Chapter **3**

Magnitude Approximations

3.1 Introduction

Filters are generally classified in the frequency domain in terms of the amplitude and phase response of their transfer function; though sometimes they are expressed in the time domain as well. The typical characteristics of an ideal LPF (low pass filter) in terms of its variation of attenuation with frequency shown in Figure 1.6(a) is redrawn in Figure 3.1. The transition of the filter from being a pass band to being a stop band occurs abruptly at $\Omega = 1$.

It is well known that the transfer function of an ideal filter, in which transition between pass band and stop band is instant, is physically realizable only by using an infinite number of elements [3.1]. For a practically realizable filter, the transfer function is always expressed by a *real rational function* $H(s)$, which is a ratio of polynomials in complex variable, $s = (\sigma + j\omega)$ as already given in Section 1.2.1 and repeated here as equation (3.1).

$$H(s) = \frac{N(s)}{D(S)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_2 s^2 + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_2 s^2 + b_1 s + b_0} \tag{3.1}$$

In a real rational transfer function, coefficients a_i and b_j are real numbers and the degree of the numerator and the denominator is m and n, respectively. Moreover, the degree of the denominator, n, should be more than or equal to the numerator degree, m, for the physical realization of the transfer function using finite number of elements to be feasible. The condition $n \geq m$ is necessary because ideal filters are non-causal and, therefore, cannot be implemented practically.

To realize a practical form of an LPF, shown as approximated LPF characteristics using a dotted line in Figure 3.1, values of the coefficients a_i and b_j in equation (3.1) are to be determined. The next step will be to find the topology of the filter and values of element to

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be used, applying the coefficients of equation (3.1). Not only are different methods available for the realization of an arbitrary transfer function, but different forms of approximating the magnitude or phase of the transfer function are also available. Some classical methods of approximating the magnitude function are discussed in this chapter.

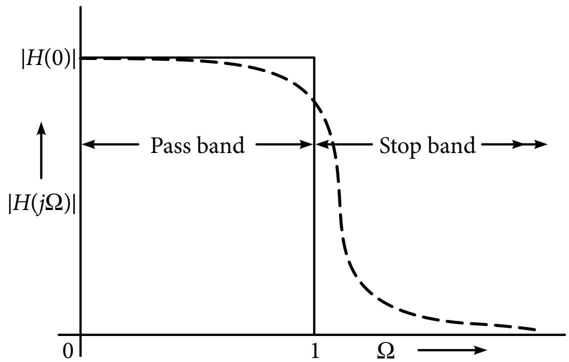

Figure 3.1 Magnitude characteristics of an ideal normalized low pass filter shown by solid line, and that of a practical or real filter shown by dotted line.

The procedure of magnitude approximation which begins by comparing an ideal LPF mathematically, with that of an approximated response is discussed in Section 3.2. One of the most commonly used approximations, namely the maximally flat Butterworth approximation and the design of a Butterworth approximation based LPF is studied in Section 3.3. Also included here is the utilization of a circuit structure in the form of a ladder, called a *lossless ladder*, containing only inductors and capacitors. Equal-ripple approximation is another very important class of magnitude approximation, whose sub-classifications – Chebyshev approximation, inverse Chebyshev approximation and Cauer approximation – have been found to realize filter sections rather economically. In the rest of the chapter, we will describe the development of prototype LPFs using these approximations. Examples have been included of filters of average level order ($n \sim 5,6$) filters. An example of a maximally flat pass band with finite zeros, the significance of which shall be seen later, is also included.

3.2 Magnitude Approximations

Response of the LPF shown by the dotted line in Figure 3.1 represents an approximation to the ideal LPF in terms of the magnitude of the transfer function. In the pass band region, gain of the transfer function is close to the ideal value at low frequencies; the gain reduces to a low value in the stop band region with a finite slope. In practice, the dotted line of the approximated response can take shapes other than the monotonic drop. Other important types of gain variation are discussed in brief in the following sections. Approximation can also be performed for the phase response of the ideal filter which shall be discussed in Chapter 4. In all the magnitude approximations of LPF, the transition band is finite instead of the abrupt transition from the pass band to stop band of the ideal filter. This means that there

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will be some deviation from the ideal, and hence, some error in the response. However, the amount of intentionally made error, shown in Figure 3.2, can be bounded, as the response has to remain restricted within the shaded region. The maximum allowable attenuation in the pass band is α_{max} and the minimum allowable attenuation in the stop band is α_{min}. The transition band separating the pass and stop band extends from ω_1 to ω_2. Depending on the specifications of the LPF in terms of α_{max}, α_{min}, ω_1 and ω_2, the next step is to find the topology of a network and the element values which satisfy these specifications. It is important to note the term *normalized angular frequency* $\Omega = (\omega/\omega_c)$; by convention, this means normalized cut-off frequency $\Omega_c = 1$.

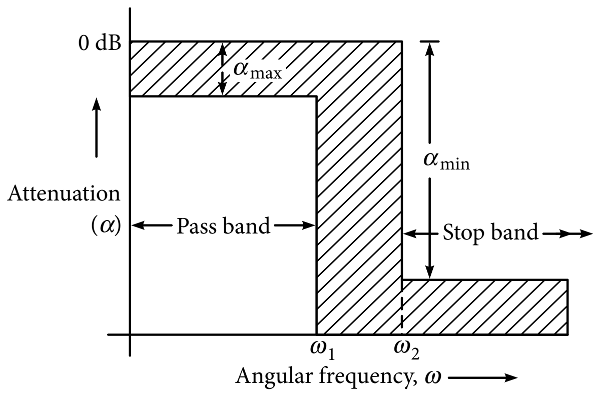

Figure 3.2 Approximated low pass characteristics lie within the shaded region.

Initially, an LP prototype transfer function is considered with all transmission zeros (zeros of the numerator) at infinity, that is, $N(s) = 1$; this is also commonly known as *all pole function*. A number of solutions can be obtained from the general amplitude function of the transfer function. Let us consider the magnitude squared transfer function: $|H(j\omega)|^2$:

$$|H(j\omega)|^2 = \frac{N(j\omega)N(-j\omega)}{D(j\omega)D(-j\omega)} = \left|\frac{N(j\omega)}{D(j\omega)}\right|^2 = \frac{A(\omega^2)}{B(\omega^2)} \tag{3.2}$$

$$= \frac{1}{|D(j\omega)|^2} \rightarrow \frac{1}{B(\omega^2)} \text{for } N(s) = 1 \tag{3.3}$$

Here, $|H(j\omega)|^2$ is an even rational function for which $|H(j\omega)|$ must be close to $|H(j0)|$ within the frequency range $0 \leq \omega \leq \omega_1$ in the pass band and close to zero for $\omega \geq \omega_2$ in the stop band. Using suitable frequency normalization with respect to pass band edge frequency ω_1, the normalized pass band edge frequency shall be $\Omega = (\omega/\omega_1)$. Hence, the pass band range is up to $\Omega = 1$, and since $N(s)$ has been selected as unity, $|H(j0)| = 1$ for all values of n. The function $|H(j\omega)|$ now modifies into a normalized function $|H(j\Omega)|$. For a mathematical understanding, it is preferable to express $|H(j\Omega)|^2$ in terms of another rational function $|K(j\Omega)|$, such that:

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$$\left|H(j\Omega)\right|^2 = \frac{1}{1+\left|K(j\Omega)\right|^2} \tag{3.4}$$

From equation (3.4), the following relation is obtained:

$$|K(j\Omega)|^2 = \{1/|H(j\Omega)|^2\} - 1 = |D(j\Omega)|^2 - 1 \tag{3.5}$$

Expression for the nth order magnitude squared function modifies from equation (3.3) as given in equation (3.6):

$$\left|H_n(j\Omega)\right|^2 = \frac{1}{B_{2n}\Omega^{2n} + B_{2n-2}\Omega^{2n-2} + \ldots + B_4\Omega^4 + B_2\Omega^2 + 1} \tag{3.6}$$

Therefore, the nth order function, $|K_n(j\Omega)|^2$ of equation (3.5) will transform to the following:

$$|K_n(j\Omega)|^2 = B_{2n}\Omega^{2n} + B_{2n-2}\Omega^{2n-2} + \ldots\ldots + B_4\Omega^4 + B_2\Omega^2 \tag{3.7}$$

This means that the squared magnitude of the characteristic function is a polynomial in Ω. It is the nature of $|K_n(j\Omega)|^2$ which give different forms of approximations for the ideal LPF (and as a consequence for other types of filter sections also) like maximally flat, Chebyshev, inverse Chebyshev or Cauer type.

3.3 Maximally Flat – Butterworth Approximation

A maximally flat response means that at $\Omega = 0$, not only is its slope (or its first derivative) zero, but the maximum number of derivatives are also equal to zero [3.2]. This stated condition requires that in equation (3.7), the maximum derivatives of K_n are zero, as shown in equation (3.8):

$$\frac{d^k\left|K_n(j\Omega)\right|^2}{d^k\left(\Omega^2\right)}\Big|_{\Omega=0} = 0 \text{ for } k = 1,2,\ldots..,n-1 \tag{3.8}$$

It means that we are required to make $B_{2n-2} = \ldots\ldots = B_4 = B_2 = 0$, resulting in the following expression, where ε is a characteristic term (a significant term effecting approximation):

$$|K_n(j\Omega)|^2 = B_{2n}\Omega^{2n} = \varepsilon^2\Omega^{2n} \tag{3.9}$$

Hence, for a maximally flat response, the magnitude of the squared nth order transfer function is expressed as:

$$\left|H_n(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2\Omega^{2n}} \to \left|H_n(j\Omega)\right| = \left(1+\varepsilon^2\Omega^{2n}\right)^{-1/2} \tag{3.10}$$

Response given by equation (3.10) is shown in Figure 3.3. Its magnitude decreases monotonically, and at $\Omega = 1$, the loss becomes $10\log_{10}(1+\varepsilon^2)$ dB, as its magnitude drops from $|H_n(0)|$ to $|H_n(0)|/(1+\varepsilon^2)^{0.5}$. Therefore, the expression for the maximum specified loss of $\alpha_{\max}$ in the pass band shall be as follows:

$$\alpha_{\max} = 10\log_{10}(1+\varepsilon^2) \tag{3.11}$$

This gives an important relation for the characteristic term ε as:

$$\varepsilon = \left(10^{0.1\alpha_{\max}} - 1\right)^{1/2} \tag{3.12}$$

In the normalized magnitude form of the LP function with maximum flatness at dc ($\Omega = 0$), when $\varepsilon = 1$, it is also called the *Butterworth approximation*; the characteristics being very similar, the terms *maximally flat* and *Butterworth approximation* are sometimes used synonymously.

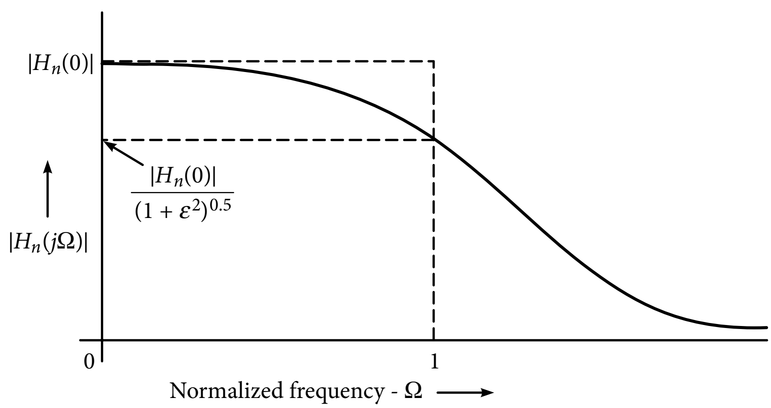

Figure 3.3 Maximally flat normalized low pass response.

For the Butterworth approximation if $\varepsilon = 1$, attenuation at the edge of the pass band, obtained from equation (3.11), is simply:

$$\alpha_{\max} = 3 \text{ dB} \tag{3.13}$$

Substituting $\varepsilon = 1$ in equation (3.10), the nth order frequency de-normalized Butterworth response is obtained using the following relation, as mentioned here:

$$|H_n(j\omega)|^2 = \frac{1}{(1+\omega^{2n})} \tag{3.14}$$

As all pole responses were selected in the beginning with $N(s) = 1$, the Butterworth response has zeros only at $\omega = \infty$. The response, shown in Figure 3.4, has the following important properties as well.

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i. Magnitude of the transfer function at $\omega = 0$ is unity for all values of n.

ii. For all values of n, magnitude $|H_n| = 1/\sqrt{2}$ at $\Omega = 1(\omega = \omega_c)$, corresponding to the attenuation of 3 dBs.

iii. In the stop band, for $\omega > \omega_c$ ($\Omega > 1$), $|H_n|$ decreases at the rate of $20n$ dBs per decade.

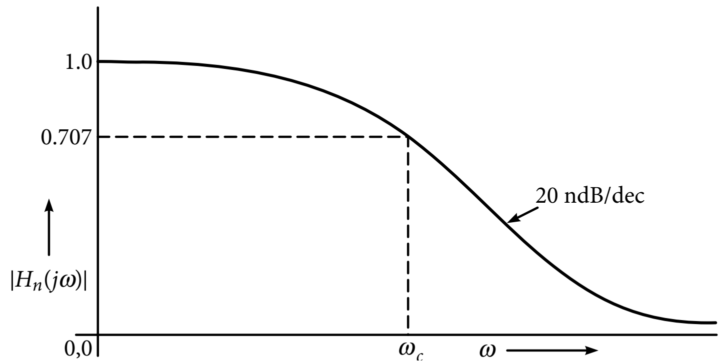

Figure 3.4 Butterworth response having loss of 3 dB at cutoff frequency ω_c.

The transfer function $H(s)$ of equation (3.1) will have n poles. In order to find poles with the Butterworth approximation ω is replaced by (s/j) in equation (3.14). Hence, the poles can be obtained by the roots in the left half-plane of the following relation:

$$D(s)D(-s) = 1 + (-s^2)^n \tag{3.15}$$

These poles have been found to be located on a semicircle in the s-plane whose value (location) can be evaluated from the following:

$$S_k = -\sin\left(\frac{2k-1}{n}\right)\pi \pm j\cos\left(\frac{2k-1}{n}\right)\pi \tag{3.16}$$

The pole locations for $k = 1, 2, \ldots, n$ (up to $n = 8$) are shown in Table 3.1. Coefficients of the Butterworth polynomial $D(s)$ can be obtained from the following recursive relation, with $b_0 = 1$.

$$b_k = b_{k-1}\left\{\cos\left(\frac{k-1}{n}\pi\right)\right\} \bigg/ \left\{\sin\left(\frac{k}{n}\pi\right)\right\} \tag{3.17}$$

Table 3.2 shows the calculated values of the coefficient b_k up to $n = 8$. Coefficient values in Table 3.2 and for any larger values of n can be calculated from equation (3.17).

Table 3.1 Pole locations for the Butterworth responses

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
–0.7071068	–0.5000000	–0.3826834	–0.8090170	–0.2588190	–0.900968	–0.1950903
±*j*.7071068	±j0.8660254	±*j*0.9238795	±*j*0.587785	±*j*0.9659258	±*j*.4338837	±*j*0.9807853
	–1.0000000	–0.9238795	–0.3090170	–0.7071068	–0.2225209	–0.5555702
		±*j*0.3826834	±*j*0.9510565	±*j*0.7071068	±*j*.9649279	±*j*0.8314696
			–1.0000000	–0.9659258	–0.6234898	–0.8314696
				±*j*0.2588190	±*j*.7818315	±*j*0.5555702
					–1.0000000	–0.9807853
						±*j*0.1950903

Table 3.2 Coefficients of the Butterworth polynomial $B_n(s) = s^n + \sum_{k=0}^{n-1} b_k s^k$

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
2	1.000	1.4142136						
3	1.000	2.0000000	2.0000000					
4	1.000	2.6131259	3.4142136	2.6131259				
5	1.000	3.2360680	5.2360680	5.2360680	3.2360680			
6	1.000	3.8637033	7.4641016	9.1416202	7.4641016	3.8637033		
7	1.000	4.4939592	10.0978347	14.5917939	14.5917939	10.0978347	4.4939592	
8	1.000	5.1258309	13.137071	21.846151	25.688355	21.846151	13.1370712	5.1258309

3.3.1 Design of low pass Butterworth filter

For designing an LPF (low pass filter), specifications are given in different ways. For example, along with the value of cutoff frequency ω_c (for which $\varepsilon = 1$), $\alpha_{\min}$ is given beyond the stop band corner frequency ω_2. Alternatively, specification can also be given in terms of $\alpha_{\max}$ up to the corner frequency of the pass band ω_1 and $\alpha_{\min}$ beyond the stop band corner frequency ω_2, as shown in Figure 3.2. In order to get a suitable topology and the values of elements used in it, pole locations for the Butterworth response or coefficients of the Butterworth polynomial are to be obtained using Table 3.1 and 3.2, respectively. However, to get either of the values, the order n is to be determined first; the other variable ε has already been given a value of unity for the Butterworth response – if $\varepsilon \neq 1$ for the general maximally flat response, it has to be calculated from the specifications.

At the pass band corner and stop band corner, respectively, we can write:

$$\alpha_{\max} = 10\log_{10}\left(1+\varepsilon^2\omega_1^{2n}\right) \tag{3.18}$$

$$\alpha_{\min} = 10\log_{10}\left(1+\varepsilon^2\omega_2^{2n}\right) \tag{3.19}$$

From equations (3.18) and (3.19), we get the following expressions:

$$\varepsilon^2\omega_1^{2n} = \left(10^{0.1\alpha_{\max}} - 1\right) \tag{3.20}$$

$$\varepsilon^2\omega_2^{2n} = \left(10^{0.1\alpha_{\min}} - 1\right) \tag{3.21}$$

Dividing equation (3.21) by equation (3.20), we get:

$$(\omega_2 / \omega_1)^{2n} = \frac{10^{0.1\alpha_{\min}} - 1}{10^{0.1\alpha_{\max}} - 1} \tag{3.22}$$

Taking log on both sides of equation (3.22), the expression for the degree n is obtained as follows:

$$n = \frac{\log\left[(10^{0.1\alpha_{\min}} - 1) / (10^{0.1\alpha_{\max}} - 1)\right]}{2\log(\omega_2 / \omega_1)} \tag{3.23}$$

Solution of equation (3.23) yields the value of n which should be able to satisfy the given filter specifications. In almost all cases, it is not possible to obtain integers for the calculated value of n – n then has to be rounded off to the next higher integer value for obvious reasons.

To utilize the large amount of data available for the Butterworth response, in terms of pole locations and transfer function for any order n of the filter [1.2], it is useful to find the normalized cutoff frequency ω_{CB} at which attenuation is 3 dB (Table 3.1 and Table 3.2 are small subsets of such information). For α = 3 dB, replacing ω_1 by ω_{CB} in equation (3.18) means:

$$3 = 10\log\left(1 + \varepsilon^2\omega_{CB}^{2n}\right) \tag{3.24}$$

or $\omega_{CB} = [(10^{0.3} - 1)/\varepsilon^2]^{1/2n}$ (3.25)

Equation (3.25) is an important relation between Butterworth and maximally flat responses.

3.3.2 Use of lossless ladder

In a large number of cases while realizing active filters, the starting point is a passive structure. Though different passive structures are available, one of the most used structures is a doubly terminated lossless ladder. Hence, the topic of lossless ladders and their utilization is important and a matter of serious study. In this section, we will discuss the basics of lossless ladders in order to understand their use in developing an all-pole LPF structure. In its most simple form, a terminated lossless ladder is as shown in Figure 3.5(a). The ladder consists of only inductors and capacitors connected in a ladder form with input and output terminating resistances. This ladder structure has been studied extensively for its utilization in realizing passive filters with different magnitude approximations like Butterworth, Chebyshev, or Cauer. Element values for the ladder structure for all the common approximation methods have been calculated and made available for filter orders starting from $n = 2$ to higher n values. Figure 3.5(b) and (c) show the structure of a lossless ladder. The element values for Butterworth approximated filters of order $n = 2$ to 8 for the ladders shown in Figure 3.5 are presented in Table 3.3. The last element in Figure 3.5(b) and (c) differs depending on if n is odd or even for all pole LPFs. The ladder will be either minimum capacitor or minimum inductor when n is odd as the number of inductors will be one more than a capacitor or vice versa. When n is even, the number of inductors and capacitors will be equal and their total will be equal to n as a doubly terminated ladder is a *canonic* structure using the minimum number of dynamic elements.

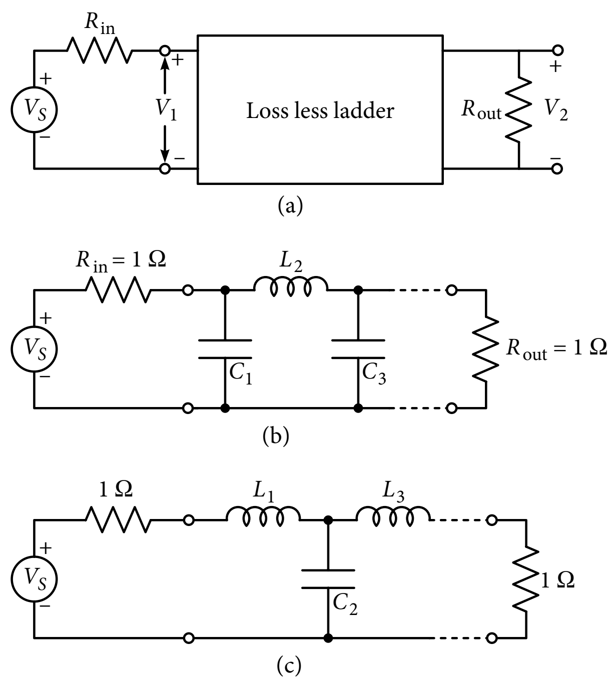

Figure 3.5 (a) Basic structure of a doubly terminated lossless ladder; (b) and (c) Two normalized forms of lossless ladders.

In the normalized low pass doubly terminated ladder, the terminating resistors $R_{in} = R_{out} = 1\ \Omega$ and the frequency normalization is assumed to be done with respect to its 3 dB frequency ω_c. If ω_c is not at 3 dB frequency, a different de-normalizing frequency is to be used, which is given by equation (3.25) as discussed earlier.

Table 3.3 Element values for a doubly terminated lossless ladder for an all-pole LPF using Butterworth approximation

n	C_1	L_2	C_3	L_4	C_5	L_6	C_7	L_8
2	1.414	1.414						
3	1.000	2.000	1.000					
4	0.7654	1.848	1.848	0.7654				
5	0.6180	1.618	2.000	1.618	0.618			
6	0.5176	1.414	1.932	1.932	1.414	0.5176		
7	0.4450	1.247	1.802	2.000	1.802	1.247	0.445	
8	0.3902	1.111	1.663	1.962	1.962	1.663	1.111	0.3902
n	L_1	C_2	L_3	C_4	L_5	C_6	L_7	C_8

Example 3.1: Find the order of the maximally flat LPF which will satisfy the following specifications. Also find the corresponding transfer function.

$$\alpha_{max} = 1\text{dB},\ \alpha_{min} = 40\text{dB},\ \omega_1 = 2000 \text{ rad/s, and } \omega_2 = 6000 \text{ rad/s.}$$

Solution: To find the order of the filter and its transfer function, equations (3.12) and (3.23) are used for calculating ε and n, respectively.

$$\varepsilon^2 = (10^{0.1} - 1) = 0.25892 \rightarrow \varepsilon = 0.50884$$

$$n = \frac{\log\left[(10^{0.1\times 40} - 1)/(10^{0.1\times 1} - 1)\right]}{2\log(6000/2000)} = 4.807$$

which is rounded to the next integer, $n = 5$.

For the fifth-order Butterworth filter, the values of the pole locations from Table 3.1 gives the following frequency-normalized transfer function.

$$H(S) = \frac{N(S)}{D(S)} = \frac{1}{(S+1)\left(S^2 + 0.618036S + 1\right)(S^2 + 1.6186S + 1)} \tag{3.26}$$

$$= \frac{1}{S^5 + 3.236S^4 + 5.236S^3 + 5.236S^2 + 3.236S + 1} \tag{3.27}$$

For using equations (3.24) and (3.25), which are valid for the Butterworth response (and not for a maximally flat response), S shall be replaced by $(j\omega_{CB})$. Hence, using equation (3.25), the normalized cutoff frequency is as follows.

$$\omega_{CB} = \{(10^{0.3} - 1)/0.25892\}^{0.5 \times 5} = 1.144$$

The de-normalized cutoff frequency is given as:

$$\omega_c = \omega_{CB} \times \omega_1 \cong 1.144 \times 2000 = 2288 \text{ rad/s}$$

Hence, equations (3.26) and (3.27) can be modified to the following for the de-normalized frequency

$$H(s) = \frac{1}{(s+2288)(s^2+1414s+2888^2)(s^2+3701.98s+2288^2)} \tag{3.28}$$

$$= \frac{1}{s^5+7.408\times10^3 s^4+2.741\times10^7 s^3+6.2714\times10^{10} s^2+9.11478\times10^{13} s+6.27018\times10^{16}} \tag{3.29}$$

Obviously, the next step is to find an active network topology containing the suitable active devices and the values of the passive elements used. One of the most commonly used architecture employs operational amplifiers (OAs) as the active device along with resistances and capacitances (forming the active RC structure). A large variety of procedures are available which lead to the active RC topology and the passive element values for the transfer function given in the form of equations (3.26) to (3.29). These procedures will be discussed later after studying other forms of approximations.

In this section, we make use of Table 3.3 and the lossless ladder of Figure 3.5. For n = 5, if the minimum inductor configuration of Figure 3.5(b) is used, the structure's normalized element values from Table 3.3 will be as follows:

$$C_1 = C_5 = 0.618\text{F},\ C_3 = 2.0\text{ F and } L_2 = L_4 = 1.618\text{H}$$

De-normalization of the elements is done by using a frequency scaling factor of 2288 rad/s and an impedance scaling factor of 1 kΩ. The de-normalized element values are as follows:

$$C_1 = C_5 = 0.2701\ \mu\text{F}, C_3 = 0.8741\ \mu\text{F},\ L_2 = L_4 = 0.707\text{ H and } R_{in} = R_{out} = 1\text{ k}\Omega.$$

The passive ladder shown in Figure 3.6 is simulated and the magnitude response is shown in Figure 3.7. At 318.018 Hz (1998.97 rad/s), attenuation was found to be 0.997 dB and at 955.85 Hz (6008 rad/s), attenuation was 41.9 dBs – an excellent response. The cutoff frequency was found to be at 364.15 Hz (2288.9 rad/s) against the theoretical value of 2288 rad/s. The phase response of the passive filter is also shown in Figure 3.7; it has a phase shift of 180° at the cutoff frequency.

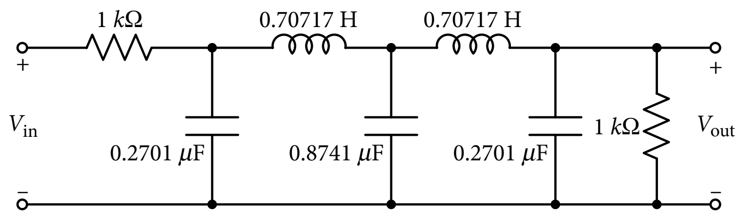

Figure 3.6 Fifth-order Butterworth doubly terminated de-normalized lossless ladder for Example 3.1.

Active realization of this passive fifth-order filter shall be taken up in Chapter 10 using the cascade technique.

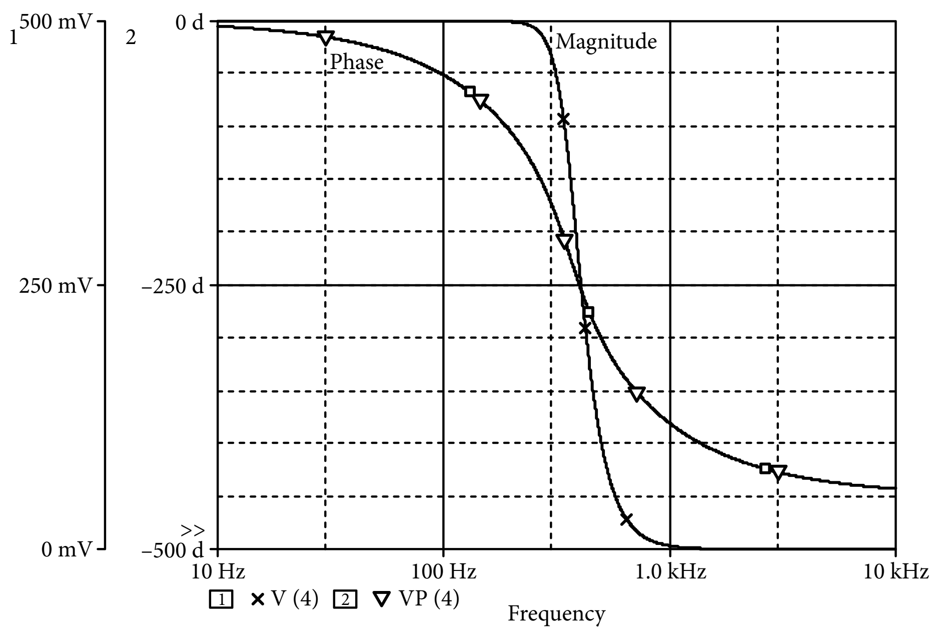

Figure 3.7 Simulated response of the fifth-order low pass Butterwoth filter shown in Figure 3.6 for Example 3.1.

3.4 Equal-ripple Approximations

It is often desirable to obtain a faster attenuation rate beyond the pass band corner frequency – as fast as is practically and economically feasible with lesser number of elements. In the maximally flat Butterworth response, the order of the filter is n and hence, the number of elements used becomes large in order to achieve larger attenuation. Hence, to improve on the value of n, the condition of being maximally flat in the pass band can be dropped. The magnitude characteristic is allowed to ripple between a series of maxima and minima. Ripples can be obtained only in the pass band or stop band, or in both, resulting in following further classifications.

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3.5 Chebyshev Approximation

The Chebyshev approximation of a magnitude function is obtained when ripples of equal height appear in the pass band of the transfer function along with a sharp decrease in the gain beyond it. To get such an approximation, the characterizing function of equation (3.5) is selected in the normalized frequency range of $0 \le \Omega \le 1$ as:

$$\left|K\left(j\Omega\right)\right|^2 = \varepsilon^2 C_n^2\left(\Omega\right) = \varepsilon^2 \cos\mathrm{h}^2\left\{n \cos\mathrm{h}^{-1}\left(\Omega\right)\right\} \tag{3.30}$$

Once again, ε is a real constant which is less than 1 and the Chebyshev polynomials are evaluated from the following recursive relation:

$$C_n\left(\Omega\right) = 2\Omega C_{n-1}\left(\Omega\right) - C_{n-2}\left(\Omega\right) \tag{3.31}$$

where $C_0(\Omega) = 1$ and $C_1(\Omega) = \Omega$ and for $\Omega \ge 1$, Chebyshev polynomial is given as follows:

$$C_n(\Omega) = \cos\mathrm{h}\{n \cos\mathrm{h}^{-1}(\Omega)\} \tag{3.32}$$

The amplitude response of the Chebyshev approximation can be obtained from equations (3.2), (3.4), and (3.30). For example, Figure 3.8 shows such an approximation for $n = 4$ (not to the scale), where ripples are shown for an even value of n. For odd values of n, the ripple height remains the same depending on the value of ε; however, at $\Omega = 0$, the function magnitude $|H(0)| = 1$ and for even value of n, $|H(0)| = (1 + \varepsilon^2)^{-½}$.

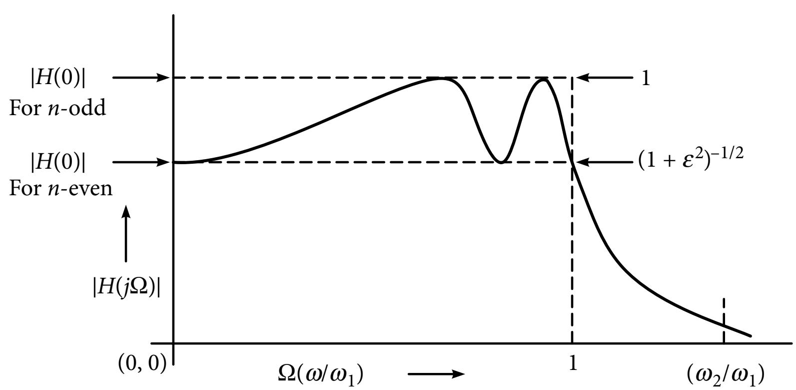

Figure 3.8 Magnitude function of a normalized Chebyshev approximation for order $n = 4$.

3.5.1 Low pass Chebyshev filter design

In order to design a low pass Chebyshev filter, we proceed like the Butterworth case, with ω_1 and ω_2 being the pass band corner frequency and stop band corner frequency, respectively.

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At the pass band corner frequency $\omega_{1,}$ use of equations (3.4), (3.10), and (3.30) gives the following expression for maximum attenuation:

$$\alpha_{\max} = 10\log\left\{1+\varepsilon^2 C_n^2(\omega_1)\right\} \tag{3.33}$$

Since at $\omega = \omega_1$ (Ω = 1), $C_n^2(\omega_1) = 1$, from equation (3.33), we get:

$$\varepsilon^2 = (10^{0.1\alpha_{\max}} - 1) \tag{3.34}$$

and at normalized ω_2, that is, at $\left(\dfrac{\omega_2}{\omega_1}\right)$ or $\left(\dfrac{\omega_s}{\omega_p}\right)$, $\alpha_{\min}$ being the minimum attenuation reached in stop band, its expression is obtained in the same way as equation (3.33) was obtained: use of equation (3.19) gives the expression for minimum attenuation in the stop band $\alpha_{\min}$ as:

$$\alpha_{\min} = 10\log\left\{1+\varepsilon^2 C_n^2(\omega_2/\omega_1)\right\} \tag{3.35}$$

Use of equation (3.32) modifies the expression for $\alpha_{\min}$ as follows:

$$\alpha_{\min} = \{10\log\{[1 + \varepsilon^2 \cosh^2 \{n \cosh^{-1}(\omega_2/\omega_1)\}\}] \tag{3.36}$$

Substituting ε^2 from equation (3.34) in equation (3.36), we get:

$$\cosh\left(n \cosh^{-1}\left(\frac{\omega_2}{\omega_1}\right)\right) = \left(\frac{10^{0.1\alpha_{\min}} - 1}{10^{0.1\alpha_{\max}} - 1}\right)^{0.5} \tag{3.37}$$

which gives the expression for the order n for the Chebyshev case as follows:

$$n = \frac{\cosh^{-1}\sqrt{\left[(10^{0.1\alpha_{\min}} - 1)/(10^{0.1\alpha_{\max}} - 1)\right]}}{\cosh^{-1}(\omega_2/\omega_1)} \tag{3.38}$$

However, a more convenient form of expression is given in equation (3.39) if cosh function is replaced with the natural log function:

$$n \cong \frac{l_n\left[4(10^{0.1\alpha_{\min}} - 1)/(10^{0.1\alpha_{\max}} - 1)\right]^{½}}{l_n\left[(\omega_2/\omega_1) + ((\omega_2/\omega_1)^2 - 1)^{½}\right]} \tag{3.39}$$

The value of n obtained from equation (3.39) has to be rounded up to the next integer. For order n, analysis has given the location of the left half pole the required transfer function as:

$$s_k = \sigma_K + j\Omega_k \tag{3.40}$$

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where,

$$\sigma_k = -\sin h(a) \sin \frac{(2k-1)}{n}\frac{\pi}{2} \tag{3.41a}$$

$$\Omega_k = j\cos h(a) \cos \frac{(2k-1)}{n}\frac{\pi}{2} \qquad k = 0, 1, 2\ (2n-1) \tag{3.41b}$$

$$a = (1/n)\ \sinh^{-1}(1/\varepsilon) \tag{3.42}$$

It is observed that the poles lie on an ellipse in the complex frequency s plane and substituting ε and n in equation (3.42) and equation (3.41) gives the location (values) of poles. There are extensive tables available that provide the location of poles for various combinations of ε and n. Table 3.4 is a subset of such a table for ε = 0.5dB, 1.0 dB, and 2.0 dB only up to $n = 6$.

In Table 3.4, the second-order factor for the Chebyshev function in terms of α and β, that is, $(s^2 + 2\alpha s + \alpha^2 + \beta^2)$ is given. It results in the pole frequency $\omega_o = (\alpha^2 + \beta^2)^{1/2}$ and the pole quality factor $Q = (\alpha^2 + \beta^2)^{1/2}/(2\alpha)$. However, in general, the pole frequency and the pole quality factor in terms of the real and imaginary parts of the pole are given as follows:

$$\omega_{ok} = \left(\sigma_k^2 + \omega_k^2\right)^{½}, Q_k = \left(\omega_{ok} / 2\sigma_k\right) \tag{3.43}$$

Table 3.4 Pole locations for the Chebyshev approximation, $s = (-\alpha + j\beta)$

N	α_{max} = 0.5 dB		α_{max} = 1 dB		α_{max} = 2 dB	
	α	β	α	β	α	β
1	2.8628	0	1.9625	0	1.3076	0
2	0.7128	1.0040	0.5489	0.8951	0.4019	0.8133
3	0.3132	1.0219	0.2471	0.9660	0.1845	0.9231
	0.6265	0	0.4942	0	0.3689	0
4	0.1754	1.0163	0.1395	0.9834	0.1049	0.9580
	0.4233	0.4209	0.3369	0.4073	0.2532	0.3968
5	0.1120	1.0116	0.0895	0.9901	0.0675	0.9735
	0.2931	0.6252	0.2342	0.6119	0.1766	0.6016
	0.3623	0	0.2895	0	0.2183	0
6	0.0777	1.0085	0.0622	0.9934	0.0470	0.9817
	0.2121	0.7382	0.1699	0.7272	0.1283	0.7187
	0.2898	0.2702	0.2321	0.2662	0.1753	0.2630

Example 3.2: Determine the pole location for the Chebyshev response for n = 3 and α_{max} = 0.5 dB.

Solution: From equation (3.34): $\varepsilon^2 = (10^{0.05} - 1) = 0.122$, $\varepsilon = 0.3493$

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The value of the parameter a from equation (3.42) is obtained as follows:

$$a = (1/3)\ \sinh^{-1}(1/0.3493) = 0.5913$$

which gives sin (ha) = 0.6264 and cos (ha) = 1.18. The location of poles is obtained from equation (3.40) and (3.41) as follows:

$$s_1 = -0.6264, s_2, s_3 = -0.3132 \pm j1.0219 \tag{3.44a}$$

Therefore, the denominator of the transfer function shall be as follows:

$$D(s) = (s + 0.6264)(s^2 + 0.6264s + 1.1424) \tag{3.44b}$$

Example 3.3: Find the order of the Chebyshev LPF for the following specifications. Also find the corresponding transfer function.

$$\alpha_{\max} = 0.5\,\text{dB},\ \alpha_{\min} = 40\,\text{dB}, \omega_1 = 2000\frac{\text{rad}}{\text{s}} \text{and } \omega_2 = 6000\,\text{rad / s} \tag{3.45}$$

Solution: Using equation (3.39), order n is evaluated as follows:

$$n = \frac{l_n\left[4(10^4 - 1)/(10^{0.05} - 1)\right]^{½}}{l_n\left[(6000/2000) + (6000/2000)^2 - 1)^{½}\right]} = 3.6 \tag{3.46}$$

This needs to be rounded up to the next integer as 4.

Pole locations can be found as in Example 3.3 or directly using Table 3.4, which are as follows:

$$s_1, s_2 = -0.1754 \pm j1.0163 \text{ and } s_3, s_4 = -0.4233 \pm j0.4209 \tag{3.47}$$

Hence, the normalized transfer function shall be given as follows:

$$H(s) = \frac{0.3377}{\left(s^2 + 0.3508s + 1.0636\right)(s^2 + 0.8466s + 0.3563)} \tag{3.48}$$

For an even-order transfer function $H(0) = H(1) = \alpha_{\max} = 0.5$ dB or 0.944 (normalized), the numerator in $H(s) = (0.944 \times 1.0636 \times 0.3563) = 0.3577$. The obtained transfer function can be realized by direct form synthesis or as a cascade of two second-order non-interactive filter sections. However, its frequency level needs to be de-normalized with respect to 2000 rad/s. The de-normalized transfer function will be as follows:

$$H(s) = \frac{0.3577 \times 2000^2}{\left(s^2 + 0.3508 \times 2000s + 1.0636 \times 2000^2\right)(s^2 + 0.8466 \times 2000s + 0.3563 \times 2000^2)} \tag{3.49}$$

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As in the case of a Butterworth approximated filter, the doubly terminated lossless ladder is also a starting point for active filters when the Chebyshev approximation is used. However, in this case as the corner frequency depends on the ripple width, separate tables are needed for element values for different values of ripple widths. With reference to the lossless ladders of Figure 3.5(b) and (c), Table 3.5 is a small subset containing some commonly used data. For filter requirements not appearing in Table 3.5, we can either consult literature [1.2] or element values can be derived. It is important to note that normalized $R_{in} = 1\Omega$, but it is not equal to R_{out} for even n; its expression is given as $R_{out} = \left\{1 + 2\varepsilon^2 \pm 2\varepsilon\sqrt{(1+\varepsilon^2)}\right\}R_{in}$.

Table 3.5 LPF element values for Chebyshev approximated response

n	C_1	L_2	C_3	L_4	C_5	L_6	C_7	L_8	R_{out}
				(a) Ripple width = 0.1 dB					
2	0.84304	0.62201							0.73781
3	1.03156	1.14740	1.03156						1.00000
4	1.10879	1.30618	1.77035	0.81807					0.73781
5	1.14681	1.37121	1.97500	1.37121	1.14681				1.00000
6	1.16811	1.40397	2.05621	1.51709	1.90280	0.86184			0.73781
7	1.18118	1.42281	2.09667	1.57340	2.09667	1.42281	1.18118		1.00000
8	1.18975	1.43465	2.11990	1.60101	2.16995	1.58408	1.94447	0.87781	0.73781
				(b) Ripple width = 0.5 dB					
3	1.5963	1.0967	1.5963						1.0000
5	1.7058	1.2296	2.5408	1.2296	1.7058				1.0000
7	1.7373	1.2582	2.6383	1.3443	2.6383	1.2582	1.7373		1.0000
				(c) Ripple width = 1 dB					
3	2.0236	0.9941	2.0263						1.0000
5	2.1349	1.0911	3.0009	1.0911	2.1349				1.0000
7	2.1666	1.1115	3.0936	1.1735	3.0936	1.1115	2.1666		1.0000
	L_1	C_2	L_3	C_4	L_5	C_6	L_7		

3.6 Inverse Chebyshev Approximations

Instead of a maximally flat response in the pass band, having equal ripples in it enables us to realize an active filter having the same specification with a lesser order network; hence, an equal ripple response is more economical than a flat response. It is expected that equal ripples in the stop band structure may further improve response realization. Such an approximation is known as an *inverse Chebyshev approximation*. Further, if there are equal ripples in both the pass band and the stop band responses, it is known as an *elliptic or Cauer approximation*. First, considering the inverse Chebyshev approximation, we can see that allowable attenuation at the edge of the stop band is α_{min}. It is obvious that this may serve no useful purpose for further

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reduction in $\alpha_{\min}$ with frequency, as shown in Figure 3.9. For this kind of approximation, its magnitude function $|H_n(j\Omega)|$ is given by the following relation using equations (3.4) and (3.30):

$$|H_n(j\Omega)|^2 = \frac{1}{1+1/\left[\varepsilon^2 c_n^2(1/\Omega)\right]} \tag{3.50}$$

where its stop band edge frequency Ω_s is normalized to 1 as shown in Figure 3.9. The significant difference between the inverse Chebyshev and the Chebyshev function is that the frequency normalization in the inverse Chebyshev case is done with respect to the stop band edge frequency (ω_s or ω_2), whereas normalization was done with respect to the pass band edge frequency (ω_p or ω_1) in the Chebyshev approximation. Since $C_n(1) = 1$ for all values of n, the magnitude squared function at $\Omega = 1$ is given as follows:

$$|H_n(j1)|^2 = \frac{1}{\left(1+1/\varepsilon^2\right)} \rightarrow \left|H_n(j1)\right| = \varepsilon/\left(1+\varepsilon^2\right)^{½} \tag{3.51}$$

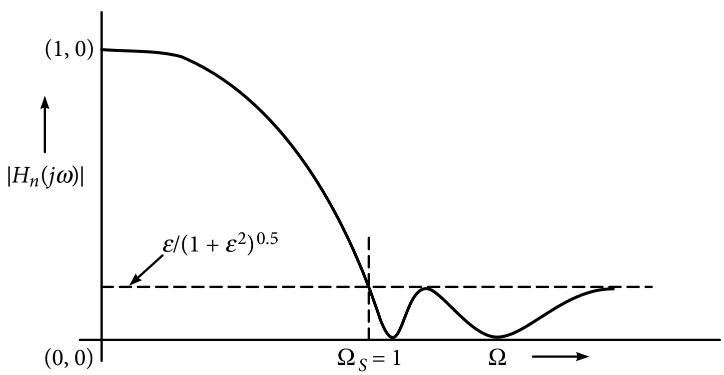

Figure 3.9 Magnitude function variation in a normalized inverse Chebyshev approximation.

The magnitude given by equation (3.51) is the upper limit of the inverse Chebyshev function in the stop band extending from $\Omega = 1$ to ∞, as shown in Figure 3.9. The nature of the magnitude of the ripples in the stop band is the same as it was in the pass band of the Chebyshev function. The number of maxima and minima are also equal to the order of the inverse Chebyshev function as in the pass band. To find the nature of variation of magnitude in the pass band, investigation has to be done at $\Omega = 0$ or $\Omega \ll 1$.

$$\text{As } C_n(1/\Omega) \simeq 2^{n-1}(1/\Omega)^n \text{ for } \Omega \ll 1 \tag{3.52}$$

$$\left|H_n(j\Omega)\right|^2 \cong \frac{1}{1+1/\varepsilon^2\left\{(2^{n-1})(1/\Omega^n)\right\}^2} = \frac{1}{1+\Omega^{2n}/(\varepsilon^2 2^{2n-2})} = \frac{1}{1+(\Omega/\Omega_k)^{2n}} \tag{3.53}$$

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where $\Omega_k = (\varepsilon 2^{n-1})^{1/n}$ (3.54)

The nature of equation (3.53) is the same as that of the maximally flat function of equation (3.10), which means that pass band of the inverse Chebyshev response is a maximally flat type of response.

3.6.1 Design of an inverse Chebyshev filter

In the previous section, it was shown that for the inverse Chebyshev response, ripples in the stop band extend from $\Omega = 1$ to ∞, where $\Omega = 1$ corresponds to the edge of the stop band and the function remains maximally flat in the pass band. This means that for an LPF, attenuation specifications will be as shown in Figure 3.10, where α_{max} is the allowable attenuation in the pass band; the attenuation in the stop band has to be at least α_{min}. Hence, the attenuation (in dB) given by the following equation (3.55) can be used.

$$A = -20 \log|H(j\Omega)| \tag{3.55}$$

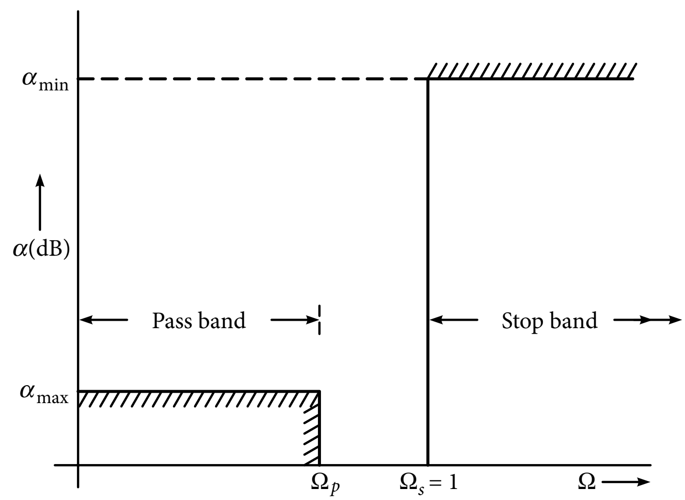

Figure 3.10 Attenuation characteristics of a low pass inverse Chebyshev function.

The equation gives the minimum value of attenuation, α_{min} as follows:

$$\alpha_{min} = -10 \log|H(j\Omega)|^2 = 10 \log(1 + 1/\varepsilon^2) \tag{3.56}$$

It gives the expression for the constant ε as:

$$\varepsilon = (10^{0.1\alpha_{min}} - 1)^{-1/2} \tag{3.57}$$

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As in Figure 3.10, limit of the attenuation is α_{max} at the edge of the pass band Ω_p. Hence, using equation (3.50), we get the expression for α_{max} in terms of Ω_p and the de-normalized pass band edge frequency as given here.

$$\alpha_{max} = 10\log\left[1 + 1/\left\{\varepsilon^2 C_n^2\left(1/\Omega_p\right)\right\}\right] \rightarrow \varepsilon^2 C_n^2\left(1/\Omega_p\right) = \left(10^{0.1\alpha_{max}} - 1\right) \tag{3.58}$$

Substituting ε from equation (3.57) in equation (3.58),

$$C_n^2(1/\Omega_p) = \frac{10^{0.1\alpha_{max}} - 1}{10^{0.1\alpha_{min}} - 1} \tag{3.59}$$

As $\Omega_p < 1$ and equation (3.59) is applicable for the pass band, use of equation (3.30) for the value of $C_n^2(\Omega)$ gives:

$$\cos\mathrm{h}\left[n\cos\mathrm{h}^{-1}(1/\Omega_p)\right] = \frac{(10^{0.1\alpha_{min}} - 1)^{1/2}}{(10^{0.1\alpha_{max}} - 1)} \tag{3.60}$$

Solving for the order of the function n,

$$n = \frac{\cos\mathrm{h}^{-1}\left[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)\right]^{1/2}}{\cos\mathrm{h}^{-1}(1/\Omega_p)} \tag{3.61}$$

If the de-normalized pass band edge frequency is Ω_1 rad/s and the stop band edge frequency is Ω_2 rad/s, $\Omega_p = \left(\dfrac{\Omega_1}{\Omega_2}\right)$, then and equation (3.61) can be modified as:

$$n = \frac{\cos\mathrm{h}^{-1}\left[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)\right]^{1/2}}{\cos\mathrm{h}^{-1}(\Omega_2/\Omega_1)} \tag{3.62}$$

Equation (3.62) is the same as that for the Chebyshev function given in equation (3.38); this means that for the same specifications, the order of the inverse Chebyshev function will be the same as that for the Chebyshev response. Value of the parameter ε will also be the same; however, a major difference between the two responses is that the frequency normalization is done with respect to the stop band edge frequency Ω_S in the inverse Chebyshev function.

To find the transfer function of the inverse Chebyshev function, either an appropriate Table like 3.4 can be used, or the location of poles and zeros needs to be determined as follows.

The magnitude squared function of equation (3.40) can be expressed as follows:

$$\left|H_n(j\Omega)\right|^2 = \frac{\varepsilon^2 C_n^2(1/\Omega)}{1 + \varepsilon^2 C_n^2(1/\Omega)} = \left.\frac{z(s)\,z(-s)}{p(s)\,p(-s)}\right|_{s=j\Omega} \tag{3.63}$$

Hence, zeros are found in the stop band for $\Omega > 1$, when $C_n^2(1/\Omega)$ is expressed through trigonometric functions, and equalized to zero, that is, $C_n^2(1/\Omega_k) = 0$, or:

$$\cos n \cos^{-1}(1/\Omega_k) = 0 \tag{3.64}$$

Equality in equation (3.64) is valid when k is odd (it equals 1 when k is even). Then, with $\varphi_k = n \cos^{-1}(1/\Omega_k)$, we get:

$$\cos n\varphi_k = 0 \text{ when } n\varphi_k = k(\pi/2). \tag{3.65}$$

Equation (3.65) gives

$$\cos^{-1}(1/\Omega_k) = \varphi_k = k\pi/2n \tag{3.66}$$

Therefore, zero frequencies are obtained as follows:

$$\Omega_k = \sec(k\pi/2n) \text{ for } k = 1, 3, 5, \ldots, n \tag{3.67}$$

We will now find the pole location of the inverse Chebyshev function. It can be observed that the denominator of equation (3.63) is the same as that for the Chebyshev function, with a difference that Ω is now replaced by $(1/\Omega)$. This means that to find the pole location for the inverse Chebyshev function, we can first determine the Chebyshev poles using equations (3.40)–(3.42) and then take its reciprocal. However, the value of ε to be used shall be the one obtained for the inverse Chebyshev function using equation (3.47). It is observed that the pole quality factor, Q for the inverse Chebyshev remains the same as that for the Chebyshev function.

Example 3.4: Find the order of the inverse Chebyshev filter for the following specifications. Also find the corresponding transfer function.

$$\alpha_{\max} = 1\,\text{dB},\ \alpha_{\min} = 40\,\text{dB}, \omega_1 = 2000\,\text{rad/s and } \omega_2 = 6000\,\text{rad/s}.$$

Solution: By calculation, the order n of the filter is 3.36; this can be approximated to 4.

Zeros of the transfer function shall be found using equation (3.67) with $\Omega_{zk} = \sec(k\pi/2 \times 4)$. Hence, for $k = 1$,

$$\Omega_{z1} = \sec(\pi/8) = 1.08239 \text{ and for } k = 3,\ \Omega_{z2} = \sec(3\pi/8) = 2.6131 \tag{3.68a}$$

The de-normalized value of the zeros is as follows:

$$z_1 = 6000 \times \Omega_{z1} = 6494.34\,\text{rad/s} \text{ and } z_2 = 6000 \times \Omega_{z2} = 15678.7\,\text{rad/s} \tag{3.68b}$$

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The first step to finding the pole values is to find the pole of the Chebyshev function with ε obtained using equation (3.47) as follows:

$$\varepsilon = (10^{0.1\times 40} - 1)^{-1/2} = 0.01 \tag{3.69}$$

Using equation (3.42),

$$a = \left(\frac{1}{4}\right)\sin\text{h}^{-1}\left(1/0.01\right) = 1.32458 \tag{3.70}$$

This yields sin h a = sin h 1.32458 = 1.74733 and cos h a = cos h 1.32458 = 2.0132

Hence, the real parts of the pole can be obtained as follows:

$$\sigma_1 = -\sin\text{h}a \times \{\sin(1/4)(\pi/2)\} = -1.74733 \sin(\pi/8) = -0.66867 \tag{3.71a}$$

$$\sigma_2 = -\sin\text{h}a \times \{\sin(3/4)(\pi/2)\} = -1.6143 \tag{3.71b}$$

$$\sigma_3 = -\sin\text{h}a \times \{\sin(5/4)(\pi/2)\} = -1.6143 \tag{3.71c}$$

$$\sigma_4 = -\sin\text{h}a \times \{\sin(7/4)(\pi/2)\} = -0.66867 \tag{3.71d}$$

The imaginary components can be obtained from equation (3.41b) as:

$$\Omega_k = j\cos\text{h}a \times \cos(2k-1)(\pi/2n)$$

$$\Omega_1 = j2.0132 \cos(\pi/8) = j1.8599, \; \Omega_2 = j2.0132 \cos(3\pi/8) = j0.7704$$

$$\Omega_3 = j2.0132 \cos(5\pi/8) = -j0.7704, \; \Omega_4 = j2.0132\cos(7\pi/8) = -j1.859 \tag{3.72}$$

For the Chebyshev function, if we know the value of the real and imaginary parts of the pole, its magnitude and the quality factor shall be given as follows:

$$\Omega_{0k} = \left(\sigma_k^2 + \Omega_k^2\right)^{½} \text{ and } Q_{kC} = \left(\Omega_{0k}/2\sigma_k\right) \tag{3.73}$$

Then, the pole location for the inverse Chebyshev response case, $p_k = x_k + jy_k$ is given by:

$$p_k = \frac{(\sigma_k - j\Omega_k)}{(\sigma_k^2 + \Omega_k^2)} \tag{3.74}$$

This gives the magnitude and quality factor for the inverse Chebyshev response as follows:

$$\left|p_k\right| = \left(x_k^2 + y_k^2\right)^{½} = \left(1/\Omega_{\text{okIC}}\right) \text{ and } Q_{k\text{IC}} = Q_{kC} \tag{3.75}$$

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Using equations (3.73)–(3.75), we get the following parameters:

$$\Omega_{01} = \left(\sigma_1^2 + \Omega_1^2\right)^{½} = \left\{\left(-0.66867\right)^2 + (1.8599)^2\right\}^{½} = \left(3.90646\right)^{½} = 1.9764 \tag{3.76a}$$

$$\Omega_{02} = \left\{\left(-1.6143\right)^2 + (0.7704)^2\right\}^{½} = \left(3.19948\right)^{½} = 1.7887 \tag{3.76b}$$

$$\Omega_{03} = \left\{\left(-1.6143\right)^2 + (-0.7704)^2\right\}^{½} = \Omega_{02} \text{ and } \Omega_{04} = \Omega_{01} \tag{3.76c, d}$$

$$Q_{1C} = \frac{\Omega_{01}}{2\sigma_1} = \frac{1.9764}{2 \times 0.66867} = 1.47786,\ Q_{2C} = \frac{\Omega_{02}}{2\sigma_2} = \frac{1.7887}{2 \times 1.6143} = 0.554\ , \tag{3.77a, b}$$

$$Q_{3C} = \frac{\Omega_{02}}{2\sigma_1} = Q_{2C} \quad \text{and } Q_{4C} = \frac{\Omega_{01}}{2\sigma_2} = Q_{1C} \tag{3.77c, d}$$

$$p_1 = \frac{\sigma_1 - j\Omega_1}{\sigma_1^2 + \Omega_1^2} = \frac{-0.66867 - j1.8599}{\Omega_{01}^2} = -0.17117 - j0.47611 \tag{3.78a}$$

$$p_2 = \frac{\sigma_2 - j\Omega_2}{\sigma_2^2 + \Omega_2^2} = \frac{-1.6143 - j0.7704}{\Omega_{02}^2} = -0.50455 - j0.2408 \tag{3.78b}$$

$$p_3 = \frac{\sigma_2 - j\Omega_2}{\Omega_{02}^2} = \frac{-1.6143 + j0.7704}{\Omega_{02}^2} = -0.50455 + j0.2408 \tag{3.78c}$$

$$p_4 = \frac{-0.66867 + j1.8599}{\Omega_{01}^2} = -0.17117 + j0.47611 \tag{3.78d}$$

$$\Omega_{01IC} = (x_1^2 + y_1^2)^{½} = \{(.17117)^2 + (.47611)^2\}^{½} = 0.5059 \tag{3.79a}$$

$$\Omega_{02IC} = \{(.50455)^2 + (.2408)^2\}^{½} = 0.55906 \tag{3.79b}$$

$$\Omega_{03IC} = \Omega_{02IC},\ \Omega_{04IC} = \Omega_{01IC} \tag{3.79c, d}$$

Instead of following the steps from equations (3.74) to (3.79), the pole location for the inverse Chebyshev function can also be found by taking the inverse of the pole locations of the Chebyshev response obtained from equations (3.71)–(3.73), while using the value of ε obtained using equation (3.47); the quality factors remain the same.

To obtain the transfer function of the inverse Chebyshev response, the pole-pair is associated with the zero nearer to it. Hence, the transfer function is obtained as follows:

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$$H(s)=\frac{\left\{s^2+\Omega_{z1}^2\right\}\left\{s^2+\Omega_{z3}^2\right\}}{\left\{s^2+\dfrac{\Omega_{01IC}}{Q_{1IC}}s+\Omega_{01IC}^2\right\}\left\{s^2+\dfrac{\Omega_{03IC}}{Q_{3IC}}s+\Omega_{03IC}^2\right\}}$$

$$=\frac{\left(s^2+1.1757\right)\left(s^2+6.8283\right)}{(s^2+0.3423s+0.2559)(s^2+1.0091s+0.31255)} \tag{3.80}$$

Since the calculation of poles and zero was done in the normalized form with normalization done with respect to the stop band edge frequency, the transfer function is also to be de-normalized with respect to it. Finally, the filter can be realized by using any of the cascade or direct form synthesis procedures. If we wanted to realize the filter as two second-order sections, it can be done by using two notch filters. Since notch filter realization shall be studied later, this transfer function shall also be taken up later.

3.7 Cauer or Elliptic Approximation

The use of Chebyshev or inverse Chebyshev approximation result in an economical or optimal filter section rather than a filter with maximally flat response. It was expected that equal ripples in both the pass band and the stop band will further decrease the required order n of the filter section for the same specification; this assumption was indeed shown to be correct by William Cauer [3.1]. Such an approximation, shown in Figure 3.11, is called a *Cauer approximation* or *elliptic approximation*. As the solutions for this approximation lead to elliptic functions that are not easy to solve, exhaustive tables and design graphs are used instead of solving the elliptic functions.

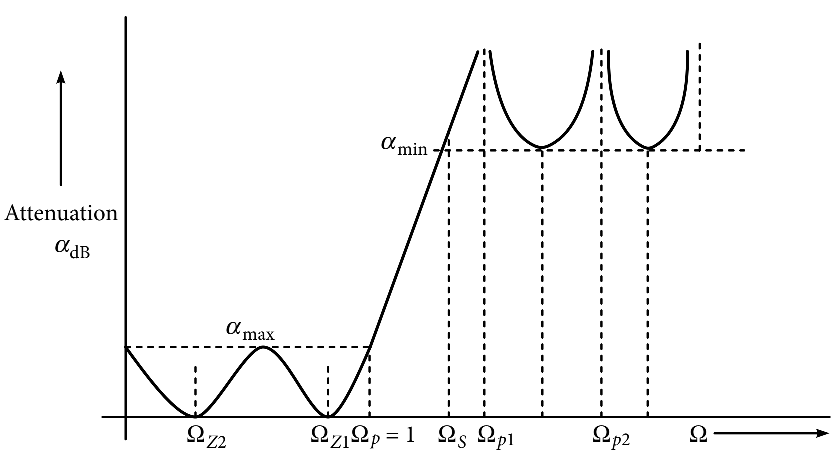

Figure 3.11 Variation of attenuation for a typical Cauer or elliptic response.

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In the previous magnitude approximations, different expressions were assigned to the term $|K(j\Omega)|$ of equation (3.4) so that the pass band would be maximally flat, or have equal ripples in the stop band or pass band. To have equal ripples in both the frequency bands, the characteristics function $K(S)$ is selected to be a ratio of polynomials whose poles and zero lie on the imaginary axis of the s plane. For $K(S)$ to be such a new function of $E_n(S)$ of order n, let $E_n(S) = N(S)'/D(S)'$. Then equation (3.4) can be written as:

$$|H_n(j\Omega)|^2 = 1/\left[1+\varepsilon^2 E_n^2(\Omega)\right]$$

$$= \frac{D'(j\Omega)D'(-j\Omega)}{D'(j\Omega)D'(-j\Omega)+\varepsilon^2 N'(j\Omega)N'(-j\Omega)} \tag{3.81}$$

In equation (3.81), ε is multiplied with $E_n(S)$ as its value is not unity in maximally flat and equal ripple approximations, instead of being unity in case of Butterworth approximation.

This means that the poles of $E_n(\Omega)$ will be the zeros of $|H_n(j\Omega)|$. Analysis of the function assumes that frequency is normalized at the edge of the pass band, that is, $\Omega_p = 1$and $E_n(\Omega = 1) = 1$. Then, maximum attenuation at the pass band edge from equation (3.81) shall be:

$$\alpha_{\max} = 10\log_{10}\left(1+\varepsilon^2\right) \tag{3.82}$$

This equation gives the same expression for ε which was obtained for the maximally flat or the Chebyshev approximation of equations (3.11) and (3.33), respectively. Further, in order to have equal ripples in the stop band (including at the stop band edge frequency Ω_s with attenuation of $\alpha_{\min}$, it is required that $|E_n| = \pm F$. Hence, in the stop band, the expression of the minimum attenuation shall be:

$$\alpha_{\min} = 10\log_{10}(1+\varepsilon^2 F^2) \tag{3.83}$$

Substitution of ε from equation (3.82) in equation (3.83) results in the expression for F which is a familiar expression for the maximally flat as well as Chebyshev response vide equations (3.23) and (3.39), respectively, in connection with the evaluation of the filter order n.

3.7.1 Design of Cauer filters

The first step in the design of a Cauer filter is to find its order from the same four specifications: $\alpha_{\max}$, $\alpha_{\min}$, Ω_p, and Ω_s. While finding n has been rather straightforward and simple in the approximation methods discussed so far, for the Cauer approximation, is the calculations become quite involved and requires solution of elliptic functions. One way out from this complexity is by using the fact that, invariably, the value of n obtained through calculations is not an integer and, therefore, the next higher integer value is selected. This approximation amounts to a bit of over-designing; however, it is customary that the given values of the specifications can be marginally changed. It allows using a lesser complex graphic process in which n is obtained from a set of curves drawn for the variation of E_n with respect to Ω_s. However, the approximation also requires finding an expression for E_n that needs to be a rational function meeting the requirements of the given specifications. Alternatively, we can use the following simpler method [3.3].

From the given specifications, a *modulator constant* q is calculated from the following relation:

$$q = u + 2u^5 + 15u^9 + 150u^{13} \tag{3.84a}$$

Here, $u = \dfrac{1-(1-k^2)^{1/4}}{\left\{2(1+(1-k^2)^{1/4}\right\}}$ and the selectivity factor,

$$k = \Omega_p/\Omega_s \tag{3.84b}$$

The next step is to find the *discrimination factor* D from the following relation:

$$D = \left(10^{0.1\alpha_{\min}} - 1\right) / \left(10^{0.1\alpha_{\max}} - 1\right) \tag{3.85}$$

Then, the order of the elliptic filter n is obtained from the following relation:

$$n = \{\log 16D/\log(1/q)\} \tag{3.86}$$

In equation (3.86), the obtained value may not be an integer; the value then has to be rounded off to the next higher integer. Due to the change in the value of n to the next higher integer value, the actual $\alpha_{\min}$ in the stop band is changed to the following:

$$\alpha_{\min} = 10\log\left(1 + \frac{10^{0.1\alpha_{\max}} - 1}{16q^n}\right) \tag{3.87}$$

Obviously, we should ascertain that the value of $\alpha_{\min}$ obtained from equation (3.87) satisfies the specifications of the design.

Example 3.5: For the following specifications, find the order of an elliptic filter:

$$\alpha_{\max} = 1 \text{ dB},\ \alpha_{\min} = 40 \text{ dB},\ \omega_1 = 2000 \text{ rad/s and } \omega_2 = 6000 \text{ rad/s}$$

Solution: Selectivity factor, $k = 2000 / 6000 = 1/3$. Using equations (3.84)–(3.86) for q, u, D and n, we get the following:

$$u = 0.5\frac{1-(1-1/9)^{1/4}}{(1+(1-1/9)^{1/4}} = 0.00736$$

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$$q = 0.00736 + 2(0.00736)^5 + 15(0.00736)^9 + 150(0.00736)^{13} \cong 0.00736$$

Value of the discrimination factor D is obtained as:

$$D = (10^{0.1\times 40} - 1)/(10^{0.1\times 2} - 1) = 9999/0.2589 = 38621$$

Then, the order of the filter is obtained from the equation (3.86):

$$n = \log(16 \times 38621)/\log(1/0.00736) = 2.714$$

Hence, the selected value of $n = 3$.

It may be noted that for the same specifications, the required filter order was 5 for the Butterworth approximation, 4 for equal-ripple filters. In practice, the difference becomes more prominent when there is a narrower transition band or selectivity factor with a large value.

The actual minimum stop band attenuation with $n = 3$ from equation (3.87) will be as follows:

$$\alpha_{\min} = 10\log\left\{1 + \frac{10^{0.1} - 1}{16(.00736)^3}\right\} = 46.08\,\text{dB}$$

In this expression, the obtained theoretical value of $\alpha_{\min}$ is well under control. After finding the order of the filter, the normalized transfer function is to be obtained. Once again, the solution requires elliptic functions, which are quite complex. Algorithms have been developed for the purpose; however, the simpler option is to use the available design tables. In the vast literature pertaining to filters, these tables and the data arranged in the tables have been presented in different ways. Only the specific table corresponding to the stated specifications is to be used to get the location of poles and zeros and the transfer function.

An alternate method for finding the order of the elliptic LPF is to use nomographs. Figure 3.12 is such a nomograph, wherein the attenuation at the pass band edge frequency is $\alpha_{\max}$ (normalized to 1 rad/s) and $\alpha_{\min}$ is the attenuation at some (normalized) stop band edge frequency Ω_s. To use the nomograph in Figure 3.12, a straight line is drawn through the specified $\alpha_{\max}$ and $\alpha_{\min}$, shown as points 1 and 2 in Figure 3.13. Intersection of this line with the ordinate of the nomograph determines point 3. A horizontal line is drawn from point 3 until it meets a vertical line drawn from point 4 which corresponds to the specified frequency Ω_s. The resulting intersection at point 5 decides the order of the required elliptic filter. Almost every time, point 5 lies between two curves corresponding to the loci of the filter orders; the higher value of the filter order is selected for obvious reasons.

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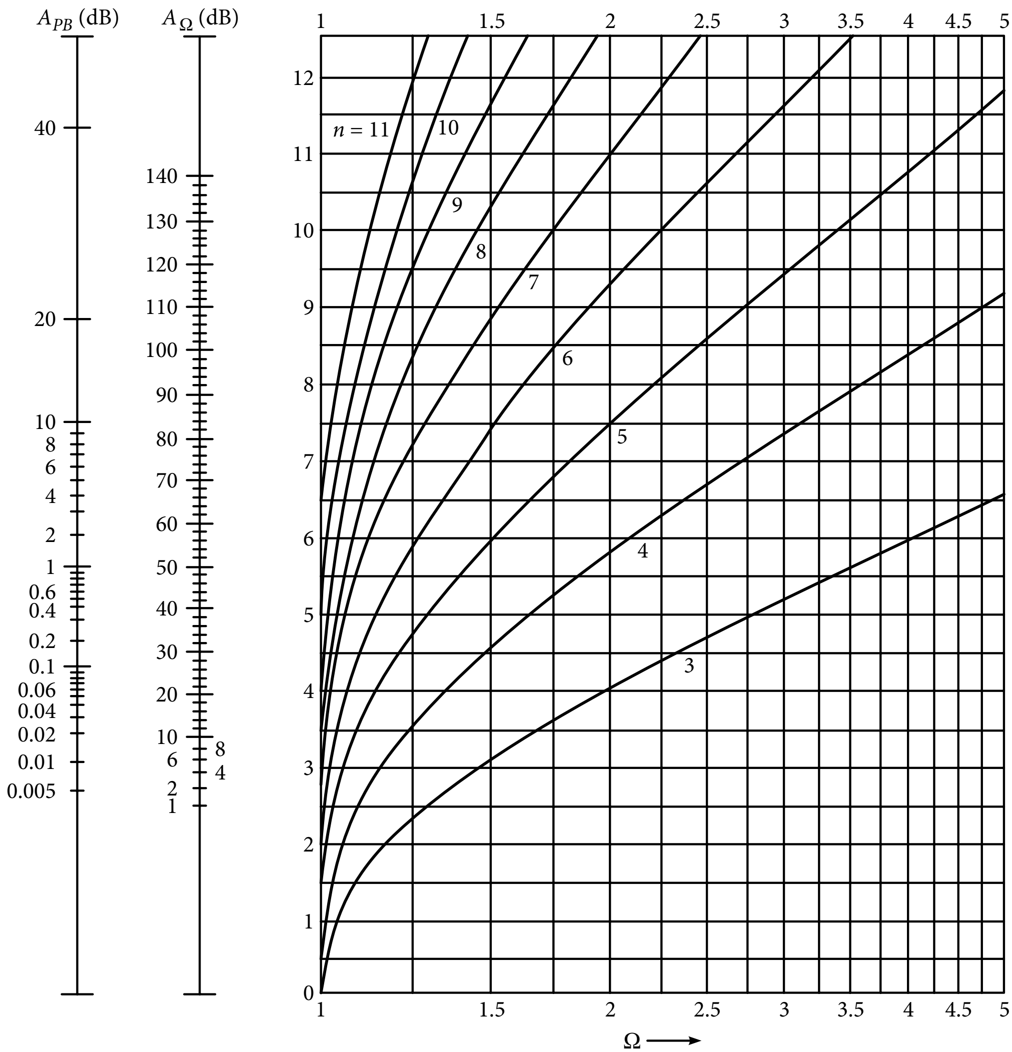

Figure 3.12 A nomograph for determining the order of an elliptic magnitude function.

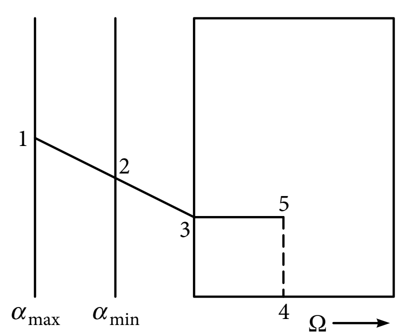

Figure 3.13 Method of using the nomograph in Figure 3.12.

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Normalized element values of a doubly terminated lossless ladder can be obtained from other tables. Figure 3.14(a) and (b) show these ladders and Tables 3.6 and 3.7 show the values of the elements for these ladders. Obviously, these ladders and tables are useful when some direct form of filter synthesis is used instead of the cascade form. However, if it is preferred to use the cascade form of synthesis, poles and zeros, and then the transfer function can be found through analyzing the ladder itself.

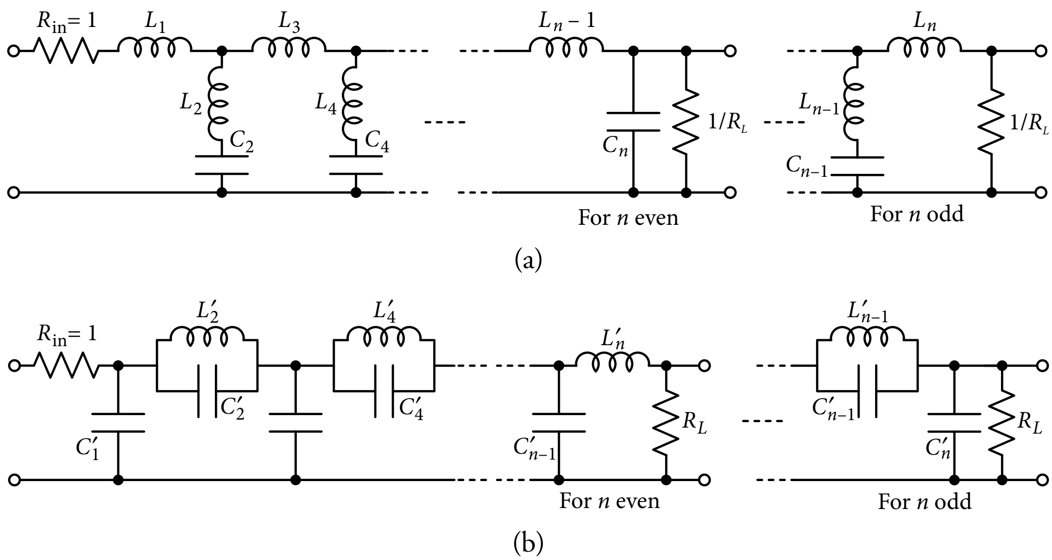

Figure 3.14 (a) Network configuration for Table 3.6 and Table 3.7. (b) Alternate configuration.

Example 3.6: Find a doubly terminated lossless ladder which gives an elliptic response while satisfying the following specifications. Verify the response using PSpice.

$$\alpha_{max} = 0.1 \text{ dB } \alpha_{min} = 54 \text{ dBs}, \omega_p = 100 \text{ krad/s and } \omega_s = 150 \text{ krad/s}$$

Solution: Normalizing the frequency by 100 krad/s, we get $\Omega_s = 1.5$ rad/s. Using the nomograph of Figure 3.12, intersection of the line from the given attenuations of 0.1 dB and 54 dB and the vertical line for $\Omega_s = 1.5$ rad/s falls between $n = 5$ and 6; hence, the required filter order will be 6. For $n = 6$ and $\Omega_s = 1.5$ rad/s, Table 3.7 gives the normalized values of the elements as:

$$R_{in} = R_L = 1\ \Omega,\ L_1 = 0.86595 \text{ H},\ L_2 = 0.18554 \text{ H},\ L_3 = 1.43106 \text{ H},\ L_4 = 0.33007\text{H},$$
$$L_5 = 1.28253 \text{ H},\ C_2 = 1.27403 \text{ F},\ C_4 = 1.27255 \text{ F},\ C_6 = 1.03317 \text{ F}$$

Using frequency scaling, elements are de-normalized by a factor of 100 krad/s and an impedance scaling factor of 10^4. The de-normalized element values are as follows:

$$R_{in} = R_L = 10 \text{ k}\Omega,\ L_1 = 0.086595 \text{ H},\ L_2 = 0.018554 \text{ H},\ L_3 = 0.143106 \text{ H},\ L_4 = 0.33007 \text{ H},$$
$$L_5 = 0.128253 \text{ H},\ C_2 = 1.27403 \text{ nF},\ C_4 = 1.27255\text{nF},\ C_6 = 1.03317 \text{ nF}$$

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Table 3.6 Element values of a doubly terminated ladder for elliptic filters with pass band ripples of 1 dB

n	ω_s	K_p	L_1	C_2	L_2	L_3	C_4	L_4	L_5	C_6	L_6	L_7
3	1.05	8.134	1.05507	.25223	3.28904	1.05507						
	1.10	11.480	1.22525	.37471	1.94752	1.22525						
	1.20	16.209	1.42450	.52544	1.11977	1.42450						
	1.50	25.176	1.69200	.73340	.48592	1.69200						
	2.00	34.454	1.85199	.85903	.22590	1.85199						
4	1.05	11.322	.63708	.35277	2.41039	1.11522	1.39953				1.0-dB pass band ripple	
	1.10	15.942	.80935	.54042	1.40015	1.18107	1.45001					
	1.20	22.293	1.00329	.77733	.79634	1.26621	1.49217					
	1.50	34.179	1.25675	1.11431	.34362	1.38981	1.53225					
	2.00	46.481	1.40677	1.32367	.15960	1.46762	1.55071					
5	1.05	24.135	1.56191	.67560	.83449	1.55460	.26584	3.31881	.88528			
	1.10	30.471	1.69691	.77511	.58827	1.79892	.39922	1.98907	1.12109			
	1.20	38.757	1.82812	.87005	.38720	2.09095	.56347	1.16672	1.38094			
	1.50	53.875	1.97687	.97694	.18824	2.49161	.79362	.51950	1.71889			
	2.00	69.360	2.05594	1.03392	.09152	2.73567	.93561	.24486	1.91939			
6	1.05	29.133	1.07458	.80116	.81300	.92735	.51753	1.71498	.92186	1.60511		
	1.10	36.680	1.22059	.94235	.57746	1.10900	.75718	1.05819	1.01676	1.64682		
	1.20	46.571	1.37146	1.08633	.38284	1.32610	1.05110	.63354	1.12484	1.68498		
	1.50	64.661	1.55425	1.25876	.18779	1.62529	1.46557	.28655	1.26961	1.72482		
	2.00	83.221	1.65661	1.35450	.09179	1.80860	1.72376	.13586	1.35729	1.74424		
7	1.05	40.926	1.82156	.86343	.42668	1.67632	.34381	2.60271	1.23696	.46779	1.63392	1.22362
	1.10	49.816	1.91040	.92662	.30705	1.93579	.48016	1.68753	1.55276	.59277	1.10699	1.41994
	1.20	61.422	1.99168	.98474	.20446	2.22804	.64444	1.04856	1.92724	.73012	.70551	1.62539
	1.50	82.588	2.07882	1.04761	.10016	2.61372	.87393	.48973	2.44021	.90483	.33349	1.87717
	2.00	104.268	2.12329	1.07993	.04884	2.84446	1.01638	.23538	2.75306	1.00567	.16034	2.01924
n	ω_s	K_p	C'_1	L'_2	C'_2	C'_3	L'_4	C'_4	C'_5	L'_6	C'_6	C'_7

Table 3.7 Element values of a doubly terminated ladder for elliptic filters with pass band ripples of 0.1 dB

n	ω_s	K_p	L_1	C_2	L_2	L_3	C_4	L_4	L_5	C_6	L_6	L_7
3	1.05	1.748	.35550	.15374	5.39596	.35550						
	1.10	3.374	.44626	.26993	2.70353	.44626						
	1.20	6.691	.57336	.44980	1.30805	.57336						
	1.50	14.848	.77031	.74561	.47797	.77031						
	2.00	24.010	.89544	.93759	.20697	.89544						
4	1.05	3.284	.00442	.17221	4.93764	1.01224	.84445				0.1 dB pass band ripple	
	1.10	6.478	.17279	.32758	2.30986	1.04894	.89415					
	1.20	12.085	.37139	.56638	1.09294	1.11938	.92440					
	1.50	23.736	.62815	.94009	.40730	1.24711	.93518					
	2.00	36.023	.77554	1.17646	.17957	1.33473	.93382					
5	1.05	13.841	.70813	.76630	.73572	1.12761	.20138	4.38116	.04985			
	1.10	20.050	.81296	.92418	.49338	1.22445	.37193	2.13500	.29125			
	1.20	28.303	.91441	1.06516	.31628	1.38201	.60131	1.09329	.52974			
	1.50	43.415	1.02789	1.21517	.15134	1.63179	.93525	.44083	.81549			
	2.00	58.901	1.08758	1.29322	.07317	1.79387	1.14330	.20038	.97720			
6	1.05	18.727	.44177	.71651	.90905	.83142	.36274	2.44680	.80463	.99857		
	1.10	26.230	.57630	.88798	.61282	.97304	.59060	1.35666	.94305	1.01381		
	1.20	36.113	.70984	1.06266	.39136	1.15974	.87407	.76185	1.09176	1.02462		
	1.50	54.202	.86595	1.27403	.18554	1.43106	1.27235	.33007	1.28253	1.03317		
	2.00	72.761	.95131	1.39297	.08926	1.60132	1.51866	.15421	1.39521	1.03621		
7	1.05	30.470	.91937	1.07659	.34220	1.09623	.40518	2.20850	.84335	.50342	1.51827	.41098
	1.10	39.357	.98821	1.16726	.24374	1.27743	.59720	1.35681	1.04029	.67881	.96669	.58282
	1.20	50.963	1.05029	1.24872	.16124	1.48377	.82869	.81542	1.28723	.87428	.58918	.75395
	1.50	72.129	1.11593	1.33554	.07857	1.75687	1.15174	.37160	1.63827	1.12502	.26822	.95588
	2.00	93.809	1.14910	1.37979	.03822	1.92026	1.35221	.17692	1.85664	1.27023	.12694	1.06720
n	ω_s	K_p	C'_1	L'_2	C'_2	C'_3	L'_4	C'_4	C'_5	L'_6	C'_6	C'_7

Figure 3.15(a) shows the magnitude response of the passive ladder with ordinates on the linear scale and Figure 3.15(b) shows the magnitude on the log scale (magnitude on the log scale is shown to get a better view of both the pass band and the stop band). The simulated pass band edge frequency is 15.92 kHz (100.06 krad/s) and the stop band edge frequency is 23.86 kHz (149.977 krad/s) giving the normalized Ω_s = 1.5 rad/s. Maximum attenuation in the pass band 0.100 dB and the minimum attenuation in the stop band is 54.2 dB.

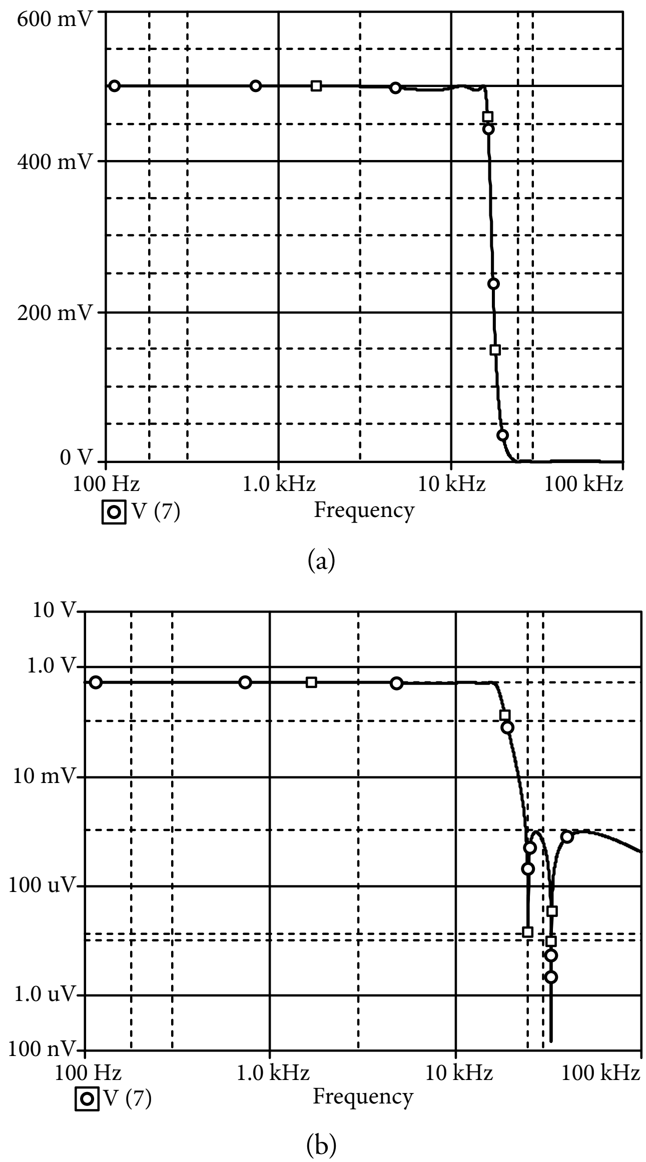

Figure 3.15 (a) Magnitude response of the elliptic filter of Example 3.6 with ordinates on the linear scale. (b) with ordinates on the log scale.

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3.8 Maximally Flat Pass Band with Finite Zeros

It has been observed that in the maximally flat Butterworth response case, it takes a higher filter order to satisfy the same specifications compared to other approximations. This limitation caused by the slower transition from pass to stop band in the maximally flat Butterworth response is minimized by the introduction of finite zeroes in the otherwise all-pole LPF, where all the zeroes were at infinity.

If finite zeroes are introduced in the maximally flat response, instead of $N(s) = 1$, $N(s)$ will be a polynomial in terms of the complex frequency s in equation (3.3), and the degree of the polynomial will depend on the number of the zeroes to be added. Equation (3.5) will become modified while using equations (3.2) as:

$$\left|K(j\Omega)\right|^2 = \frac{B(\Omega^2)}{A(\Omega^2)} - 1 = \frac{B\left(\Omega^2\right) - A(\Omega^2)}{A(\Omega^2)} \tag{3.88}$$

However, for the response to remain maximally flat, equation (3.88) has to remain satisfied even when transmission zeroes are introduced. This condition implies that the following relation is satisfied for as many derivatives as possible:

$$\frac{d^k\left(\left|K(j\Omega)\right|^2\right)}{d^k(\Omega^2)} = \frac{d^k}{d^k(\Omega^2)}\left\{\frac{B\left(\Omega^2\right) - A(\Omega^2)}{A(\Omega^2)}\right\} = 0 \tag{3.89}$$

Application of chain rule, differentiation of equation (3.89) gives following relation:

$$B_{2i} = A_{2i} \text{ for } i = 0, 1, \ldots, (n-1) \tag{3.90}$$

Hence, in equation (3.2) or equation (3.88), $A(\Omega^2)$ is selected in such a way that the desired zeroes are realized and the denominator is the sum of $A(\Omega^2)$ and $B_{2n}\Omega^{2n}$. With a modified transfer function, at $\Omega = 1$, and $|H_n(j1)|^2$ with equation (3.10), we get:

$$\left|H_n(j1)\right|^2 = \frac{A(1)}{A(1) + B_{2n}} = \frac{1}{1+\varepsilon^2} \to B_{2n} = \varepsilon^2 A(1) \tag{3.91}$$

The transfer function can now be found by multiplying $H_n(j\Omega)$ with $H_n(-j\Omega)$, and substituting $S = j\Omega$:

$$H_n(j\Omega)H_n(-j\Omega)|_{j\Omega=S} = \frac{A(-S^2)}{A\left(-S^2\right) + (-1)^n * B_{2n}S^{2n}} \tag{3.92}$$

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It is important to note that in an all pole transfer function, the gain drops at a rate of $20n$ dB per decade, but after the addition of finite zeroes, the rate of fall of gain in the transition band increases but the rate of fall of gain at higher frequencies will be at the rate of $20(n - m)$ dB per decade.

The following example is provided to help understand the design process while introducing finite zeroes in a flat pass band transfer function.

Example 3.7: In a maximally flat LPF, it is desired that the dc gain of the filter remains as unity and its gain drops by 1 dB at 20 krad/s. Introduce transmission zeroes at 40 krad/s and 50 krad/s to increase the rate of fall of attenuation in the transition band and find its transfer function.

Solution: As gain is dropping by 1 dB at 20 krad/s, this is taken as the normalizing frequency. Then the transmission zeroes will become $\Omega = 2$ and 2.5, respectively, and with the dc gain as unity, the normalized transfer function is obtained using equation (3.92):

$$H_n(j\Omega)^* H_n(-j\Omega) = \frac{\left\{\left(\dfrac{\Omega_n}{2}\right)^2 - 1\right\}^2 \left\{\left(\dfrac{\Omega_n}{2.5}\right)^2 - 1\right\}^2}{\left\{\left(\dfrac{\Omega_n}{2}\right)^2 - 1\right\}^2 \left\{\left(\dfrac{\Omega_n}{2.5}\right)^2 - 1\right\}^2 + B_{2n}\Omega^{2n}} \tag{3.93}$$

The numerator is of degree 4 and for a minimum rate of fall of attenuation of 40 dB at *higher frequencies*, the denominator should have degree $n = 6$; hence, B_{12} is to be determined for equation (3.93).

The value of B_{12} can be found from equation (3.91), as at $\Omega = 1$, the gain drops by 1 dB, and we can write:

$$\left|H_6(j1)\right|^2 = 1/\left\{1 + \frac{B_{12}}{\left(\dfrac{3}{4}\right)^2 \left(\dfrac{5.25}{6.25}\right)^2}\right\} \tag{3.94}$$

which is equal to 1 dB of attenuation or we can calculate that the output will drop by a factor of $10^{-(1/10)} = 0.7943$; hence, comparing this equation with equation (3.94), we get $B_{12} = 0.102767$. The value of B_{12} is substituted in equation (3.94), and while S is replaced by $j\Omega$, the transfer function is obtained from the following:

$$H_6(S)H_6(-S) = \frac{\left(\left(\dfrac{S}{2}\right)^2 + 1\right)^2 \left(\left(\dfrac{S}{2.5}\right)^2 + 1\right)^2}{\left(\left(\dfrac{S}{2}\right)^2 + 1\right)^2 \left(\left(\dfrac{S}{2.5}\right)^2 + 1\right)^2 + 0.102767 S^{12}} \tag{3.95}$$

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The numerator has four roots, which are easily identifiable; however, the root finder is used to find 12 roots of the denominator. Along with a multiplying factor of 64.229, the following roots are obtained:

$$\pm 1.256 \pm j0.41,\ \pm 0.762 \pm j0.932,\text{ and } \pm 0.238 \pm j1.085 \tag{3.96}$$

In equation (3.96), the roots are in all the quadrants. Selecting the roots on the left half of the s plane, for a real rational function, the following factors (normalized) become available.

$$S^2 + 2.511S + 1.744,\ S^2 + 1.524S + 1.449 \text{ and } S^2 + 0.475S + 1.234 \tag{3.97}$$

As result of the root multiplying factor of 64.229, the numerator coefficient will become $4 \times 6.25/(64.229)^{1/2} = 3.119$. The resulting normalized transfer function will be as follows:

$$H_6(S) = \frac{3.119\left\{\left(\frac{S}{2}\right)^2 + 1\right\}\left\{\left(\frac{S}{2.5}\right)^2 + 1\right\}}{\left(S^2 + 2.511S + 1.744\right)\left(S^2 + 1.524S + 1.449\right)\left(S^2 + 0.475S + 1.234\right)} \tag{3.98}$$

Realization of the transfer function of equation (3.98) using cascade technique will be discussed in Chapter 10.

References

[3.1] Kuo, F. F. 1966. *Network Analysis and Synthesis.* New York: Wiley.

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[3.3] Cauer, W. 1958. *Synthesis of Linear Communication Networks*. New York: McGraw Hill.

[3.4] Zverev, A. I. 1967. *Handbook of Filter Synthesis*. New York: Wiley.

Practice Problems

3-1 (a) Find the pole location and the coefficients of the Butterworth polynomial for order n = 5, 8 and 10. Compare the answers for n = 5 and 8 from Table 3.1 and 3.2.

(b) Plot the pole values calculated in part (a) on the s plane.

(c) Factorize the Butterworth polynomials found in part (a) using a root finder or any other method.

3-2 (a) Determine the transfer function for an LP filter having a maximally flat magnitude characteristic, which is 2 dB down at 2 rad/s and 32 dBs down at 7.5 rad/s.

(b) Find a doubly terminated lossless ladder realization for the LP filter.

(c) Find the element values for the LP ladder filter with its 3 dB frequency at 3.4 kHz.

(d) Simulate the ladder structure used and verify the results.

3-3 Design an LP filter for the following specifications

a_{max}, dB	a_{min}, dBs	ω_1, rad/s	ω_2, rad/s
1.0	40	2000	4500

(a) Determine the degree n of the required maximally flat magnitude response.

(b) Determine the location of poles on the s plane.

(c) Find the quality factor of each pole.

Determine the actual loss $\alpha_{max}(\omega_1)$ and $\alpha_{min}(\omega_2)$at the pass band and the stop band edge frequencies.

3-4 Repeat problem 3-3 for the following specifications:

$\alpha_{max} = 2.0$ dBs, $\alpha_{min} = 50$ dBs, $\omega_1 = 2000$ rad/s and $\omega_2 = 6000$ rad/s.

3-5 Repeat problem 3-3 for the following specifications:

$\alpha_{max} = 1.0$ dB, $\alpha_{min} = 30$ dBs, $\omega_1 = 2000$ rad/s and $\omega_2 = 3600$ rad/s.

3-6 Repeat problem 3-3 for the following specifications:

$\alpha_{max} = 2.0$ dBs, $\alpha_{min} = 40$ dBs, $\omega_1 = 2000$ rad/s and $\omega_2 = 5000$ rad/s.

3-7 Consider the following set of specifications:

(i) $\alpha_{max} = 0.5$ dB, $\alpha_{min} = 32$ dBs, $\omega_1 = 1500$ rad/s, $\omega_2 = 3600$ rad/s

(ii) $\alpha_{max} = 1.0$ dB, $\alpha_{min} = 25$ dBs, $\omega_1 = 2000$ rad/s, $\omega_2 = 7000$ rad/s

(a) Find the required value of order n of the LP filter with maximally flat response.

(b) Determine the actual attenuation at the edge of the pass band and stop band.

(c) Determine the attenuation at $2.5\omega_1$ and $5\omega_1$.

3-8 Determine the Chebyshev polynomial $C_4(\Omega)$, $C_5(\Omega)$ and $C_6(\Omega)$, using equation (3.31)

3-9 Determine the pole location for the Chebyshev response for:

(a) $n = 5$ and $\alpha_{max} = 0.5$ dB,

(b) $n = 5$ and $\alpha_{max} = 1.0$ dB,

3-10 (a) Determine the order n, the pole location and the transfer function of an LP filter having 1.0 dB ripple width from 0 to 2.5 rad/s and a maximum of 30 dB attenuation beyond 5.0 rad/s.

(b) Find a resistance terminated lossless ladder for the filter realization in part (a).

3-11 Find the transfer function, ω_o and Q values for the following specifications with the help of Table 3.4.

(i) $\alpha_{max} = 1$ dB, $n = 6$,

(ii) $\alpha_{max} = 2$ dB, $n = 5$,

(iii) $\alpha_{max} = 0.5$ dB, $n = 4$

3-12 A sixth-order LP Chebyshev filter was realized with three options $-\alpha_{max} = 0.5$ dB, 1 dB and 2 dBs. Determine a relationship between ripple width and respective quality factor. Which option shall be preferred and why? (Use Table 3.4)

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3-13 In a Chebyshev filter of order 5, (de-normalized) $\omega_{CB} = 1$ Krad/s and $\alpha_{max} = 1.0$ dB. Determine: (a) the value of ε, (b) the value of the pass band edge frequency ω_1, (c) the value of $|H(j\omega)|^2_{min}$, (d) the frequencies of the peaks in pass band, and (e) the frequencies of the valleys in the pass band. (f) Accurately sketch the magnitude response, using only a calculator for the necessary calculations. Use a vertical scale in dBs and a linear radian frequency scale.

3-14 An anti-aliasing filter is needed for an A to D converter working at a sampling rate of 6000 samples/s. Hence, the anti-aliasing filter is to have a minimum attenuation of 60 dBs at 3 kHz using a Chebyshev filter.

(a) If $\omega_1 = 5$ krad/s and $\alpha_{max} = 1$dB, what is the required minimum order?

(b) If $\alpha_{max} = 1$dB and $n = 7$, what is the maximum value of ω_2?

(c) If $\omega_1 = 5\pi$ krad/s and $n = 7$, what is the minimum value of α_{max}.

3-15 Determine the transfer function and give numerical values of poles for part (c) of Problem 3-14. What shall be the value of center frequency and pole-Q of the second order sections?

3-16 (a) Find the required order for a maximally flat magnitude function which is down 1 dB at 1 rad/s and down 34 dBs at 1.5 rad/s.

(b) Repeat part (a) for an equal-ripple pass band filter.

(c) Repeat part (a) for an equal-ripple stop band filter.

3-17 For the attenuation characteristics shown in Figure P3.1, find the attenuation α at the frequency which is 2.5 times the pass band edge frequency.

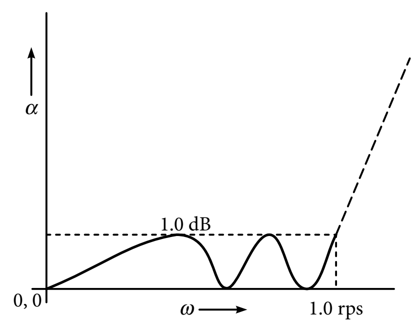

Figure P3.1

3-18 Specifications of an inverse Chebyshev function as shown in Figure P3.2 are as follows:

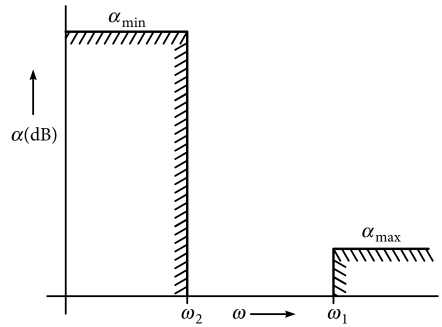

Figure P3.2

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$\alpha_{max} = 0.5$ dB, $\alpha_{min} = 20$ dBs, $\omega_1 = 36$ krad/s and $\omega_2 = 80$ krad/s

(a) Determine the order of the filter

(b) Determine the location of poles and zeros.

(c) Determine the frequency of the peaks and the valleys in the stop band.

(d) Find the transfer function satisfying the specifications in terms of the product of second-order (a first-order also if needed) sections.

3-19 Repeat sections (a) to (d) of Problem 3-18 for the following specifications:

$\alpha_{max} = 0.5$ dB, $\alpha_{min} = 30$ dBs, $\omega_1 = 2$ krad/s and $\omega_2 = 3.45$ krad/s

3-20 Find order of an elliptic HP filter using two alternate methods for the following specifications:

(i) $\alpha_{max} = 1.0$ dB, $\alpha_{min} = 30$ dBs, $\omega_1 = 80$ krad/s and $\omega_2 = 50$ krad/s

(ii) $\alpha_{max} = 0.1$ dB, $\alpha_{min} = 20$ dBs, $\omega_1 = 30$ krad/s and $\omega_2 = 15$ krad/s

3-21 Find the passive ladder structures for the elliptic filters of Problem 3-20, with (a) inductors and capacitors in series occurring in the shunt branches, and (b) inductor and capacitors in parallel occurring in the series branches of the networks.

3-22 Find practically suitable values of the elements while integrating for the filters obtained in Problem 3-21 and test the circuits using PSpice.

3-23 Find order of an elliptic filter using two alternate methods for the following specifications:

$\alpha_{max} = 1.0$ dB, $\alpha_{min} = 50$ dBs, $\omega_1 = 20$ krad/s and $\omega_2 = 24$ krad/s.

Find passive ladder structures for the obtained filter with (a) inductors and capacitors in series occurring in the shunt branches, and (b) inductor and capacitors in parallel occurring in the series branches of the networks. Find the actual minimum attenuation in the stop band in both cases.

3-24 It is desired that the dc gain of the maximally flat LP filter with the specifications given in Problem 3-7(ii) remains unity when a zero is introduced at 2750 rad/s. Find the modified transfer functions.