltage V_{out} is the same as the input, it can be written as:

$$V_{out} = V_{in} \sin(\omega t + \varphi + \theta) \tag{4.1a}$$

$$= V_{in} \sin(\omega(t - D) + \theta) \tag{4.1b}$$

In equation (4.1), D is obviously the delay, which is associated with phase shift in the following relation:

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$$D = -(\varphi/\omega) \tag{4.2}$$

In equation 4.2, D (in seconds) is a frequency dependent term. Hence, if a signal comprising components having different frequencies is applied to a filter section, these signal components will be delayed by different amounts. Figure 2.14(b) of Chapter 2, redrawn here as Figure 4.2, shows how the phase changes from the input to the output for a second-order LPF section. Delay D can be evaluated at different frequencies employing the expression in equation (2.42), which represents Figure 4.2. Delay can be evaluated not only for LPFs, but for any type of filter from its phase variation plot. It is important to note that a signal comprising a large number of components will produce output signals with a different amount of phase shift/delay resulting in distorted signals. The problem can be solved by observing that the delay D is inversely proportional to the frequency ω. Hence, if the delay, expressed as a derivative of phase shift (equation (4.3)), is a constant, all components of a signal shall be delayed by the same amount. In that case the final output signal will not be distorted, but only delayed by certain amount.

$$D(\omega) = d\varphi/d\omega \tag{4.3}$$

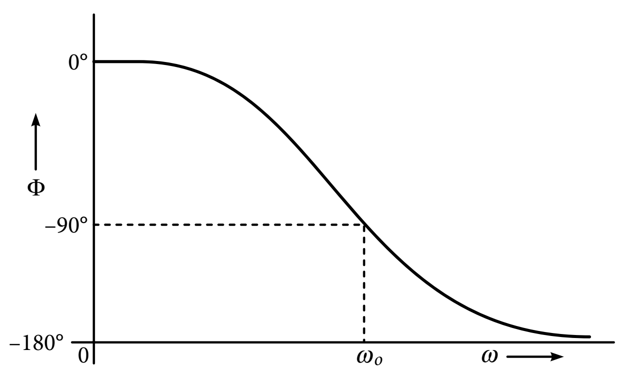

Figure 4.2 Change in phase from input to output for a second-order low pass section.

It is to be noted that the delay $D = -(\varphi/\omega)$ in equation (4.2) is called *phase delay* and the delay given by equation (4.3) is called *group delay*. In practice, the term $D(\omega)$ of equation (4.3) is called *delay* and φ is termed *phase shift*. Hence, in this book, delay $D(\omega)$ will be considered the derivative of phase with frequency.

The output voltage in equation (4.1) and its input in phasor form are represented in the following equation.

$$V_{out} = V\angle(\varphi - \omega D) \text{ and } V_{in} = V\angle\varphi \tag{4.4a}$$

Hence, the transfer function of the filter will be as follows:

$$H(j\omega) = V_{out}(j\omega)/V_{in}(j\omega) = 1.0e^{-j\omega D} \tag{4.4b}$$

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Normalizing equation (4.4b) with a frequency $\omega_d = (1/D)$, $H(j\omega)$ will become $e^{-j\omega/\omega_d} = e^{-j\Omega}$. Generalizing the transfer function in the complex frequency s plane by replacing $j\omega$ with s, the desired transfer function shall be as follows:

$$H(s) = V_{out}(s)/V_{in}(s) = e^{-s} \tag{4.5}$$

A number of methods are available for obtaining a real rational function for the physical realization of the transfer function of equation (4.5) with normalized delay $D = 1$ s. However, the most commonly used method is the *Bessel–Thomson (BT) method*, employing the Bessel function [4.1, 4.2]. Therefore, before discussing the BT method of filter design, the Bessel function and some of its basic properties are discussed in brief.

4.3 Bessel Polynomial

Using the known identity $e^{-s} = \sin h\ s + \cos h\ s$, the transfer function in equation (4.4) can be expressed in terms of hyperbolic functions. As the denominator of $H(s)$ is the sum of sin h s + cos h s, it can be expressed in terms of the series expansion of hyperbolic functions: an infinite polynomial in the s plane. It was shown by Storch (1954) that the infinite series for the hyperbolic can be truncated after the nth term and the truncated terms give the maximally flat delay at $\omega = 0$ [4.3]. In addition, such a denominator polynomial was obtained with a class of Bessel polynomials given by the following recursive formula.

$$B_n = (2n - 1)B_{n-1} + s^2 B_{n-2} \tag{4.6}$$

In general, the BT function is formed in such a way that the selected numerator results in $H_n(j0) = 1$. Hence, the form of the transfer function is as follows:

$$H_n(s) = B_n(0)/B_n(s) \tag{4.7}$$

From equation (4.6), the Bessel polynomial for order n can easily be found. For example, the following relations are given up to $n = 4$.

$$B_0 = 1,\ B_1 = s + 1,\ B_2 = s^2 + 3s + 3,\ B_3 = s^3 + 6s^2 + 15s + 15,$$
$$B_4 = s^4 + 10s^3 + 45s^2 + 105s + 105. \tag{4.8}$$

We know that there are different methods available for filter realization. However, in a commonly used approach, the filter design uses pole frequency ω_o and pole Q, which are related with the pole location of the transfer function. In the present case, it is required to factorize the Bessel polynomial in order to find the location of poles to find its roots. Unfortunately, there is no simple method to find the polynomial's roots and computer methods have to be employed. Data tables giving roots of the polynomial, location of poles and resultant pole frequency ω_o

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and pole Q have been made available for different values of n. Table 4.1 gives the values of ω_o and pole Q for $n = 1$ to 6, which has been obtained from the calculated values of the roots of the Bessel polynomial. Roots of the Bessel polynomial can easily be obtained from root finding programs. Along with Table 4.1, Tables 4.2 and 4.3 give the factored form of $B_n(s)$ for $n = 1$ to 6. Pole-Q and pole location are also given.

Table 4.1 Roots of the Bessel polynomial $B_n(s) = 0$ for $n = 1$ to 6

N	Roots					
1	–1.000000					
2	–1.500000	$\pm j0.866025$				
3	–2.322185	–1.8389073	$\pm j1.754381$			
4	–2.896210	$\pm j0.867234$	–2.1037894	$\pm j2.657418$		
5	–3.646738	–3.3519561	$\pm j1.742661$	–2.3246743	$\pm j3.571022$	
6	–4.248359	$\pm j0.867509$	–3.7357084	$\pm j2.626272$	–2.5159322	$\pm j4.492673$

Table 4.2 Factored form of $B_n(s)$ for $n = 1$ to 6

n	$B_n(s)$	
1	$s + 1$	1
2	$s^2 + 3s + 3$	3
3	$(s^2 + 3.67782s + 6.45944)(s + 2.232219)$	15
4	$(s^2 + 5.79242s + 9.14013)(s^2 + 4.20758 + 11.4878)$	105
5	$(s^2 + 6.70391s + 14.2725)(s^2 + 4.64943s + 18.15631)(s + 3.64674)$	945
6	$(s^2 + 8.49672s + 18.8013)(s^2 + 7.74142s + 20.85282)(s^2 + 5.03186s + 26.51402)$	10,395

Table 4.3 ω_o, pole-Q and real pole σ for the transfer function $H_n(s)$ of equation (4.7)

n	ω_o, and pole-Q are in pairs, real pole $(-\sigma)$ is a single entry			$B_n(s)$
2	1.732;0.577			3
3	2.542;0.691	2.322		15
4	3.023;0.522	3.389;0.806		105
5	3.778;0.564	4.261;0.916	3.647	945
6	4.336;0.510	4.566;0.611	5.149;1.023	10,395

Before moving on to the design method of BT filters, a brief comparison with Butterworth filters is useful. Figure 4.3 shows the variation of the normalized delay D_n with ω up to $n = 10$ in BT filters. It is observed that D_n is maximally flat at $\omega = 0$ with magnitude unity. The delay remains flat for even higher values of ω for larger values of n.

Figure 4.4 shows the attenuation variation of Butterworth filters ($n = 4$ and 8) and the BT approximation filter which also leads to low pass characteristics. Rate of attenuation for both the responses at high frequencies is 20 n dB/dec, but BT response has a much greater transition band and attenuation is at a much slower rate at lower frequencies.

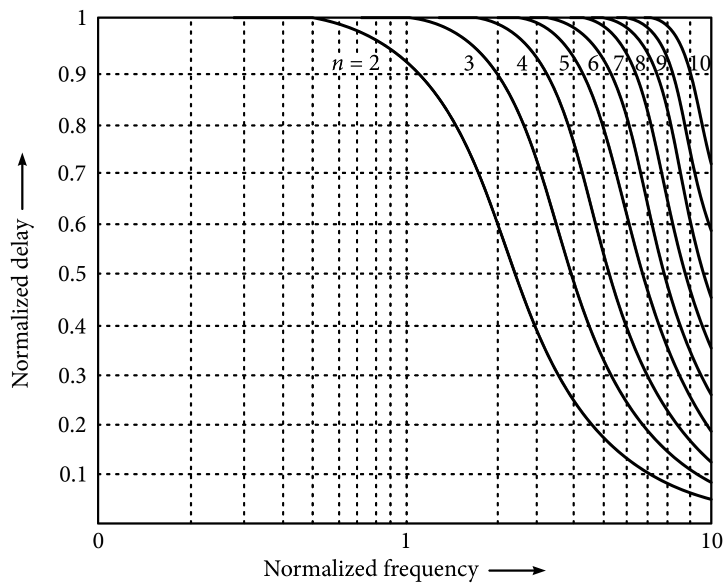

Figure 4.3 Normalized delay in Bessel–Thomson filters of order 2 to 10.

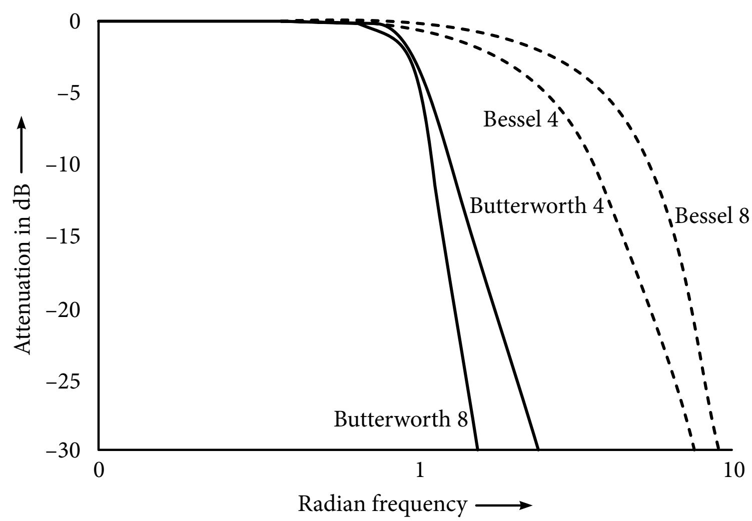

Figure 4.4 Comparison between Bessel–Thomson and Butterworth characteristics for filter of order 4 and 8.

4.4 Design of Bessel Filters

While designing magnitude-approximated filters, it is necessary to determine the values of the attenuation in the pass band, the stop band, and the band edge frequencies. However,

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for the delay-approximated filter, the amount of permissible deviation in delay compared to the ideal value $D = 1$, from $\omega = 0$ to a certain frequency (say ω_1) is the standard form of specification. However, for the Bessel filter, no closed form solution is available to find the order of the filter for a given delay deviation. Using computer programs, design curves (Figure 4.5(a)) have been obtained showing permissible error (in percent) with normalized frequency as function of the order of the BT filter. Such design curves are used to find the required order of the BTF for the permissible maximum delay error. If, along with the delay requirements, there is constraint on the magnitude as well, curves in Figure 4.5(b) are also used: these curves show the relation between permissible attenuation and order n. After finding the filter order (selecting the greater n obtained from the two specifications), computer program generated pole values or pole frequency ω_o and pole-Q of second-order sections are obtained. Filters are then realized using a suitable network employing any selected approach.

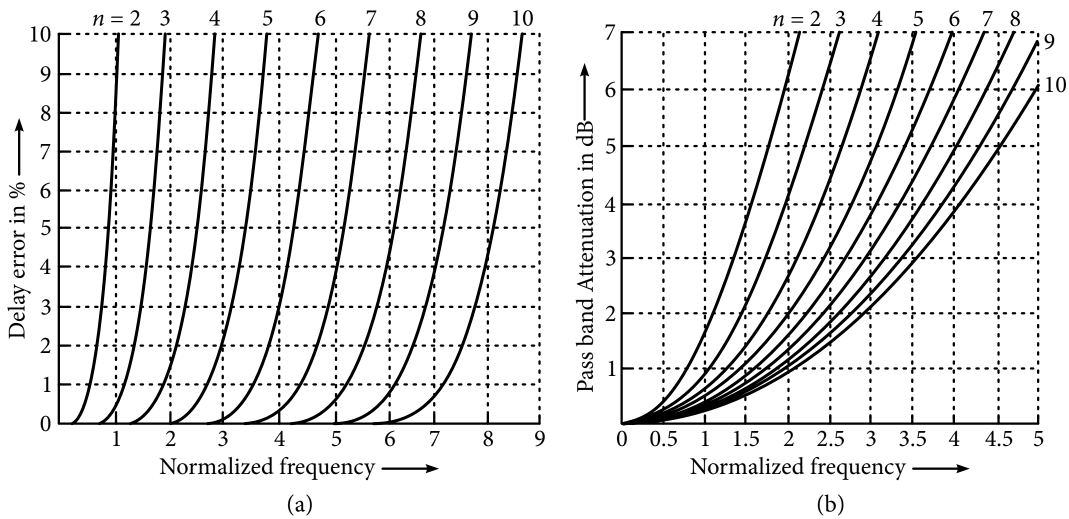

Figure 4.5 Design curves for obtaining the degree n of the Bessel–Thomson filter to meet (a) the maximum delay error and (b) permissible attenuation in pass band.

It can be observed from Figure 4.4 that the use of BT approximation for satisfying attenuation specification requires higher-order filters compared to the case when it is used for approximating delay specification. The problem becomes worse, that is, the required order of the filter becomes larger, if the transition band is narrow or the stop band has to have larger attenuation. It is because of this reason, and the involved process of finding filter networks, that the use of BT approximation only is limited. Other alternatives, for example, the use of a cascading magnitude-approximated filter section and a delay equalizer, are preferred. Such alternatives are discussed in the next section.

Example 4.1: Find the order and the transfer function of a delay filter which can provide delay $D = 200$ μs with a permissible error of 10% in delay at the frequency $\omega_1 = 18$ krad/s. The permissible loss in magnitude at a frequency of $\omega_2 = 10$ krad/s is 2.0 dBs.

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Solution: The order of the BTF shall be found using Figures 4.5(a) and (b). For the utilization of these figures, normalized frequency Ω_c is to be found. Its value is obtained as: $\Omega_c = 1/D = 1/200\ \mu s = 5$ krad/s.

Hence, normalized pass band edge frequency (for delay) $\omega_1 = 3.6$ and normalized pass band edge frequency (for magnitude) $\omega_2 = 2$. For normalized $\omega_1 = 3.6$ with 10% allowable deviation in delay required, order is $n = 5$ from Figure 4.5(a). For normalized $\omega_2 = 2$, the required filter order from Figure 4.5(b) is $n = 5$. Therefore, the selected order of the BTF is 5 for which the transfer function is as follows from Table 4.2.

$$\begin{aligned} H(s) &= 945/\{(s^2 + 6.7039s + 14.2725)(s^2 + 4.64943s + 18.1563)(s + 3.64674)\} \\ &= 945/(s^5 + 15s^4 + 105s^3 + 420s^2 + 945s + 945) \end{aligned} \tag{4.9}$$

Example 4.2: Design a delay filter with specifications as given in Example 4.1.

Solution: A fifth-order filter will need one first-order and two second-order sections. From equation (4.9), pole of the first-order section is at $s = -3.6467$. A simple OA–RC (operational amplifier with resistors and capacitors) circuit shown in Figure 4.6(a) will have the following transfer function:

$$H_0(\Omega) = (1/RC)/(s + 1/RC) \tag{4.10}$$

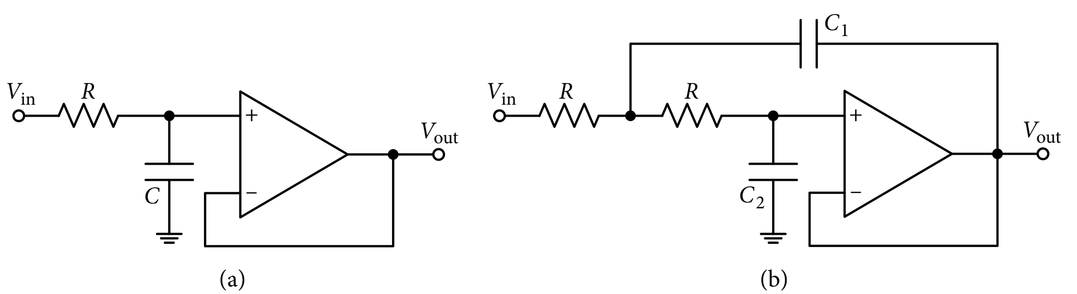

Figure 4.6 (a) Active first-order low pass as a first-order passive filter followed by a buffer and (b) a second-order low pass section.

For normalized 3 dB frequency of 1 rad/s and a dc gain of unity, the elements will be as follows:

$$C = 1\ \text{F and } R = 0.2742\ \Omega \tag{4.11}$$

For the remaining two second-order sections, the LP circuit shown in Figure 4.6(b) is selected (any other second-order circuit can also be selected), for which the transfer function is given as follows:

$$H(s) = \frac{(G_1G_2/C_1C_2)}{s^2 + \left(\frac{G_1 + G_2}{C_1}\right)s + \frac{G_1G_2}{C_1C_2}} \tag{4.12}$$

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From equation (4.12), the following expressions of the parameters are easily obtainable:

Voltage gain at dc equals unity (4.13a)

$$\omega_o = \left(\frac{G_1G_2}{C_1C_2}\right)^{1/2} \text{ and } \frac{\omega_o}{Q} = \frac{G_1+G_2}{C_1} \tag{4.13b}$$

From Table 4.1 or Table 4.3 value of the required two sets of parameters for $H_1(s)$ and $H_2(s)$ are as follows:

$$\omega_{o1} = 3.778,\ Q_1 = 0.564 \text{ and } \omega_{o2} = 4.261,\ Q_2 = 0.916 \tag{4.14}$$

Selecting C_1 = 1F and C_2 = 0.1 F, equation (4.13b) gives normalized conductance values for the two respective circuits as follows:

$$G_{11} = 6.4776,\ G_{12} = 0.22034 \text{ and } G_{21} = 4.2109,\ G_{22} = 0.44075 \tag{4.15}$$

To bring the resistances to a practical level, impedance scaling factor of 10^3 is used. For frequency normalization, the scaling factor of Example 4.1 (5×10^3) is used. The respective element values for the two second-order sections are as follows:

$$R_{11} = 0.1543 \text{ k}\Omega,\ R_{12} = 4.538 \text{ k}\Omega,\ C_{11} = 0.2\ \mu\text{F and } C_{12} = 0.02\ \mu\text{F} \tag{4.16a}$$

$$R_{21} = 0.23747 \text{ k}\Omega,\ R_{22} = 2.2688 \text{ k}\Omega,\ C_{21} = 0.2\ \mu\text{F and } C_{22} = 0.02\ \mu\text{F} \tag{4.16 b}$$

Similarly, for $H_0(s)$ values of elements after de-normalization are as follows:

$$R = 0.2742 \text{ k}\Omega \text{ and } C = 0.02\ \mu\text{F} \tag{4.17}$$

The complete fifth-order delay filter is shown in Figure 4.7: here the three sections are connected in cascade. Figure 4.8 shows the simulated group delay response of the filter, where its value at low frequencies is 200 μs. It gives the delay D at 2.873 kHz (18.085 krad/s) as 185.3 μs:

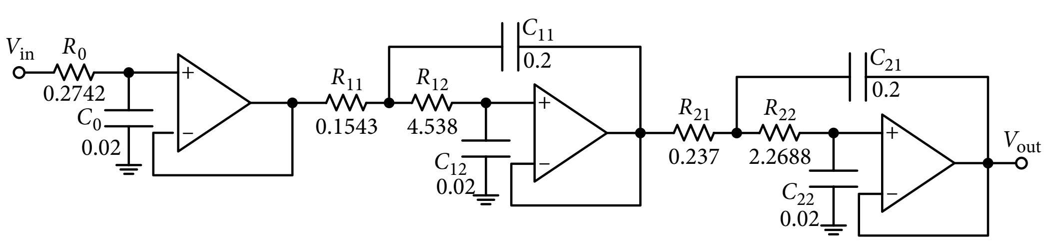

Figure 4.7 Fifth-order delay filter for Example 4.2. All resistors are in kΩ and capacitors are in μF.

Value of D is well within 10% of the specification D = 200 μs up to 18 krad/s. Figure 4.8 also shows the magnitude response, where the voltage gain is 0.866 at 1.272 kHz (8 krad/s). Hence, attenuation is 1.24 dB, which is also satisfied being less than 2 dB of specification.

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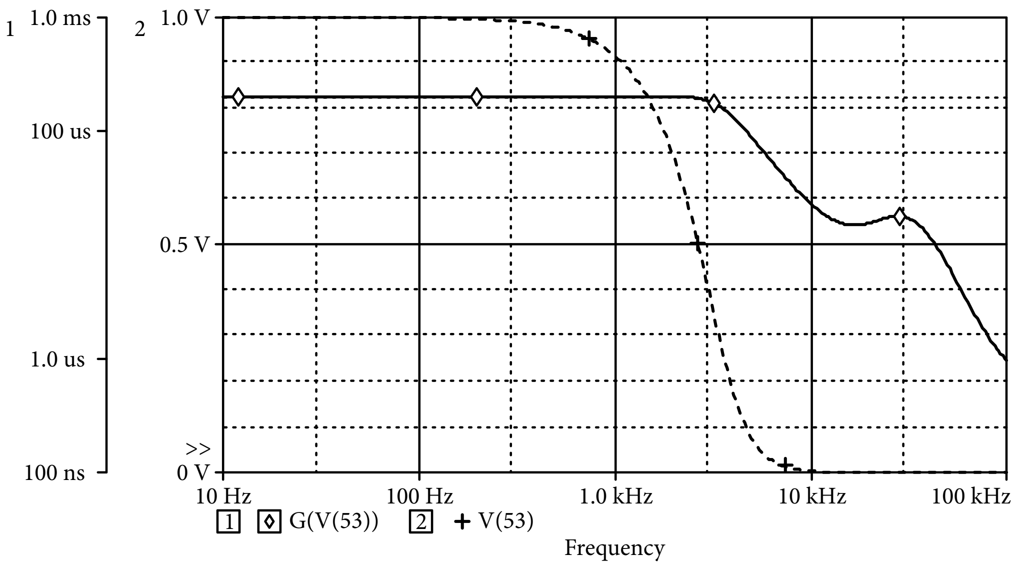

Figure 4.8 Magnitude and delay variation of the fifth-order BT filter for Example 4.2.

4.5 Delay Equalization

By now, the advantages and limitations of magnitude-approximated and delay-approximated filters are obvious. With magnitude-approximation filters, it is not possible to obtain constant delay at all frequencies; whereas for delay/phase-approximated filters, not only is the magnitude roll-off very slow, its design is also rather involved. Simultaneous satisfaction of magnitude and delay mostly requires extremely high-order filters; hence, they are expensive if only BTF is used. In many applications, especially for signals having a sharp transition in the time domain, like a pulse or a pulse train, not only should the magnitude be retained, equal delay is necessary for a very wide range of frequencies. If a sufficiently high-order filter is not used, the pulse output will be distorted. One preferred solution to the problem, as suggested in the last section, is to cascade a magnitude-approximated filter, which satisfies the magnitude response, with an AP (all pass) filter that least affects the magnitude with frequency but controls delay as per the requirement. The simple arrangement is shown in Figure 4.9. Figure 4.10 shows a typical combination of delay provided by a magnitude-approximated filter and an AP filter that is almost constant in the pass band range. As phase or delay equalizers generally use first- or second-order AP sections, their arrangement will be discussed in the following sections.

4.6 Delay Equalization Using First-order Sections

For less challenging cases of delay equalization, a first-order section with the following transfer function can be employed.

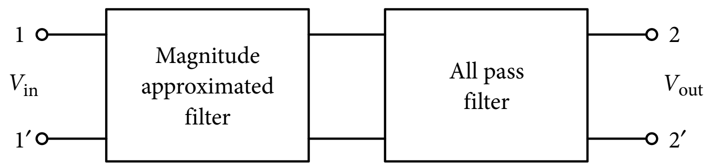

Figure 4.9 Cascading a magnitude-approximated filter and an all pass filter.

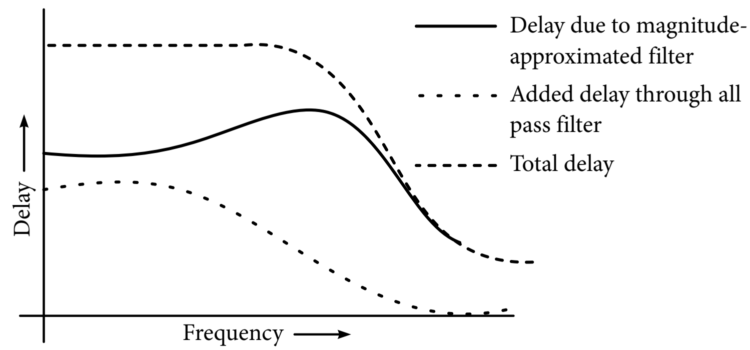

Figure 4.10 Obtaining a constant delay over most of the pass band with the help of an all pass filter.

$$H_{AP1} = k\frac{s - a_0}{s + a_0} \tag{4.18}$$

For sinusoidal inputs, magnitude and phase for the first-order transfer function for equation (4.18) are as follows:

$$|H_{AP1}(j\omega)| = k \text{ and } \varphi_1 = -2\tan(\omega/a_0) \tag{4.19}$$

Corresponding delay is obtained as:

$$D_1 = -(d\varphi_1/d\omega) = (2/a_0)/\{1 + (\omega/a_0)^2\} \tag{4.20}$$

Magnitude of the delay D_1 at $\omega = 0$ is $(2/a_0)$ and goes on decreasing monotonically with ω as given by equation (4.20) and illustrated in the next example.

Example 4.3: Figure 4.11(a) shows a first-order AP section realizing one real pole and one real finite zero. Its transfer function is as follows:

$$H_{AP1}(s) = \frac{V_{out}}{V_{in}} = \frac{s - (R_2/R_1R_3C)}{s + (1/R_3C)} \tag{4.21}$$

In order to equalize the magnitude of real pole and zero at $s = |a_0|$, a convenient choice is $R_1 = R_2 = R_3 = 1\Omega$ and $C = 1/a_0$. Frequency scaling by 10^4 and impedance scaling by 10^3 is done to bring component values within the practical range: $C = 0.1$ μF and $R_1 = R_2 = R_3 = 1$ kΩ; the values are also shown in Figure 4.11(a). Simulated response for the AP filter is shown in Figure 4.11(b), and some of the delay values with respect to frequency are shown in Table 4.4.

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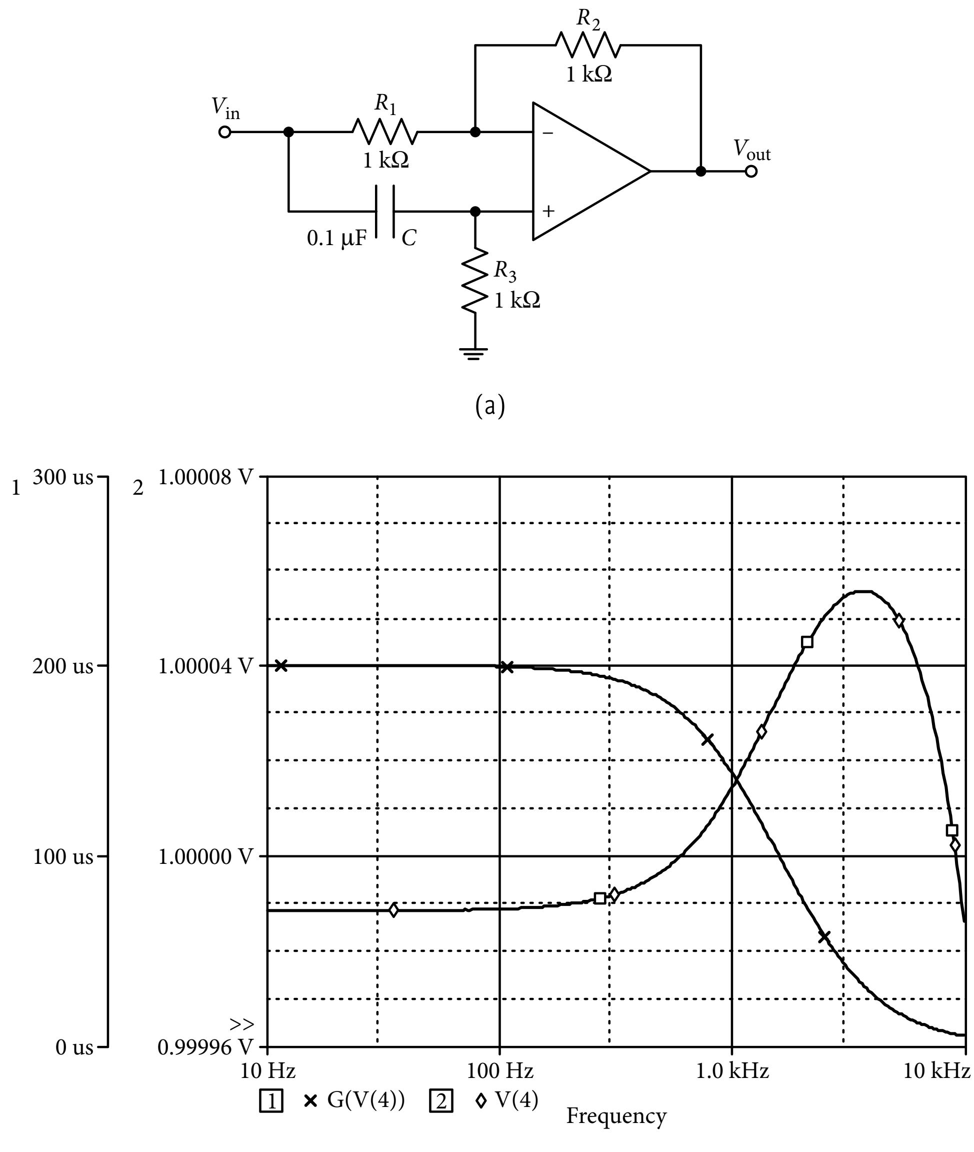

Figure 4.11 (a) A first-order all pass section for Example 4.3. (b) Variation of delay and magnitude for the first-order all pass filter of Figure 4.11(a).

Table 4.4 Variation of delay for the first-order AP filter of Figure 4.11(a)

Freq(Hz)	10	100	200	400	600	800	1 k	1.3 k	1.5 k	1.8 k	2 k	3 k	5 k
Delay(μs)	200.3	199.5	197	188.3	175.1	160.1	143.7	120.2	105	87.5	78.5	44.5	18.7

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The shape of the delay curve of the first-order AP filter verifies the statement made earlier that it decreases monotonically with frequency. It is thus obvious that the delay equalization through the use of a first-order AP section is limited to the cases where delay to be added is highest at $\omega = 0$ and decreases with ω. Moreover, there is only one parameter a_0, which decides the amount of introduced delay; hence, delay optimization is less flexible and if delay optimization is done without using a computer program, it will require some trial and error. In this example, as the selected RC product = 10^{-4}, theoretical value of delay $D = 0.20$ ms; the simulated result is in conformity with this value.

Example 4.4: Design a third-order LP filter using Chebyshev approximation with a 0.5 dB ripple in the pass band and plot its delay response. Use a frequency scaling factor of 10^4 and an impedance scaling factor of 10^3. Connect a suitable first-order AP filter in cascade with the LP filter and observe its effect on the delay differential of the overall composite filter.

Solution: Pole locations of a third-order Chebyshev filter with 0.5 dB pass band ripples are obtained from Table 3.3, which are as shown here:

$$s_1 = -0.62565,\ s_{2,3} = -0.3132 \pm j1.0219 \tag{4.22}$$

The normalized transfer function expression is obtained from equation (4.22) as:

$$H_{\text{ch0}} = \frac{0.6265}{(s+0.6265)}\frac{1.1424}{(s^2+0.6264s+1.1424)} \tag{4.23}$$

The transfer function H_{ch0} is realized as a cascade of the following first-order and second-order functions.

$$H_{\text{ch1}} = \frac{0.6265}{(s+0.6265)} \tag{4.24}$$

$$H_{\text{ch2}} = \frac{1.1424}{(s^2+0.6264s+1.1424)} \tag{4.25}$$

For the first-order section, the circuit shown in Figure 4.6(a) is used and comparison of equation (4.10) with equation (4.24) gives normalized element values as

$$R = 1.0\ \Omega \text{ and } C = 1/0.6265\ \text{F} \tag{4.26}$$

For the second-order section, the circuit shown in Figure 4.6(b) is used and comparison of the equation (4.25) with equations (4.12) and (4.13) gives the following normalized element values:

$$C_1 = 1\ \text{F},\ C_2 = 0.05\ \text{F},\ R_1 = 9.027\ \Omega \text{ and } R_2 = 1.939\ \Omega \tag{4.27}$$

A frequency scaling by 10^4 and an impedance scaling by 10^3 are applied on both the first- and second-order functions, and the cascaded third-order filter circuit is shown in Figure 4.12. Its

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magnitude and delay response are shown in Figure 4.13 and 4.14, respectively. The peak delay at pass band edge frequency is 0.376 ms and the lowest delay at 665 Hz is 0.194 ms. Hence, the delay differential in the designed third-order filter is 0.182 ms or 49.4% which is expected to be reduced by cascading a first-order AP filter of the type shown in Figure 4.11(a). Since we are not using a computer program to calculate the parameters of the AP filter, we employ trial and error. Three different values of the normalized parameter $a_0 = 0.8$, 1.0 and 1.25 are tried for the transfer function of equation (4.18). For the AP of Figure 4.11(a), normalized resistances R_1, R_2, and R_3 of 1 Ω each with $C = 1.25$ F, 1 F, and 0.8 F, respectively, for the three cases of a_0 are used. Using the same frequency and impedance scaling factors of 10^4 and 10^3 on the RC elements, the respective de-normalized a_0 and introduced delays at dc is easily obtained. The overall delay responses in the three cases are simulated using PSpice which are also shown in Figure 4.14. There is an improvement in the delay differential as shown in Table 4.4. Obviously, $a_0 = 1.25$ gives a better result; however, the total delay is still not constant because of the limitation of the first-order section in having only one parameter that controls the shape of the added delay.

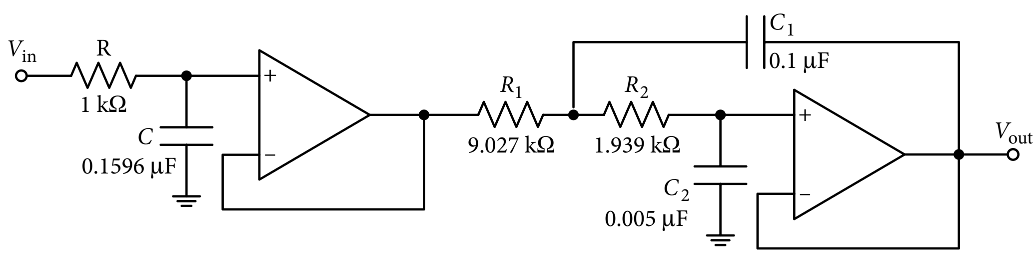

Figure 4.12 Third-order low pass Chebyshev filter for Example 4.4.

Table 4.5 Effect of using a first-order AP section in Example 4.4

Normalized capacitance value (F)	De-normalized a_0	Introduced delay (ms)	Delay differential (%)
1.25	0.8×10^4	0.25	22
1.0	1.0×10^4	0.20	25
0.8	1.25×10^4	0.16	29.5

4.7 Delay Equalization Using Second-order Sections

The transfer function of a second-order AP section in terms of pole frequency ω_o and pole-Q are given as:

$$H_2(s) = k\frac{s^2 - (\omega_o / Q)s + \omega_o^2}{s^2 + (\omega_o / Q)s + \omega_o^2} \tag{4.28}$$

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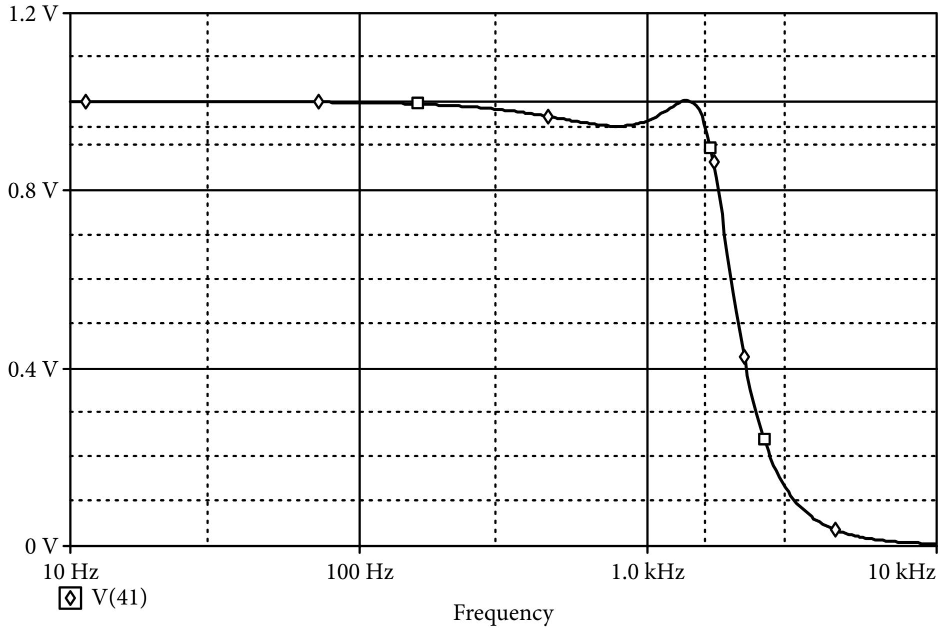

Figure 4.13 Magnitude response of the third-order Chebyshev filter of Figure 4.12.

Obviously, the gain constant k of equation (4.28) needs to be unity if the gain of the overall filter section is to remain unchanged. The important advantage of the second-order section is in its flexibility as there are two parameters, ω_o and Q, available. We can vary the shape of the delay versus frequency curve of the AP section more freely. Using normalized frequency $s_n = (s/\omega_o)$ and $k = 1$, the normalized AP function will be modified as follows:

$$H_2(s_n) = \frac{s_n^2 - s_n(\omega_n / Q) + 1}{s_n^2 + s_n(\omega_n / Q) + 1} \tag{4.29}$$

Replacing s_n by $j\omega_n$, the phase shift of the second-order function is calculated as:

$$\varphi_2(\omega_n) = -2\tan\left(\frac{\omega_n / Q}{1 - \omega_n^2}\right) \tag{4.30}$$

It corresponds to the delay in the following way:

$$D_2(\omega_n) = -\frac{d\varphi_2(\omega_n)}{d\omega} = \left\{\frac{d\varphi_2(\omega_n)}{d\omega_n}\right\}\left(\frac{d\omega_n}{d\omega}\right)$$

$$= \frac{1}{\omega_o}\left\{\frac{(2/Q)(1+\omega_n^2)}{(1-\omega_n^2)^2 + (\omega_n / Q)^2}\right\} \tag{4.31}$$

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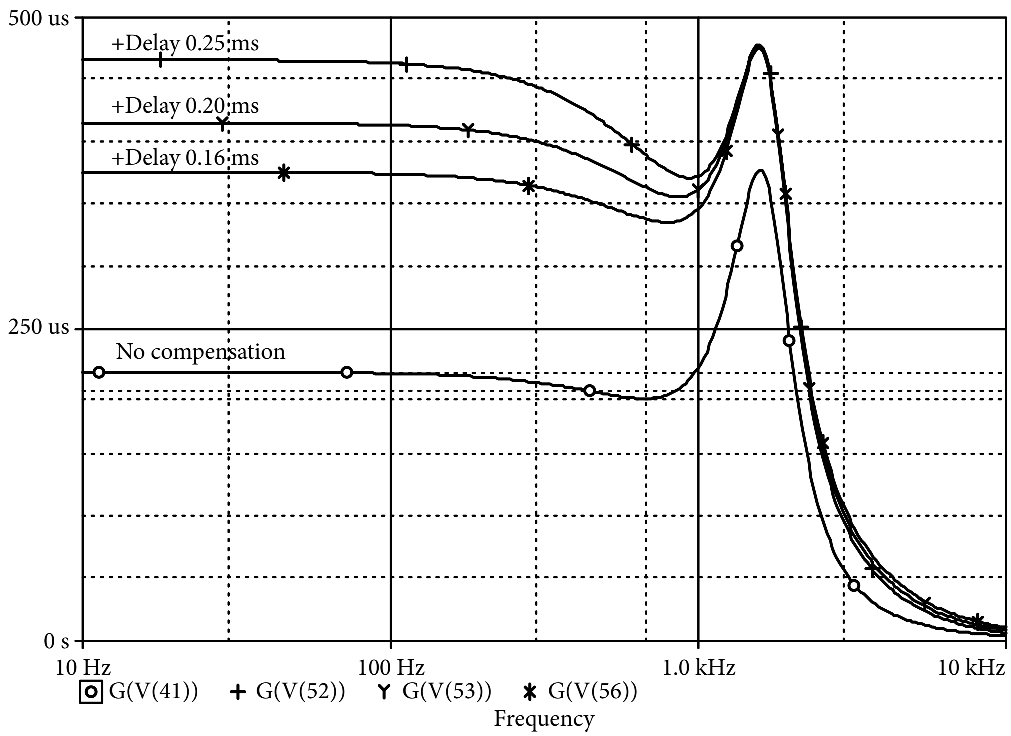

Figure 4.14 Delay in a third-order low pass Chebyshev filter of Example 4.4; the delay with three different all pass filters cascaded to it is also shown.

As mentioned earlier and shown by equation (4.31), variation of D_2 depends on ω_o and Q. Value of the delay at dc is obtained from equation (4.31) as:

$$D_2(0) = (2/\omega_o Q) \tag{4.32}$$

Maximum of the delay curve obtained by taking the derivative of equation (4.31) and equating it to zero results in the following:

$$D_{2,\max} \cong (4Q/\omega_o) \text{ for } \omega_n \cong 1 \tag{4.33}$$

The maxima occur at the frequency

$$\omega_{n,\max} = \{-1 + (4 - Q^{-2})^{0.5}\}^{0.5} \cong 1 \text{ for } Q > 1 \tag{4.34}$$

Equations (4.32) and (4.33) indicate that a larger value of pole Q means a lower delay value at dc and a higher maximum delay.

Based on the selected values of ω_n and Q, the resulting second-order section is then cascaded with the magnitude-approximated filter section. Once again, we use trial and error for simpler cases or a computer program for more demanding cases. More than one second-order AP section will be used if a single section is not able to satisfy the requirement of the delay. Any

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appropriate circuit employing operational amplifier(s), operational transconductance amplifiers (OTAs) or some other active devices can be used to realize second-order AP section(s).

If a second-order LP section is used instead of an APF, its expression of phase shift, delay, delay at dc, and maximum delay, corresponding to the equations (4.30) to (4.33), are as given below:

$$\varphi_{2\text{LP}}(\omega_n) = -\tan\left(\frac{\omega_{n/Q}}{1-\omega_n^2}\right) \tag{4.35a}$$

It corresponds to the delay as:

$$D_{2\text{LP}}(\omega_n) = -\frac{d\varphi_{2\text{LP}}(\omega_n)}{d\omega} = \left\{\frac{d\varphi_{2\text{LP}}(\omega_n)}{d\omega_n}\right\}\left(\frac{d\omega_n}{d\omega}\right) = \frac{1}{\omega_o}\left\{\frac{(1/Q)(1+\omega_n^2)}{(1-\omega_n^2)^2 + (\omega_n/Q)^2}\right\} \tag{4.35b}$$

Value of the delay at dc: $D_{2\text{LP}}(0) = (1/\omega_o Q)$ (4.35c)

Maximum of the delay curve $D_{2\text{LP,max}} \cong (2Q/\omega_o)$ for $\omega_n \cong 1$ (4.35d)

Example 4.5: Obtain the delay response of the fifth-order passive Chebyshev filter having a corner frequency of 10 krad/s (1.59 kHz) and a ripple width of 0.5 dB. Apply the delay optimization using first- and second-order AP sections.

Solution: A fifth-order Chebyshev filter is shown in Figure 4.15(a) (its design and simulation will be done in Chapter 5).

Figure 4.15(b) shows the delay caused by the fifth-order Chebyshev filter having 0.5 dB ripple width in the pass band. It shows a peak delay of 1.064 ms at the pass band corner frequency of 10 krad/s (1.59 kHz): this is normal in Chebyshev filters. At low frequency near dc, delay is considerably small having a value of 0.42 ms; it also shows a minimum of ~0.375 ms at around 445 Hz. The ratio of maxima to minima in the pass band is 2.837, which needs to be reduced.

A first-order AP section as shown in Figure 4.11(a) (which essentially has the highest delay at dc) is to be cascaded to a filter as shown in Figure 9.5(a). To introduce a delay of 0.64 ms at dc (the approximate delay of the passive LP filter cascaded with the AP section would be expected to become 1.62 ms; very close to the peak value of its delay at pass band edge). Equation (4.20) gives the characteristic constant of the first-order AP filter:

$$a_0 = (2/D) = 3125 \tag{4.36}$$

Since for the first-order filter shown in Figure 4.11(a), $a_0 = (1/RC)$, with an impedance scaling factor of 10^3, element values are $C = 0.32$ μF and $R = 1.0$ kΩ. The simulated delay response of the designed first-order AP filter (AP1-1) is shown in Figure 4.16. The filter has a maximum delay of 0.64 ms at 10 Hz and as expected, the delay drops monotonically with frequency. The designed first-order AP filter was cascaded with the Chebyshev filter and the resulting

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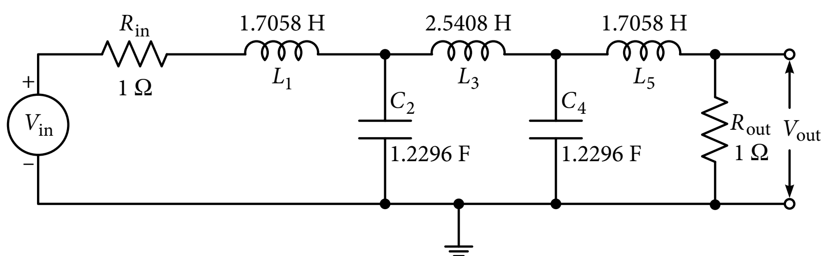
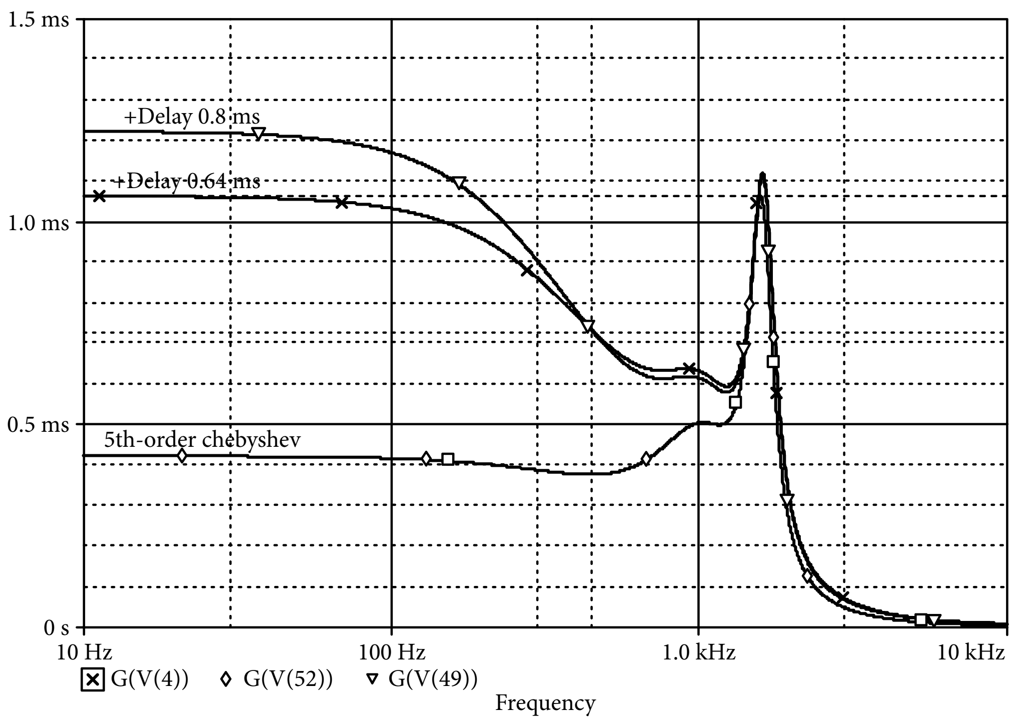
circuit is simulated for delay which is also shown in Figure 4.15. This filter has a total delay of 1.06 ms at 10 Hz, and as a result of the sum of the delay of the Chebyshev filter and the delay of the first-order AP section, the maximum delay at pass band edge is 1.12 ms. A minimum delay of 0.591 ms occurs at 1.22 kHz, resulting in a delay differential of 41.3%. This shows an improvement in the differential delay due to the filter of 64.6%. Further improvement in the delay needs to be achieved by the use of an additional second-order AP section.



(a)



(b)

Figure 4.15 (a) Fifth-order passive Chebyshev approximated filter for Example 4.5. (b) Delay response of the given passive fifth-order filter and total delay when AP1 and AP2 are cascaded with the passive filter.

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However, before using second-order AP, another first-order APF (AP1-2) with delay of 0.8 ms at dc is cascaded, (for AP1-2, the only change from AP1-1 is that $C = 0.4$ μF; resistances remain the same) which makes the total delay at dc as 1.22 ms. In this case, a dip occurs with a delay value of 0.576 ms as shown in Figure 4.15, resulting in a differential delay of 52.7%; which means that this alternate of AP1-2 is worse.

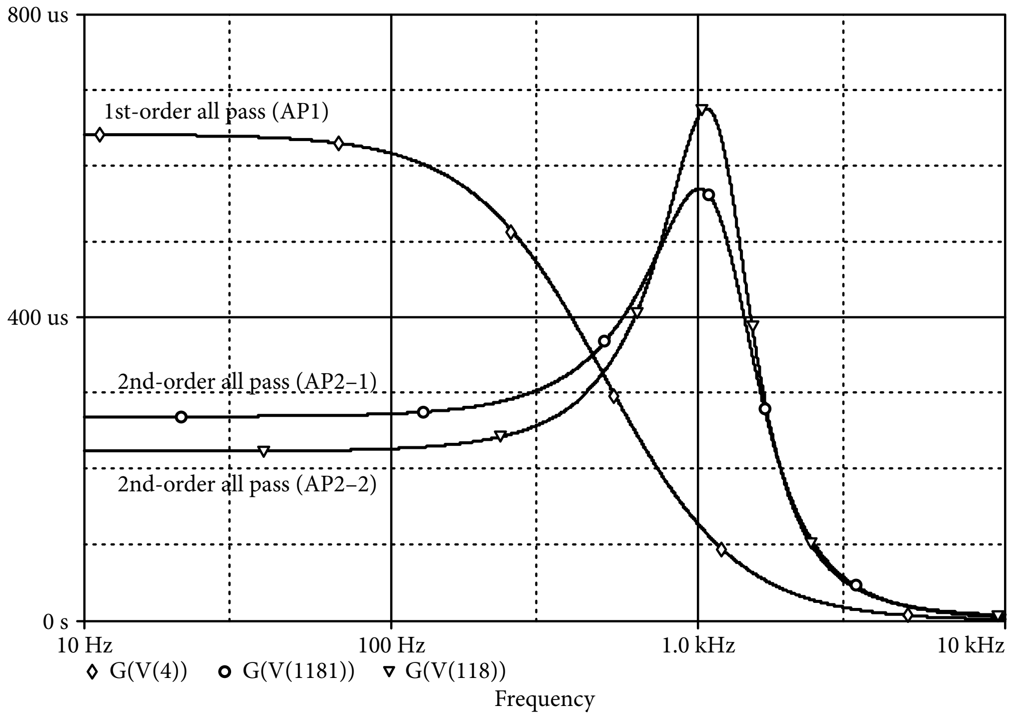

Figure 4.16 Variation of delay for the first-order AP filter (AP1) and two second-order AP filters used in Example 4.5.

For the second-order AP section, we select $\omega_o = 7.5$ krad/s (corresponding to 1200 Hz). For the desired additional delay of 0.53 ms by the second-order AP section at ω_o (an approximation looking at the delay curve after cascading the first-order AP1), equation (4.31) can be used. However, an approximate value of pole-Q can be obtained from equation (4.33) as follows:

$$Q = (0.53 \times 10^{-3} \times 7.5 \times 10^3/4) = 0.99375$$

To realize the second-order AP filter with $\omega_o = 7.5$ krad/s and $Q = 0.99375$, the circuit shown in Figure 8.13(a) was used (any other second-order AP circuit can also be used). Design of the circuit is discussed in Chapter 8, and use of equation (8.31) gives $R = 5$ kΩ, $QR = 4.968$ kΩ and $C = 26.667$ nF; for the equation (8.33b), $\beta = 2.014$, $\gamma = 0$ and $k = 1$ gives $R_f = R_\alpha = 5$ kΩ, $R_\beta = 2.484$ kΩ. Delay introduced due to the second-order AP2-1 circuit is shown in Figure 4.16. The designed AP2-1 section was also cascaded and the overall circuit is shown in Figure 4.17.

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The values of the elements of the second-order AP are also shown in Figure 4.17 and the filter's delay response is obtained through PSpice simulation.

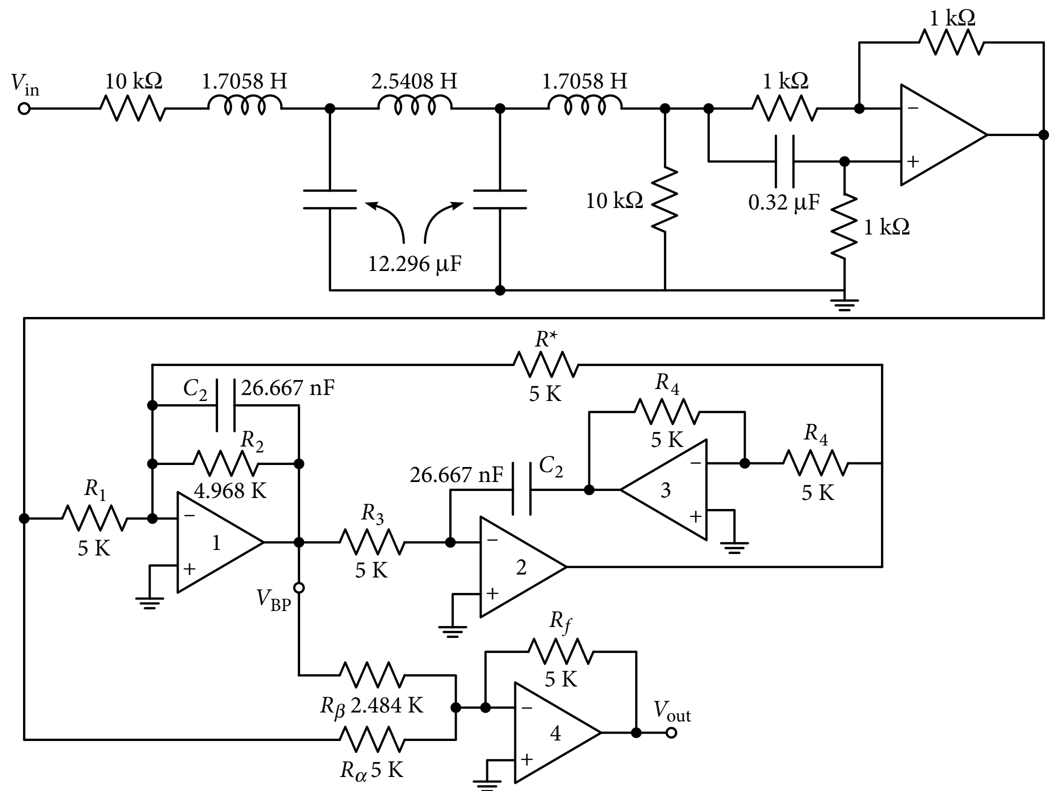

Figure 4.17 Passive fifth-order filter being delay compensated with a first-order and a second-order all pass filter.

The magnitude response of the composite filter is shown in Figure 4.18 along with the magnitude response after cascading with AP1-1 and AP2-1. It is important to note that the response magnitude remains unaffected.

The delay response of the compensated filter after cascading with AP1-1, having $D(0) =$ 0.64 ms, as well as second-order sections AP2-1 (case I) is shown in Figure 4.19. It improves the delay differential at dc (1.33 ms) and the peak delay at pass band edge frequency (1.42 ms).The peak delay and the delay at dc are close; a minimum delay of 1.06 ms occurs at nearly 520 Hz, and the ratio of minimum to maximum delay in the pass band is 1.33. The delay curve still has depressions at around 520 Hz and around 1320 Hz; but the delay differential has improved to 25.3%.

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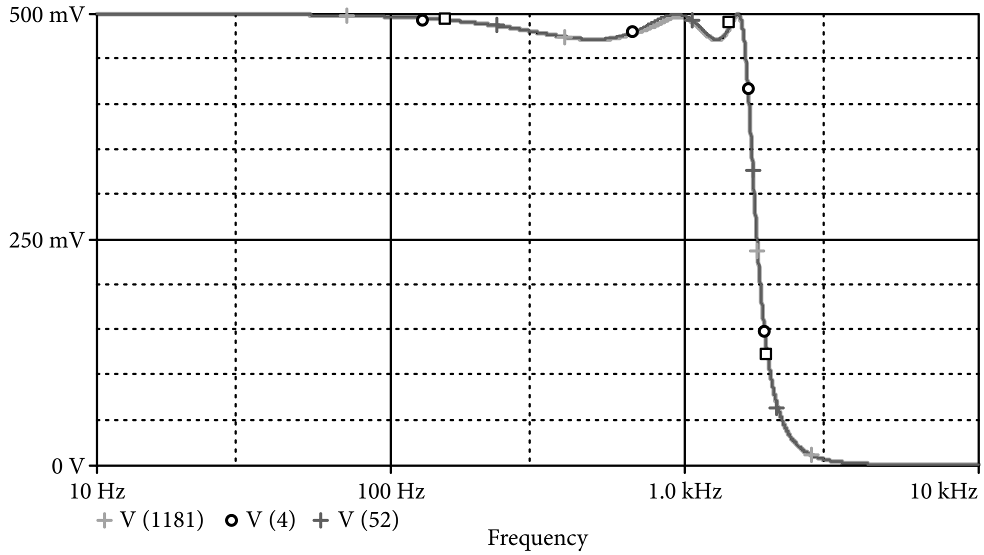

Figure 4.18 Unaffected magnitude response of the delay compensated filter of Example 4.5 along with the magnitude response before cascading with AP1 and AP2-1.

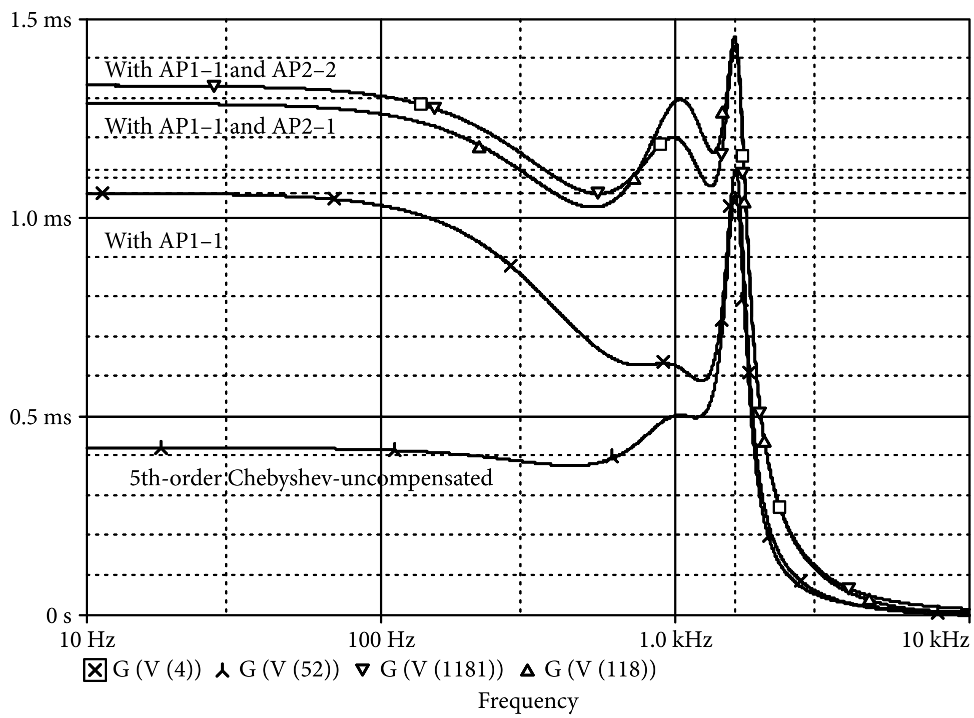

Figure 4.19 Variation of the delay for the uncompensated fifth-order Chebshev filter and delay after compensation with first- and second-order AP filters.

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A second attempt was made with another second-order section AP2-2 with Q = 1.2 (case II), for which enhanced delay by 0.64 ms is spread over a frequency range of around 1200 Hz, (another possible choice for a second-order section can be that which has a different value of ω_o). The modified filter was realized with new element values ($QR = 6$ kΩ, R_β = 3 kΩ) and the delay response was simulated (Figure 4.19). Delay of the Chebyshev filter shown in Figure 4.15 is also included here for comparison sake. It shows the ratio of the maximum to minimum delay as 1.418 in case II since the maximum value of the delay is 1.45 ms and minimum is 1.0223 ms. Case I with $Q = 0.993$ appears to be slightly better. Obviously some more trials may give even better results without resorting to computer usage.

Example 4.6: Obtain a maximally flat filter for the following specifications and use AP filter(s) to improve its delay response.

$$\alpha_{\max} = 1 \text{ dB},\ \alpha_{\min} = 40 \text{ dBs},\ \omega_1 = 2000 \text{ rad/s and } \omega_2 = 6000 \text{ rad/s} \tag{4.36}$$

Solution: In Chapter 3, Example 3.1, order of the filter was 5 for the aforementioned specifications and a passive filter structure with element values was given in Figure 3.6. The filter in active form is shown in Figure 4.20 with all element values (its design in active form shall be taken up later in Chapter 10). Figure 4.21 shows the delay due to the Butterworth filter which remains nearly constant for 1.416 ms up to 100 Hz and the peak delay of 2.25 ms near the pass band edge. Since the ratio of maxima to minima (1.56) is not very high, utilization of the first-order AP filter was considered adequate. The AP filter section shown in Figure 4.11(a) with $D(0) = 1.2$ ms and 1.0 ms, having all resistances as 1 kΩ and capacitance as 0.6 μF and 0.5 μF, respectively, were used. Delay of the compensated circuits is also shown in Figure 4.21. Case II with $D(0) = 1$ ms gives sufficiently good results as the maxima to minima ratio reduces to nearly 1.169 from 1.56. However, case I is slightly better with the maxima to minima ratio a bit less at 1.133. The example shows that the delay introduced in maximally flat filters is much less compared to equal-ripple filters; hence, it is easier to compensate their delay. When it is important to have uniform delay, then an overall compensated filter using a maximally flat approximation becomes economical rather than using an equal-ripple base filter, though its order may be less.

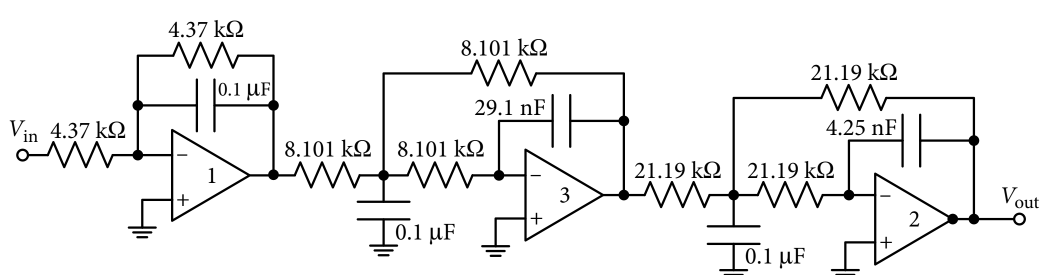

Figure 4.20 Fifth-order Butterworth filter circuit for Example 4.6.

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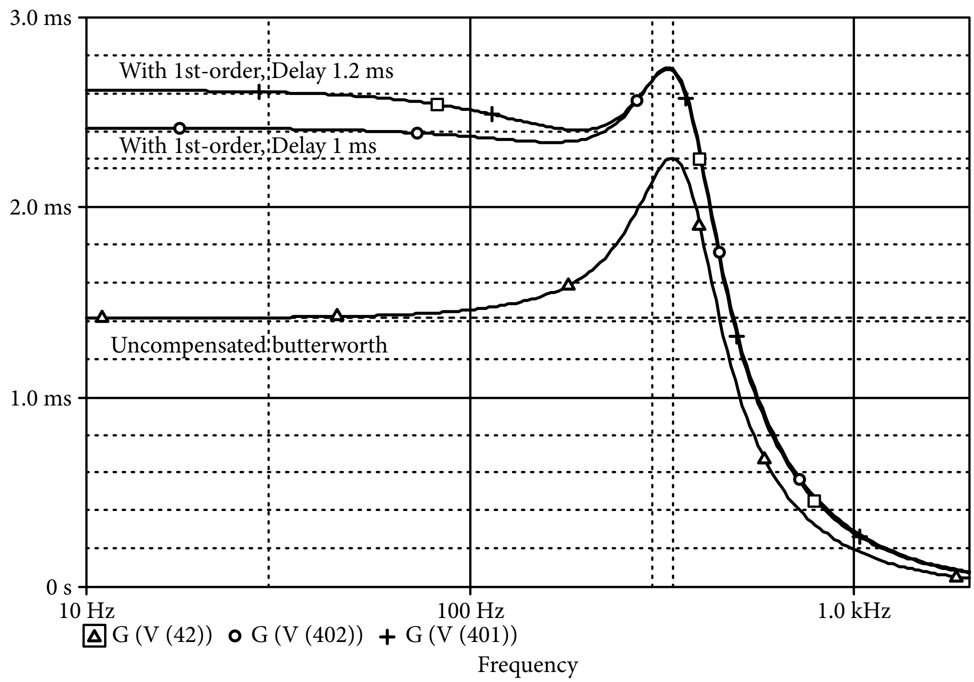

Figure 4.21 Variation of delay in a fifth-order Butterworth filter, and with delay compensation for Example 4.6.

Practice Problems

4-1 While looking at the phase response of a filter, the following phase versus frequency observations were taken. Calculate the phase delay at each frequency and the group delay at each frequency differential.

Frequency (Hz)	197.956—202.076	396.314—403.081	598.358—601.744
Phase (degrees)	71.885—74.558	−68.942—(−73.489)	−150.554—(−151.338)

4-2 Using equation (4.6), find the Bessel polynomial for filter order $n = 5$ and 6, and verify its roots given in Table 4.1.

4-3 Calculate the delay in a normalized fifth-order maximally flat and Chebyshev filter at the pass band edge frequency. In both cases, the voltage gain drops by 1 dB at the pass band edge frequency.

4-4 Find the order and the transfer function of a delay filter which can provide a delay of 0.5 ms, with a permissible delay error of 8% at 8.5 krad/s. Will the required order of the filter increase if permissible loss in magnitude is 3 dBs at 5 krad/s?

4-5 Design a delay filter which satisfies the specifications in Problem 4-4 and check the design through simulation.

4-6 Find the order and the transfer function of a delay filter for which permissible loss in magnitude is 2 dB at 6 krad/s; the filter should provide a delay of 0.25 ms with permissible delay error of 5% at 9 krad/s.

4-7 Repeat Problem 4-5 for the specifications in Problem 4-6.

4-8 Design a BT filter which provides 120 μs delay. Attenuation error is not expected to be more than 2 dBs up to $\omega = 12$ krad/s and the delay should stay below 6% in the frequency range below 12 krad/s. Realize the filter as a cascade of sections having maximally flat approximations and test using PSpice.

4-9 Obtain the delay response of a third-order passive Chebyshev filter having corner frequency of 10 krad/s and ripple width of 1.0 dB, and apply delay optimization using first- and second-order AP sections.

4-10 Design a BT delay filter with practical value of elements, which has a 4% deviation at the normalized frequency $\omega = 4$ krad/s and almost 1.5 dB attenuation at $\omega = 2.1$ krad/s. The filter has to provide 500 μs delay.

4-11 Obtain a maximally flat filter for the following specifications and use AP filter(s) for improving its delay response.

$\alpha_{max} = 0.5$ dB, $\alpha_{min} = 32$ dBs, $\omega_1 = 1500$ rad/s, $\omega_2 = 3600$ rad/s

4-12 Using the circuit of Figure 4.11(a), design a first-order AP filter having maximum delay of 10 ms. Employ practical values of components, test the circuit and find the frequency where introduced delay reduces to 9 ms.

4-13 Design a third-order maximally flat filter with 3 dB gain loss at a frequency of 1 krad/s. Find its delay at 10 Hz, 100 Hz, 500 Hz and 1 kHz. What is the ratio of maximum to minimum delay in the pass band? Cascade a first-order AP filter having maximum delay equal to the delay differential of the maximally flat filter between delays at 10 Hz and 1 kHz. What is the value of the ratio of the maximum to minimum delay in the pass band for the composite filter?

4-14 Repeat Problem 4-13 for (i) for a third-order maximally flat filter and (ii) a third-order equal-ripple filter having 1 dB ripple width.

4-15 Derive equation (4.35c) and find the delay at 1 kHz in a second-order filter if the value of the pole-Q and normalized ω_n is (i) 2 and 0.5, (ii) 2 and 0.75, (iii) 5 and 0.5.

4-16 Derive equation (4.35d) and find the value of maximum delay for each case in Problem 4-15; also find the frequency at which maximum delay occurs.

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