Chapter 6

Sensitivity of Active Networks

6.1 Introduction

In the previous chapters, comparatively simple methods have been discussed for the realization of active OA (operational amplifier) RC filters. Many more methods of filter synthesis will be discussed later, providing many more filter circuit configurations. Among this large number of available circuits, the choice of a 'best' filter circuit may depend on the specific user's requirements. However, every application ideally requires a practical filter for which performance parameters, like ω_o (center frequency or cut-off frequency) and quality (pole-Q), are needed exactly as designed, and expected to remain invariant with use in varying environment. However, in practice, the user is satisfied if the parameters remain within such limits that do not make the filter impractical. Though there are different reasons during fabrication which cause deviations in the performance parameters, there is one factor which is common to all circuits at design stage. This factor can be termed as the first consideration in connection with these deviations. It is studied under *sensitivity* and is due to the following reasons.

i. Design of a filter circuit assumes active and passive elements to be ideal, whereas in every practical fabrication process, the nominal value of the passive element has statistical variations around its mean value. In general, sophisticated, higher level fabrication processes reduce the parameter variations; elements are said to have smaller *tolerance*. However, for such advanced processes, filter fabrication cost will go up.
ii. The values of both active and passive components change with change in operating conditions like change in temperature, humidity and supply voltage. Some chemical changes due to aging also affect the element values.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

Whatever be the reason for the difference between the practical element value(s) and the original design value(s), performance of the filter gets affected, and it is said that the filter performance parameters are *sensitive* to the elements used. Sensitivity studies of filter circuit parameters provide the information whether a particular circuit will meet the given specifications under likely tolerances of the elements or not. The studies also help in establishing filter stability and hence the filter's utility in the long term.

Study of *sensitivity* begins with single-element (incremental) sensitivity, that is, the effect of change in a single-element on a certain filter parameter. Evaluation of incremental sensitivity is very important as it gives significant information about the filter. Additionally, it is also widely used for finding other advanced form of sensitivities. Hence, study of single-element sensitivity is taken up first in Section 6.2 in detail. For most applications, this study suffices the requirements for filter design. However, other significant sensitivity factors like *transfer function sensitivity* and *sensitivity of second-order*, which are important because of the greater utility of second-order filter sections, is taken up in Sections 6.3 and 6.4. Further, sensitivity of higher-order filters and advanced topics such as multi-parameter sensitivity are discussed in brief towards the end of the chapter.

6.2 Single-element (Incremental) Sensitivity

Every single parameter of a filter, say pole frequency, quality factor, poles and zeros of the transfer function depend on the parameters of the active devices and the values of the passive components and their tolerances. Let P be a performance parameter of a filter, and x be one of the elements (a passive component or parameter of an active device) which may cause change in the parameter P; we can express this relation as $P = f(x, s)$. s has been added as P is also a function of the complex frequency. However, we shall restrict our study to $P = f(x)$ only (for the sake of keeping the expressions simple); this means that we are operating at a fixed frequency or over a small band of frequency, where a small frequency change has little or no effect on the value of the element x.

Generally, the process of finding a change in P due to a change $\Delta x = x - x_o$ in the element x is done through Taylor's series expansion of P around the nominal value x_o of the element x, as shown in Figure 6.1:

$$P(x) = P(x_o) + \left.\frac{\partial P(x)}{\partial x}\right|_{x_o} dx + \frac{1}{2}\left.\frac{\partial^2 P(x)}{\partial x^2}\right|_{x_o} (dx)^2 + \ \ldots\ldots \tag{6.1}$$

If it is assumed that the change Δx in Figure 6.1 is small and at the nominal point x_o, the curvature showing the variation of P due to the variation in x is also not very large, then the second- and higher-order derivative terms in equation (6.1) can be neglected. This omission leads to the following expression for absolute change in the parameter P:

$$\Delta P(x_o) = P(x_o + dx) - P(x_o) \cong \left.\frac{\partial P(x)}{\partial x}\right|_{x_o} dx \tag{6.2}$$

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

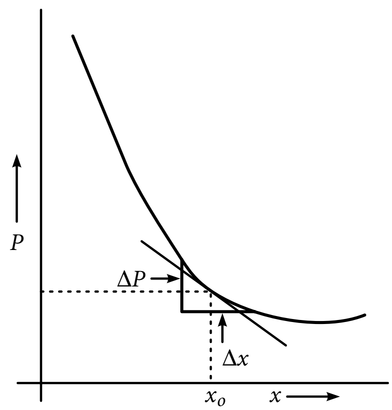

Figure 6.1 Small change in the parameter P, shown as ΔP, due to a small change Δx in x at the nominal value of the element as x_o.

In many cases, it is not very useful to find the absolute change in the parameter ΔP given by equation (6.2); instead, the useful term is the relative change in P, which is given as:

$$\frac{\Delta P(x_o)}{P(x_o)} \cong \frac{x_o}{P(x_o)} \frac{\partial P(x)}{\partial x}\bigg|_{x_o} \left(\frac{dx}{x_o}\right) \tag{6.3}$$

Part of the right-hand side of equation (6.3), which is given in equation (6.4), and expressed as S_x^P, is known as *sensitivity* of the parameter P with respect to the element x at its nominal value x_o. Sensitivity can also be expressed in the natural log form of the same equation.

$$S_x^P = \frac{x_o}{P(x_o)} \frac{\partial P(x)}{\partial x}\bigg|_{x_o} = \frac{\partial P/P}{\partial x/x}\bigg|_{x_o} = \frac{d(\ln P)}{d(\ln x)}\bigg|_{x_o} \tag{6.4}$$

This expression of single-element sensitivity in equation (6.4), in which the amount of deviation in x has been assumed to be small, also known as *incremental sensitivity*, is extremely useful while analyzing the sensitivity of electronic circuits including filter circuits. Once the value of the sensitivity S_x^P is known, the *relative change* or *variability* of the parameter P can be determined from the relative change in an element x, as follows.

$$\frac{\Delta P}{P} \cong S_x^P \frac{dx}{x} \tag{6.5}$$

As we know, the parameter P can be the pole frequency ω_o, pole-Q, transfer function $H(s)$ or its poles and zeros. Hence, if one wants to express the parameters' sensitivities with respect to (say) a resistor R, then the expression for the sensitivities will simply be:

$$S_R^{\omega_o} = \frac{R}{\omega_o} \frac{\partial \omega_o}{\partial R} = \frac{d(\ln \omega_o)}{d(\ln R)} \tag{6.6a}$$

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

$$S_R^Q = \frac{R}{Q}\frac{\partial Q}{\partial R} = \frac{d(\ln Q)}{d(\ln R)} \tag{6.6b}$$

$$S_R^{H(s)} = \frac{R}{H(s)}\frac{\partial H(s)}{\partial R} = \frac{d(\ln H(s))}{d(\ln R)} \tag{6.6c}$$

It is obvious from equation (6.5) that it is always desirable to have the sensitivity as small as possible in order to have a better option of a smaller relative change in the parameter P. At the same time, it is important to note that a larger sensitivity is acceptable with respect to those elements which are very stable. This is because the product of a larger sensitivity and components of smaller variability will result in an allowable smaller variability in the parameter P.

In general, evaluation of sensitivity is not difficult, especially for lower-order filter circuits. Though the obtained sensitivity figures with respect to a single-element do provide a fair assessment of the stability of the filter, evaluation of sensitivity figures with respect to all the active and passive individual elements (with the remaining elements considered as constants) do not give a complete picture; the reason being that in each case, sensitivity evaluation is done at the nominal value of that element, whereas the parameter P depends on other elements also. Hence, if the nominal values of other elements change, then the simple incremental sensitivity evaluation will not remain correct. Of course, the amount of incorrectness will depend upon the changes in the nominal values of the other elements. An accurate evaluation is done under the topic of multi-parameter (or multi-element) sensitivity evaluation, where account is taken of the fact that a network parameter depends on many elements which can simultaneously change by varying amounts. At this stage, consideration of simultaneous change in the parameters can also be done in a simplistic way. For example, if parameter $P = f(x_1, x_2, \ldots.., x_n)$, then the likely total change in P is found as:

$$\Delta P = \Delta P\left(x_1, x_2, \ldots.., x_n\right) = \frac{\partial P}{\partial x_1}dx_1 + \frac{\partial P}{\partial x_2}dx_2 + \ldots\ldots + \frac{\partial P}{\partial x_n}dx_n \tag{6.7}$$

The relative change in the parameter P can be written as:

$$\frac{\Delta P}{P} = \frac{x_1}{P}\frac{\partial P}{\partial x_1}\frac{dx_1}{x_1} + \frac{x_2}{P}\frac{\partial P}{\partial x_2}\frac{dx_2}{x_2} + \ldots\ldots.. + \frac{x_n}{P}\frac{\partial P}{\partial x_n}\frac{dx_n}{x_n}$$

$$= S_{x_1}^P\frac{dx_1}{x_1} + S_{x_2}^P\frac{dx_2}{x_2} + \ldots\ldots. + S_{x_n}^P\frac{dx_n}{x_n} \tag{6.8}$$

which means that the total relative change is the sum of individual sensitivities multiplied with the relative change in the elements.

It was mentioned earlier that performance parameters are functions of the complex frequency s; hence, the sensitivity expression is also a function of s. While finding the sensitivity for a filter section, the proper frequency range should be kept in mind, as the sensitivity value may get changed at different frequency levels.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

Example 6.1: Find the incremental sensitivity of the pole frequency and the quality factor for the RLC filter section shown in Figure 6.2. Also find the total relative change in ω_o and Q if the inductor and capacitor change by –5% and the resistors change by 8%.

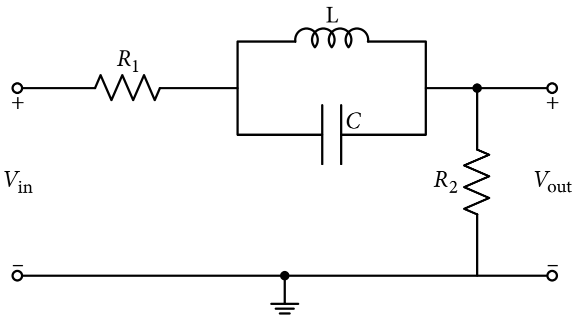

Figure 6.2 A simple RLC filter section for Example 6.1.

Solution: The transfer function of the filter section of Figure 6.2 is obtained as follows

$$H(s)=\frac{V_{out}}{V_{in}}=\frac{R_2}{R_1+R_2}\frac{(s^2+1/LC)}{s^2+\{1/C(R_1+R_2)\}+1/LC} \tag{6.9}$$

Expression of the pole frequency ω_o and the quality factor Q are as follows

$$\omega_o=1/\sqrt{(LC)}\text{, and } Q=(R_1+R_2)\sqrt{(C/L)} \tag{6.10}$$

Use of equations (6.6) and (6.7) on equation (6.10) gives the incremental or single-element sensitivity figures as:

$$S_{R_1}^{\omega_o}=S_{R_2}^{\omega_o}=0,\quad S_L^{\omega_o}=-\frac{1}{2},\ S_C^{\omega_o}=-\frac{1}{2}$$

$$S_{R_1}^{Q}=\frac{R_1}{R_1+R_2},\ S_{R_2}^{Q}=\frac{R_2}{R_1+R_2},\ S_C^{Q}=\frac{1}{2},\ S_L^{Q}=-\frac{1}{2}$$

It means that any change in the value of R_1 and R_2 does not make any difference in the value of ω_o but it affects Q, which depends on the relative values of R_1 and R_2. For example, if $R_1 = 2R_2 = 2R$, $S_{R_1}^{Q}=\frac{2}{3}$ and $S_{R_2}^{Q}=\frac{1}{3}$; ±1% change in R_1 will cause a change of $\pm\frac{2}{3}$ % in Q, whereas a change of ±1% in R_2 will cause a change of $\pm\frac{1}{3}$ % in Q.

Similarly, a change of 1% in L or C will cause a change of –0.5% in ω_o, whereas a 1% change in C will change Q by 0.5%, but a 1% change in L will change Q by –0.5%

The total relative change in ω_o, and Q are computed using equation (6.8) as follows:

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

$$\frac{\Delta\omega_o}{\omega_o} = S_L^{\omega_o}\frac{\Delta L}{L} + S_C^{\omega_o}\frac{\Delta C}{C} + S_{R_1}^{\omega_o}\frac{\Delta R_1}{R_1} + S_{R_2}^{\omega_o}\frac{\Delta R_2}{R_2} = -\frac{1}{2}\left(\frac{\Delta L}{L} + \frac{\Delta C}{C}\right)$$

$$\frac{\Delta Q}{Q} = S_L^{Q}\frac{\Delta L}{L} + S_C^{Q}\frac{\Delta C}{C} + S_{R_1}^{Q}\frac{\Delta R_1}{R_1} + S_{R_2}^{Q}\frac{\Delta R_2}{R_2} = -\frac{1}{2}\left(\frac{\Delta L}{L} - \frac{\Delta C}{C}\right) + \frac{1}{R_1 + R_2}(\Delta R_1 + \Delta R_2)$$

It can easily be observed for this example that if relative change in L and C are in the same direction, then their effects add in the case of ω_o but cancels each other in the case of Q; the opposite happens if the relative change in L and C occur in the opposite direction. For the given changes in elements, the following will be the values of the important relative changes:

$$\frac{\Delta\omega_o}{\omega_o} = 5\% \text{ and } \frac{\Delta Q}{Q} = 0 + 8\%$$

Example 6.2: Figure 6.3 shows a single amplifier biquad. Its transfer function is as shown in equation (6.11) while OA is considered as ideal. (a) Find all the incremental sensitivities for the important parameters of the biquad and (b) find the variability in Q for small changes in the active parameter.

$$H(s) = \frac{V_o(s)}{V_{\text{in}}(s)} = -\frac{(1/aR_1C_1)s}{s^2 + \left\{\frac{1}{R_2C_1} + \frac{1}{R_2C_2} - \frac{K}{R_1C_1(1-K)}\right\}s + \frac{1}{R_1R_2C_1C_2}} \tag{6.11}$$

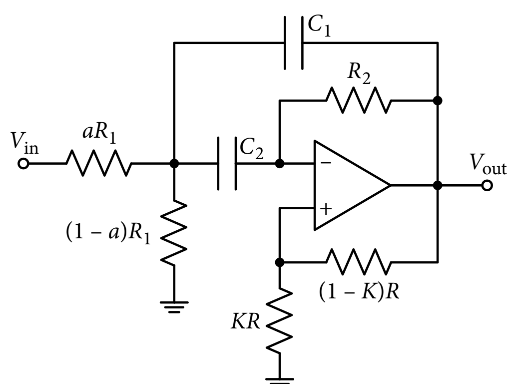

Figure 6.3 Delyiannis–Friend single amplifier biquad with Q-enhancement.

Solution: If there was no positive feedback in the circuit, the non-inverting terminal of the OA would have been connected to the ground and the value of K would have become zero. If in the beginning it is assumed that $C_1 = C_2 = C$, the transfer function in equation (6.11) will be modified as:

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

$$H(s) = -\frac{H_m(1/aR_1C)s}{s^2 + (2/R_2C)s + 1/C^2R_1R_2} \tag{6.12}$$

which can be written in the standard format of a second-order filter having parameters ω_o, Q and mid-band gain H_m as:

$$H(s) = -\frac{H_m(\omega_o/Q_o)s}{s^2 + (\omega_o/Q_o)s + \omega_o^2} \tag{6.13}$$

Comparison of equations (6.12) and (6.13) gives expressions for the important parameters: Center frequency ω_o:

$$\omega_o = 1/C\sqrt{(R_1R_2)} \tag{6.14a}$$

Quality factor without positive feedback Q_o, and mid-band H_m are as follows:

$$Q_o = \omega_o CR_2/2 = 0.5\sqrt{(R_2/R_1)} \tag{6.14b}$$

$$H_m = R_2/2aR_1 = 2Q_o^2/a \tag{6.14c}$$

For the given circuit with positive feedback expression, ω_o remains unchanged, but Q is enhanced to Q_o, and its expression is obtained from the following equations:

$$\left(\frac{\omega_o}{Q}\right) = \frac{1}{R_2C}\left\{2 - \frac{K}{(1-K)}\frac{R_2}{R_1}\right\} \tag{6.15}$$

With $K/(1-K) = \delta$, we get $Q = \pm Q_o/(1-2\delta Q_o^2)$ (6.16a)

where, $Q_o = \frac{1}{2}\sqrt{C_1R_2/C_2R_1}$ (6.16b)

The expression of the mid-band gain is:

$$H_B = \frac{H_m}{1-2\delta Q_o^2}\left(\frac{1}{1-K}\right) = \frac{H_m}{1-K}\frac{Q}{Q_o} \tag{6.17}$$

(a) Using equations (6.6) and (6.14), we get the sensitivities as

$$S_{R_1}^{\omega_o} = \frac{R_1}{\omega_o}\frac{\partial\omega_o}{\partial R_1} = -\frac{1}{2},\ S_{R_2}^{\omega_o} = -\frac{1}{2} \text{ and } S_C^{\omega_o} = \frac{C}{\omega_o}\frac{\partial\omega_o}{\partial C} = -1$$

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

Sensitivity $S_C^{\omega_o} = 1$ is a little misleading because capacitors C_1 and C_2 were assumed equal. It is important to note that while doing the sensitivity calculations, the general expression should be used without any specific ratio between element values as it was done in this case. Hence, the correct value will be obtained from the general expression of $\omega_o^2 = 1/R_1R_2C_1C_2$ as $S_{C_1,C_2}^{\omega_o} = -\frac{1}{2}$.

All the passive sensitivities are (−1/2), which is a theoretical minimum for an active filter based on the product of two RC time constants. Designers try to design filters with ω_o sensitivities as close to these minimum values as possible. As far as active sensitivity of ω_o is concerned, in this case, $S_K^{\omega_o} = 0$, which is highly desirable.

Before finding Q – sensitivities, it is desirable to note that the second-order filter section employs R_1, R_2, C_1, C_2, and a potential divider involving resistor R for deciding the value of K, whereas there are only two filter parameters, ω_o, and Q; H_B can be controlled independently by the factor a. Therefore, some assumptions have to be taken for the element values. One such assumption has already been taken in the form of having the capacitors C_1 and C_2 as equal; this is very attractive from the point of view of integrated circuit fabrication. Let another assumption be that $\sqrt{(R_2/R_1) = r}$; then, the expression for Q will become:

$$Q = \frac{Q_o}{1-2\delta Q_o^2} = \frac{\left(\frac{1}{2}\right)\sqrt{(R_2/R_1)}}{1-2(K/1-K)(1/4)(R_2/R_1)} = \frac{r}{2-r^2K/(1-K)} \tag{6.18a}$$

With $C_1 = C_2 = C$, Q is independent of C_1 and C_2; hence, the sensitivity of Q with respect to these capacitors is zero. Sensitivity of Q with respect to C_1 and C_2 can separately be obtained from the general expression of Q obtained from equation (6.11).

Sensitivity of Q with respect to K is obtained now as:

$$S_K^Q = \frac{K}{Q}\frac{\partial Q}{\partial K} = \frac{K}{Q}\frac{\partial}{\partial K}\frac{r}{\{2-r^2K/(1-K)\}} = \frac{K}{Q}\frac{\partial}{\partial K}\frac{r(1-K)}{2(1-K)-r^2K} \tag{6.18b}$$

Alternatively, expressing Q from equation (6.18a) in terms of the ratio of polynomials as $Q = N(s)/D(s)$ will give:

$$S_K^Q = \frac{K}{N(s)}\frac{\partial(1-K)r}{\partial K} - \frac{K}{D(s)}\frac{\partial\{2(1-K)-Kr^2\}}{\partial K}$$

$$= \frac{Kr^2}{(1-K)\{2(1-K)-Kr^2\}} = \frac{Kr}{(1-K)^2}Q \tag{6.19}$$

Obviously, Q sensitivity with respect to K will depend on the selected value K (or r) and the value of K (or r) will be dependent on the specified value of Q. It is obvious that sensitivity will shoot to very high values with the value of K nearing unity.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

(b) Calculation of variability in Q: For a selected value r = 1, the obtained value of K for Q = 10 is (19/29) from equation (6.19). Hence, from equation (6.19): $S_K^Q = 55.1$.

This is rather a large value for active Q sensitivity. A small 0.5% of change in the value of K means a large –27.55% change in the value of Q. Worse, if K becomes (2/3), for a mere increase of (1/57), $Q \rightarrow$ infinity and the network become unstable. Practically, it is very difficult, with this choice of element (or the value of K) to set K accurately; the circuit becomes almost non-workable.

Instead of using equation (6.19), the percentage change in Q can also be calculated directly from the expression of Q in equation (6.18a); for the same value of r = 1, we get:

$$Q = (1 - K)/(2 - 3K) \tag{6.20}$$

Equation (6.20) shows that for a change of +0.5% in K (from 19/29), the change in the value of Q will be +38.5%, and if K changes by –0.5%, Q will change by –21.4%.

It is significant to note that variations in Q obtained through different methods of calculations have big differences. The reason behind this is the fact that while deriving the sensitivity definition in equation (6.4), it is assumed that at the nominal point, change in the element with respect to which sensitivity is being calculated is small, whereas in this example, it is not so. This tells us that accuracy of calculating the variability of a performance parameter depend on the smallness, or otherwise of the rate of change of the element x at the nominal point x_o.

6.2.1 Semi-relative sensitivity

In Example 6.1, calculation of the single-element incremental sensitivity was not involved because the expressions of ω_o and Q were simple, and the parameters were in simple relations with the elements. In reality, all the cases are not so simple and even evaluation of single-element sensitivity becomes involved and cumbersome, or the value of sensitivity becomes infinite. Instead of a direct application of equation (6.4) using the involved expressions of the parameters, alternatives are available, which are in fact derived from the definition of equation (6.3) and (6.4). Some of such relations are as follows:

$$S_x^{P_1P_2} = S_x^{P_1} + S_x^{P_2} \tag{6.22a}$$

$$S_x^{P_1/P_2} = S_x^{P_1} - S_x^{P_2} \tag{6.22b}$$

With P being a function of y as $P(y)$ and y being a function of x as $y(x)$, then:

$$S_x^P = S_y^P \times S_x^y \tag{6.22c}$$

Additionally, when k and n are constant, the following relations are useful:

$$S_x^{1/P} = S_{1/x}^P = -S_x^P \text{ and } S_x^{kP} = S_x^P \tag{6.23a, b}$$

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

$$S_x^{P^n} = nS_x^P \text{ and } S_x^{kx^n} = n \tag{6.24a, b}$$

$$S_x^{k+P} = \frac{k}{k+P} S_x^P \text{ and } S_x^{\sum P_i} = \frac{1}{\sum P_i} \sum \left(P_i S_x^{P_i} \right) \tag{6.25a, b}$$

There are some cases where it is the absolute change in P, rather than the relative value which is desired; hence, the value of sensitivity itself is not important. For example, if S_x^P is to be evaluated near (or at) a nominal point $P \to 0$, then its value will tend to infinity. This result is not very useful for practical purposes. Hence, in such cases, instead of finding S_x^P directly, the following *semi-relative sensitivity* measure is evaluated:

$$Q_x^{P(x)} = x \frac{dP}{dx} \tag{6.26}$$

It will be observed that $Q_x^{P(x)}$ is very useful in many cases as will be shown soon.

6.3 Transfer Function Sensitivity

Expression of a general transfer function introduced in Chapter1 is repeated here:

$$H(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots\ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots\ldots + b_1 s + b_0} \tag{6.27}$$

Both the numerator and the denominator in equation (6.27) are functions of the elements used in the construction of the circuit for which the transfer function is obtained. Hence, their coefficients are also functions of these elements. Use of equation (6.22b) on equation (6.27) gives the following relation for the transfer function sensitivity which simplifies its evaluation considerably:

$$S_x^{H(s)} = S_x^{N(s)} - S_x^{D(s)} = x \left(\frac{1}{N(s)} \frac{\partial N(s)}{\partial x} - \frac{1}{D(s)} \frac{\partial D(s)}{\partial x} \right) \tag{6.28}$$

Most of the time, more than one coefficient in $N(s)$ as well as in $D(s)$ depends on an element x. It results in the modification of equation (6.28) as:

$$S_x^{H(s)} = \frac{x}{N(s)} \sum \frac{\partial a_i}{\partial x} s^i - \frac{x}{D(s)} \sum \frac{\partial b_i}{\partial x} s^i \tag{6.29}$$

Obviously, the two summations in equation (6.29) will comprise only those terms for which a_i or b_i, respectively, depend on element x; this leads to the understanding of the transfer function sensitivity from a different angle. When the numerator and denominator of the transfer function are factorized, zeros and poles also deviate from their nominal positions when any element gets changed. A relation between the amount of shift in any pole or zero location

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

and the resultant behavior of the transfer function can be obtained and studied. If $H(s)$ given in equation (6.27) is factorized in terms of poles and zeros and its natural log is obtained, we get:

$$\ln H(s) = \ln k + \sum_{i=1}^{m} \ln(s - z_i) - \sum_{i=1}^{n} \ln(s - p_i) \tag{6.30}$$

Here, along with the poles and zeros, the coefficient $k = (a_m/b_n)$ as well may be a function of active parameters and the passive elements. A derivative of equation (6.30) is taken and then both sides of the equation are multiplied with x, which results in the following important relation in terms of semi-relative sensitivity expressions of equation (6.26):

$$S_x^{H(s)} = S_x^k - \sum_{i=1}^{m} \frac{Q_x^{z_i}}{(s - z_i)} + \sum_{i=1}^{n} \frac{Q_x^{p_i}}{(s - p_i)} \tag{6.31}$$

Equation (6.31) clearly shows that any shift in the location of a single pole or zero will affect the transfer function and its sensitivity becomes high at frequencies close to a pole or zero of $H(s)$. But for physical frequencies, with $s = j\omega$, the transfer function sensitivity tends towards infinity when the transmission zero is on the $j\omega$-axis. In addition, for $s = j\omega$, with larger Q values, complex pole pairs are very near to the $j\omega-$ axis, resulting in $|(j\omega - p_i)|$ becoming small and the sensitivity of $H(s)$ becoming high. It is illustrated in Figure 6.4 with the help of the location of poles of an eighth-order Chebyshev filter with $\alpha_{max} = 0.5$ dB.

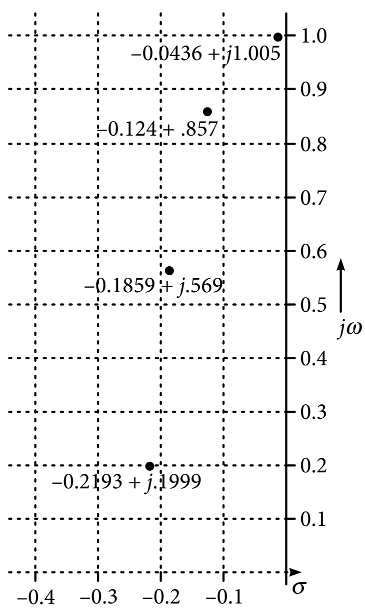

Figure 6.4 Pole location in the second ordinate for an eighth-order Chebyshev filter with α_{max} = 0.5 dB.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

Use of equation (6.31) is not very desirable as factorization of a polynomial of even order 4 (and more) is extremely difficult (without a root finder).

Significant relations are obtained in the context of the transfer function sensitivity if $H(s)$ is expressed in terms of its magnitude and phase as:

$$H(j\omega, x) = |H(j\omega, x)| e^{j\varphi(\omega, x)} \tag{6.32}$$

Using the relation (6.22a), and later equation (6.26) for the phase part, we get

$$S_x^{H(j\omega)} = S_x^{|H(j\omega)|} + jQ_x^{\varphi(\omega)} \tag{6.33}$$

A significant inference from equation (6.33) is that the real part of the transfer function is the magnitude sensitivity:

$$\mathrm{Re}\left\{S_x^{H(j\omega)}\right\} = S_x^{|H(j\omega)|} \tag{6.34}$$

and the imaginary part of the transfer function sensitivity is the semi-relative phase sensitivity:

$$\mathrm{Im}\left\{S_x^{H(j\omega)}\right\} = Q_x^{\varphi(j\omega)} \tag{6.35}$$

The right-hand side of equation (6.31) is the partial fraction expansion of $S_x^{H(s)}$ in equation (6.28); hence, $Q_x^{z_i} \text{and} Q_x^{p_i}$ are the residues. Hence, equalizing the mentioned equations for $s \to p_i$, the last term in equation (6.32) will dominate, resulting in the following relation:

$$\lim_{s \to p_i} x \left(\frac{1}{N(s)} \frac{\partial N(s)}{\partial x} - \frac{1}{D(s)} \frac{\partial D(s)}{\partial x} \right) = \lim_{s \to p_i} \frac{Q_x^{p_i}}{(s - p_i)} \tag{6.36}$$

Since p_i is the pole of $H(s)$, which is obtained through factorizing the denominator, the term $\frac{1}{D(s)} \frac{\partial D(s)}{\partial x}$ dominates $\frac{1}{N(s)} \frac{\partial N(s)}{\partial x}$; an important inference.

6.4 Second-order Filter Sensitivities

The sensitivity expressions developed so far are valid for both active and passive filters of any order. Values of the sensitivities obtained depend on the type of structure used, and these values helps in comparing the circuits from the point of view of sensitivity. However, there are certain relations which allow us to get useful information about second-order filters (with $Q > 0.5$). Since second-order active filters form an important entity in the design of any higher-order filter, this area of sensitivity study forms an useful and important topic.

Transfer function of a general biquadratic function is written as:

$$H(s) = \frac{a_2(s-z_1)(s-z_2)}{s^2 + (\omega_o / Q_o)s + \omega_o^2} = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + (\omega_o / Q_o)s + \omega_o^2} \tag{6.37}$$

Since for all practical purposes, active filters are used with $Q > 0.5$, poles are complex conjugate and their expression, from equation (6.37), is obtained as:

$$p_1,\ p_1^* = -\frac{\omega_o}{2Q_o} \pm j\omega_o(1 - 1/4Q_o^2)^{\frac{1}{2}} \tag{6.38}$$

In order to determine an important relation for the sensitivities of biquadratic sections with respect to an element x, we first proceed with the conjugate poles considering ω_o and Q_o to be functions of x by taking the derivative of p_1 as:

$$\frac{\partial p_1}{\partial x} = -\frac{\partial \omega_o}{\partial x}\left\{\frac{1}{2Q} - j(1 - \frac{1}{4Q^2})^{\frac{1}{2}}\right\} - \omega_o \frac{\partial}{\partial x}\left\{\frac{1}{2Q} - j(1 - \frac{1}{4Q^2})^{\frac{1}{2}}\right\} \tag{6.39a}$$

We first multiply both sides of equation (6.39a) with x. Then, multiplying and dividing the first term on the right-hand side by ω_o, and taking the derivative of the second term with respect to x, the following relation is obtained:

$$x\frac{\partial p_1}{\partial x} = -\frac{x}{\omega o}\omega_o \frac{\partial \omega_o}{\partial x}\left\{\frac{1}{2Q} - j(1 - \frac{1}{4Q^2})^{\frac{1}{2}}\right\} - \omega_o\left\{-\frac{1}{2Q^2} - j\frac{1/2Q^3}{2\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}}\right\} x\frac{\partial Q}{\partial x} \tag{6.39b}$$

Dividing equation (6.39b) by p_1 of equation (6.38), and after a bit of manipulation, we get:

$$\frac{x}{p_1}\frac{\partial p_1}{\partial x} = S_x^{p_1} = S_x^{\omega_o} - jS_x^Q / (4Q^2 - 1)^{\frac{1}{2}} \tag{6.40}$$

Following the same procedure, it is shown that, with p_1^*, we get:

$$\frac{x}{p_1^*}\frac{\partial p_1^*}{\partial x} = S_x^{p_1^*} = S_x^{\omega_o} + jS_x^Q / (4Q^2 - 1)^{\frac{1}{2}} \tag{6.41}$$

Equations (6.40) and (6.41) show that pole-sensitivity depends on both the sensitivities of ω_o and Q but an important observation is that pole-sensitivity is more dependent on ω_o sensitivity than that of Q. In fact, the position of pole p_1 is $(4Q^2 - 1)^{1/2} \cong 2Q$ times more sensitive to the deviations in ω_o than the deviations in the Q value. It is for this reason that the filter designer needs to care more for the ω_o sensitivity than the Q sensitivity. For a second-order active RC filter, for which ω_o depends on the product of two RC time constants (R_1C_1 and R_2C_2),

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

designers try to get equal to or as close to the ideal ω_o sensitivities of (–1/2) with respect to all the mentioned elements.

From equation (6.36), an important observation was made that while finding the transfer function sensitivity, the dominant term is the one which depends on the denominator $D(s)$, or poles of $H(s)$, and the remaining terms can be neglected. The reason for finding this expression is that the poles lie mostly in the pass band and the zeros of the transfer function lie in the stop band and it does not affect the filter output. Therefore, for most of our discussion on the sensitivity of a biquad, the following modified form of the equation (6.28) is used for finding the sensitivity of the transfer function. Later, for specific cases, the effect of the contribution due to the zeros can be added.

$$S_x^{H(s)} = -\frac{x}{D(s)}\frac{\partial D(s)}{\partial x} = -x\frac{\left(2\omega_o + \frac{s}{Q}\right)\frac{\partial \omega_o}{\partial x} - \frac{s\,\omega_o}{Q^2}\frac{\partial Q}{\partial x}}{s^2 + \left(\frac{\omega_o}{Q}\right)s + \omega_o^2} \tag{6.42}$$

Multiplying and dividing the first term in the numerator of the right-hand side term by ω_o in equation (6.42), simplifies it to the following expression:

$$S_x^{H(s)} = \left\{-\left(2\omega_o^2 + s\frac{\omega_o}{Q}\right)S_x^{\omega_o} - s\frac{\omega_o}{Q}S_x^{Q}\right\} / D(s) \tag{6.43}$$

In order to find the magnitude sensitivity and semi-relative phase sensitivity of the biquad as given in equation (6.43), s is replaced by $j\omega$ in equation (6.43), and the expression is multiplied and divided with the conjugate of the denominator $D(j\omega)$.

$$S_x^{H(j\omega)} = -\frac{\left\{\left(2\omega_o^2 + j\omega\frac{\omega_o}{Q}\right)S_x^{\omega_o} - j\omega\frac{\omega_o}{Q}S_x^{Q}\right\}}{\left\{\left(\omega_o^2 - \omega^2\right) + j\frac{\omega\omega_o}{Q}\right\}}\frac{\left\{\left(\omega_o^2 - \omega^2\right) - j\frac{\omega\omega_o}{Q}\right\}}{\left\{\left(\omega_o^2 - \omega^2\right) - j\frac{\omega\omega_o}{Q}\right\}}$$

$$= -\frac{\left\{2\omega_o^2\left(\omega_o^2 - \omega^2\right) + \frac{\omega^2\omega_o^2}{Q^2}\right\}S_x^{\omega_o} - \frac{\omega^2\omega_o^2}{Q^2}S_x^{Q}}{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\omega_o^2/Q^2} + j\frac{\frac{\omega\omega_o}{Q}\left(\omega_o^2 + \omega^2\right)S_x^{\omega_o} + \frac{\omega\omega_o}{Q}(\omega_o^2 - \omega^2)S_x^{Q}}{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\omega_o^2/Q^2} \tag{6.44}$$

The right-hand side is divided in the numerator and denominator by ω_o^4, equation (6.44) is now modified in terms of $\omega_n = (\omega/\omega_o)$.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

$$S_x^{H(j\omega_n)} = -\frac{\left\{2\left(1-\omega_n^2\right)+\dfrac{\omega_n^2}{Q^2}\right\}S_x^{\omega_o} - \dfrac{\omega_n^2}{Q^2}S_x^Q}{\left\{\left(1-\omega_n^2\right)^2+\dfrac{\omega_n^2}{Q^2}\right\}} + j\frac{\omega_n}{Q}\frac{\left(1+\omega_n^2\right)S_x^{\omega_o}+(1-\omega_n^2)S_x^Q}{\left\{\left(1-\omega_n^2\right)^2+\dfrac{\omega_n^2}{Q^2}\right\}} \tag{6.45}$$

The real part of equation (6.45) may be written as:

$$S_x^{|H(j\omega_n)|} = S_{\omega_o}^{|H(j\omega_n)|}S_x^{\omega_o} + S_Q^{|H(j\omega_n)|}\,S_x^Q \tag{6.46}$$

$$\text{where, } S_{\omega_o}^{|H(j\omega_n)|} = \frac{\left\{2\left(1-\omega_n^2\right)+\dfrac{\omega_n^2}{Q^2}\right\}}{\left\{\left(1-\omega_n^2\right)^2+\dfrac{\omega_n^2}{Q^2}\right\}} \text{ and } S_Q^{|H(j\omega_n)|} = \frac{\dfrac{\omega_n^2}{Q^2}}{\left\{\left(1-\omega_n^2\right)^2+\dfrac{\omega_n^2}{Q^2}\right\}} \tag{6.47}$$

The imaginary part of equation (6.45) may be written as:

$$Q_x^{\varphi(\omega_n)} = Q_{\omega o}^{\varphi(\omega_n)}S_x^{\omega_o} + Q_Q^{\varphi(\omega_n)}\,S_x^Q \tag{6.48}$$

$$\text{where, } Q_{\omega_o}^{\varphi(\omega_n)} = \frac{\omega_n}{Q}\frac{\left(1+\omega_n^2\right)}{\left\{\left(1-\omega_n^2\right)^2+\dfrac{\omega_n^2}{Q^2}\right\}} \text{ and } Q_Q^{\varphi(\omega_n)} = \frac{\omega_n}{Q}\frac{(1-\omega_n^2)}{\left\{\left(1-\omega_n^2\right)^2+\dfrac{\omega_n^2}{Q^2}\right\}} \tag{6.49}$$

For a given value of Q, the determination of the terms given in equation (6.47) provides information about the overall magnitude sensitivity of a biquad using equation (6.46); sensitivity of ω_o and Q should have been calculated for the biquad with respect to the element x. Such calculations are very important since for most of the filters, specifications are given in terms of the required magnitude in the pass and stop bands. Moreover, the functions given in equation (6.47) also provide useful information if these terms are plotted as shown in Figures 6.5(a) and (b), respectively, for a few different values of Q.

For example, it can be observed from Figure 6.5(a) that the maximum of the Q magnitude sensitivities of $|H(j\omega)|$ is unity for any value of Q. $S_Q^{|H(j\omega_n)|} = 1$ and it occurs at ω_n = 1; its variation with frequency is very similar to the response of a band pass (BP) filter having mid-band gain equal to unity.

However, the shape of the term $S_{\omega_o}^{|H(j\omega_n)|}$ is a bit complicated, as shown in Figure 6.5(b). For large values of Q, it is given as:

$$\max\left\{S_{\omega_o}^{|H(j\omega_n)|}\right\} \cong (Q-1)\text{at } \omega_{n2} \cong \left(1+\frac{1}{2Q}\right) \tag{6.50a}$$

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

$$\min\{S_{\omega_o}^{|H(j\omega_n)|}\} \cong (-Q-1)\,\text{at}\,\omega_{n1} \cong \left(1-\frac{1}{2Q}\right) \tag{6.50b}$$

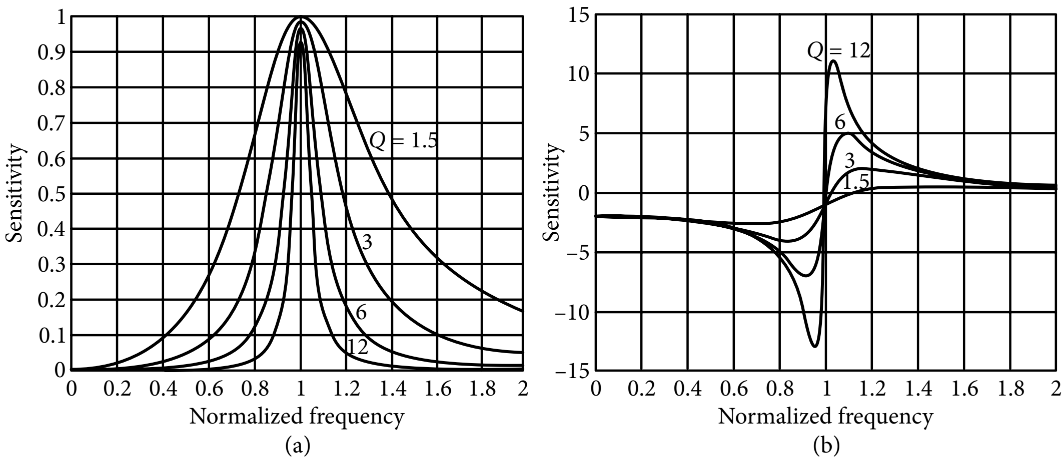

Figure 6.5 Sensitivity of the magnitude and phase function of the second-order transfer function for a few values of Q: (a) $S_Q^{|H(j\omega_n)|}$; (b) $S_{\omega_o}^{|H(j\omega_n)|}$.

From Figure 6.5(b) and equation (6.50), it is clear that the magnitude maxima (and the minima) directly depend on the value of Q and these are located close to $\omega_n = 1$ (in the pass band and close to its edge). Comparison of the maxima in the two figures, Figures 6.5(a) and (b) confirms that the effect of the first term in equation (6.46) is more pronounced, nearly $(4Q^2 - 1)^{1/2}$ or around $(2Q)$ times. It again shows that ω_o sensitivities are to be kept at minimum and nearly Q times less than Q sensitivities if their effect is to be made comparable. The peaks occur as shown by equation (6.50a, b) at a distance of $(\omega_n/2Q)$ from ω_n, which means approximately at 3 dB frequencies.

The phase sensitivity of the transfer function is also important. Not only does it evaluate deviation in phase output in a filter whose specification is given in terms of magnitude attenuation, but it is the phase in terms of which specifications are mentioned while designing phase equalizers or all pass filters. Hence, for observation of phase deviation, equation (6.48) needs to be studied.

Other than keeping $S_x^{\omega_o}$ small, it is observed that error in both magnitude and phase increases with increase in the design value of Q. This implies that filter design is easy with smaller values of Q.

6.5 Sensitivity Considerations for High-order Active Filters

In the last section, sensitivity considerations for a general second-order section were discussed. The study has its own importance, as the information obtained is not only for a standalone

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

second-order section, but is also useful while designing higher-order filters employing a few section-order sections. Before taking up the sensitivity consideration of higher-order filters employing section-order sections, an important issue needs to be considered.

In Section 6.3, it was observable from equation (6.31) that the transfer function sensitivity is very high in the vicinity of either a pole or a zero of a transfer function. This means that for any higher-order transfer function, with high selectivity (having complex–conjugate poles close to the $j\omega$-axis), its sensitivity will be high throughout the pass and stop band. Hence, designing a high-order filter will result, in general, in a high transfer function sensitivity. This means that in fabricated elements with practically obtained variability, deviations in the transfer function are likely to go beyond acceptable limits, making the design impractical.

The problem has been overcome in different ways as will be seen in the following sections.

6.5.1 Simulation of LC ladder method

Equation (6.28) shows that the overall sensitivity of a transfer function depends on the difference term $\left\{\frac{1}{N(s)}\frac{\partial N(s)}{\partial x}-\frac{1}{D(s)}\frac{\partial D(s)}{\partial x}\right\}$. If a circuit topology is such that the two terms here are equal or near equal, it will result in a zero or very small transfer function sensitivity. Such a methodology has been employed using a doubly terminated lossless ladder structure as shown in Figure 6.6 in a block form. It has been shown mathematically that for such a lossless ladder structure, the magnitude sensitivity $S_x^{|H(j\omega)|}$ of equation (6.34) becomes zero, if it is designed for maximum power transfer condition. However, in some other studies, the transfer function sensitivities are shown to be not exactly zero but small in the pass band region.

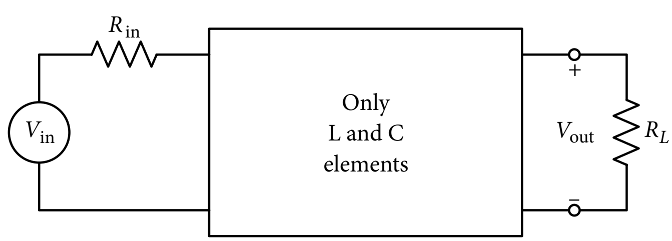

Figure 6.6 A doubly terminated lossless ladder structure in block form.

It needs to be kept in mind that though the sensitivities remain small in the pass band while using doubly terminated lossless ladders, the sensitivities with respect to inductors and capacitors can be large in the transition and stop bands.

After it was proved mathematically that a doubly terminated lossless ladder designed for maximum power transfer possesses low sensitivities in the pass band, they have been extensively used in the passive RLC filter design. Their modified versions in active forms have also been used in large numbers. Such methods include either active simulation of active components or operational simulation of passive components (Chapter 9).

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

6.5.2 Sensitivity in cascade design

The direct realization of higher-order filters means that all poles and zeros are functions of almost all the elements used, which means that each of the term $S_x^{z_i}$ and $S_x^{p_i}$ is finite. Instead of using the doubly terminated ladder structure, an alternate method of reduction in the transfer function sensitivities comes in the form of cascade realization (Chapter 10) of higher-order filters. In this method, any element x affects only one pole pair and corresponding zero(s). Sensitivity of other pole pairs and zero(s) is zero with respect to this particular element x. Under such a condition, equation (6.31) reduces to the following for a second-order section:

$$S_x^{H(s)} = S_x^K + \frac{(Q_x^{p_1})^*}{(s-p_1^*)} + \frac{Q_x^{p_1}}{(s-p_1)} - \frac{Q_x^{z_1}}{(s-z_1)} - \frac{(Q_x^{z_1})^*}{(s-z_1^*)} \tag{6.51}$$

If the second-order function represented by poles and zeros of equation (6.51) is an all pole function (or the effect of sensitivity of zeros and the gain constant K is not affective in the pass band), equation (6.51) reduces to the following.

$$S_x^{H(s)} = \frac{(Q_x^{p_1})^*}{(s-p_1^*)} + \frac{Q_x^{p_1}}{(s-p_1)} \tag{6.52}$$

Equation (6.52) informs that the transfer function sensitivity with respect to the element x near pole pair p_1, p_1^* depends only on this particular second-order function in the cascade formation. For the rest of the second-order functions in the cascade, their sensitivity with respect to x is zero. This result has led to the useful cascade design methodology of higher filter design discussed in Chapter 10. Of course, each second-order section to be connected in cascade needs to be non-interactive with its transfer function.

Since the overall sensitivity of the transfer function $H(s)$ depends on the sensitivity of each one of the constituent second-order sections, it is important that each section is designed with optimum sensitivities with respect to the elements used in that section so that variability of $H(s)$ is minimum.

6.6 Multi-parameter Sensitivity

Incremental sensitivity (or single-parameter sensitivity) has been found to be very useful as a large amount of information becomes available through it. Most of the sensitivity studies discussed so far is sufficient for comparing filter circuits. However, incremental sensitivity-based comparison has to be applied with caution, remembering the assumptions made, which include that effective change in elements is small, the nominal point of the element x is fixed and the rest of the components remain unchanged. In practice, these assumptions might not be true. Element tolerances are different and almost all elements may vary simultaneously, either by the same amount or differently. Under such conditions, single-element sensitivity measures will not give precise information; they may even mislead by providing incorrect information depending on the amount of the magnitude of approximations. Therefore, it becomes important to study multi-parameter sensitivity measures. Unfortunately, because of the large number of varying elements, many of them inter-dependent as well, finding multi-parameter sensitivity becomes very involved and a high amount of computations are required, which necessitates the use of computers and related software.

Practice Problems

6-1 Derive the voltage ratio transfer function for the circuit shown in Figure P6.1, and show that it s realizes a BP response. Determine the incremental sensitivity of the parameters ω_o, Q and the mid-band gain H with respect to the elements used.

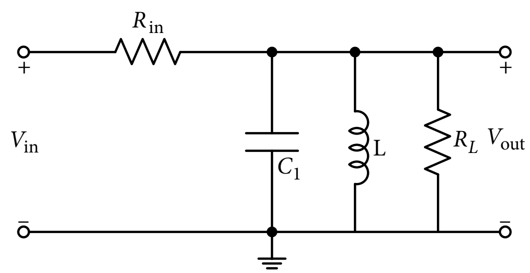

Figure P6.1

(a) Also find the changes in the parameters ω_o, Q and H, when the inductor changes by +5%, the capacitance changes by – 6% and resistors change by +10%.

(b) If all the elements are likely to change in the positive or negative direction by the same amount as in part (a), then calculate the worst case percent deviation in the filter parameters.

6-2 Repeat Problem 6-1 for the circuit displayed in Figure P6.2 after showing that it realizes a band stop response. The concerned filter parameters are ω_o, Q and attenuation α in the stop band.

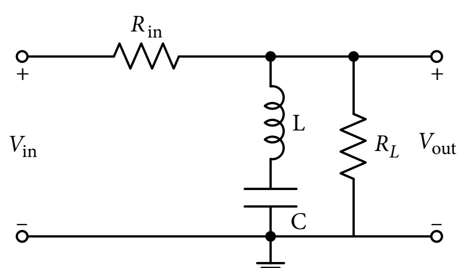

Figure P6.2

6-3 (a) Design a BP filter using the circuit diagram shown in Figure 6.3 with no positive feedback, while assuming that $\sqrt{(R_2/R_1)} = 10$, for $\omega_o = 10^4$ rad/s, and $Q = 5$. What shall be the value of mid-band gain H_m for $C_1 = C_2$. Find the sensitivities of the parameters ω_o, Q and H_m with respect to the elements R_1, R_2, C_1 and C_2.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

(b) What are the changes in the parameters when positive feedback is introduced with (i) $K = 1/10$, (ii) $K = 1/20$.

(c) Find the sensitivities for the parameters ω_o, Q and H_m with respect to the passive elements and K.

(d) With the given values of K in part (b), find the variability in Q using two approaches. Which one out of the two processes is more accurate?

6-4 Repeat Problem 6-3 for $\omega_o = 10^4$ rad/s, $Q = 2$ and $H_m = 12.5$, with the assumption that $\sqrt{(R_2 / R_1)} = 5$.

6-5 Prove the relations in equations (6.22)–(6.25) using equation (6.4), with P_i being a function of x, and n and k being constant.

6-6 Find the transfer function $H(s)$ for the circuit shown in Figure P6.3 and determine the expressions for the parameter ω_o and Q.

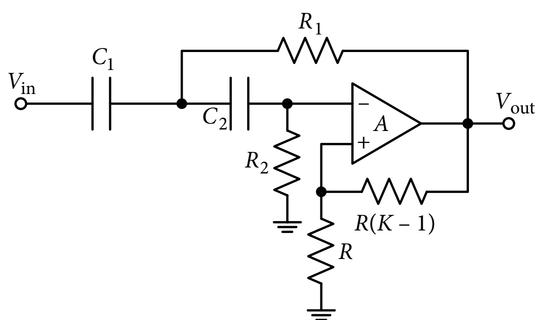

Figure P6.3

Find the sensitivities of ω_o, Q and $H(j\omega)$ with respect to all the passive and active elements. It would be better if the sensitivity expressions are given in terms of ω_o and Q.

6-7 For the circuit shown in Figure P6.3, $\omega_o = 5 \times 10^3$ rad/s and $Q = 20$.

(a) Show by calculation that the maximum of the sensitivity of $|H(j\omega)|$ with respect to Q is unity.

(b) Find the maximum and minimum sensitivity of $|H(j\omega)|$ with respect to ω_o.

(c) Find the magnitude and phase sensitivities of $|H(j\omega_n)|$ at $\omega_n = 0.25, 0.5, 1.0, 1.25$ and 1.5.

6-8 Obtain the voltage ratio transfer function for the circuit shown in Figure P6.4, and find the sensitivity of Q with respect to all the passive and active elements for:

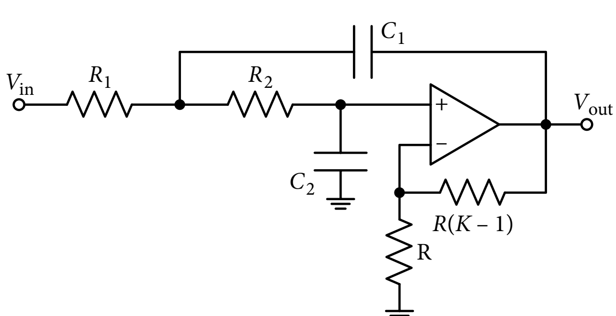

Figure P6.4

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press

(i) $R_1 = 1\Omega$, $R_2 = 1\Omega$, $C_1 = 4Q$, $C_2 = 1/4Q$, $K = 1$.

(ii) $R_1 = 1\Omega$, $R_2 = 2Q\,\Omega$, $C_1 = 1$, $C_2 = 1/2Q$, $K = 2.5$.

(iii) $R_1 = 1\Omega$, $R_2 = 1\Omega$, $C_1 = 1$, $C_2 = 1$, $K = 3 - 1/Q$.

Evaluate the sensitivity values found for $Q = 20$ and discuss the results.

6-9 Derive the transfer function for the circuit shown in Figure P6.5, while considering the OA as ideal and find the sensitivity expression for ω_o and Q with respect the resistors R_1 and R_2.

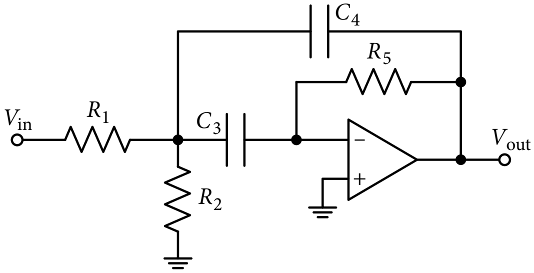

Figure P6.5

6-10 Design the filter in Figure P6.5 for ω_o = 10 krad/s and Q = 10, and determine the displacement in the pole location if (a) all resistors and capacitors increase by 5%, and (b) all resistor values increase and all capacitor values decrease by 5%.

6-11 For the filter circuit of Figure 6.3, (a) discus the effect of selecting the parameter $r = 4$ and (b) design the circuit for $\omega_o = (3.4 \times 2\pi)$ krad/s.

6-12 Find the transfer function sensitivity for the Tow–Thomas biquadratic filter circuit shown in Figure P6.6 with respect to the passive components R_1 and C_1.

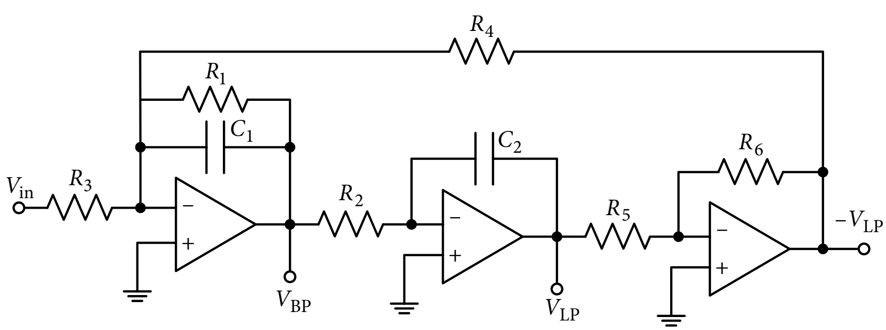

Figure P6.6 Tow–Thomas biquad.

https://doi.org/10.1017/9781108762632.007 Published online by Cambridge University Press