Chapter 2

First- and Second-order Filters

2.1 Introduction

Circuit designers/users evaluate filters and the order of the filters needed is based on what are the given specifications. The filter order n can be small or large. There are techniques by which filters of order $n \geq 2$ can be realized directly. Filters with order one or two can be used as such depending on the requirement; they can also be combined to provide filters of higher-order. Therefore, it is necessary to study the basic principles underlying the behavior of first- and second-order filter sections and the important terms used for their parameters before studying realization of higher order filters.

A first-order section can easily be realized using RC components only; but such sections suffer from certain limitations as shall be shown in Section 2.2. Hence, it is advisable to use first-order active filters with inverting or non-inverting amplifiers. A comparative study of active first-order filters, along with a discussion on the non-ideal effect of operational amplifiers (OAs) on their frequency response is given in Section 2.3 and 2.4. Terminologies used for second-order active sections and characteristics associated with low pass (LP), high pass (HP), band pass (BP), band reject (BR), and all pass (AP) responses are included in Sections 2.5 to 2.11. Constraints of the finite bandwidth of the OA on second-order filters are briefly discussed in Section 2.12. Three application examples in Sections 2.3.2, 2.3.3, and 2.7.1 are included to show the utility of these simple filter structures.

2.2 First-order Filter Sections

The transfer function of a physically realizable filter using a finite number of elements has to be a real rational function [2.1]. The rational function is a ratio of polynomials in the complex frequency s. It is repeated here from Chapter 1.

$$H(s)=\frac{N(s)}{D(s)}=\frac{a_m s^m + a_{m-1}s^{m-1}+\ldots+a_1 s+a_0}{b_n s^n + b_{n-1}+s^{n-1}+\ldots+b_1 s+b_0} \tag{2.1}$$

In this equation, the order of the transfer function is n, with $m \le n$.

For $m = n = 1$, the transfer function of equation (2.1) reduces to the following.

$$H(s)=\frac{N(s)}{D(s)}=\frac{(a_1 s+a_0)}{(b_1 s+b_0)} \tag{2.2}$$

As $H(s)$ in equation (2.2) is a ratio of two polynomials representing a straight line, it is also called a *bilinear function*. The transfer function $H(s)$ of equation (2.2) can be modified into a desirable form in terms of pole p_1 and zero z_1 zero as follows:

$$H(s)=\frac{a_1}{b_1}\frac{(s+a_0/a_1)}{(s+b_0/b_1)}=k\frac{s+z_1}{s+p_1} \tag{2.3}$$

For the transfer function to be physically realizable in stable form, its pole must not be in the right-hand side of the s plane [2.1]. Hence, b_0 and b_1 will have to be positive and finite (or both negative and finite), though a_0 and a_1 can be positive, negative, or even zero (one of the two, either a_0 or a_1). The zero z_1 can be located anywhere on the real axis. To realize the transfer function $H(s)$ with passive elements, a simple arrangement as shown in Figure 2.1 can be used. Different variations are possible depending on the choice of impedances Z_1 and Z_2. Some of the combinations are as follows:

For $Z_1 = R_1$ and $Z_2 = 1/sC_1$, as shown in Figure 2.2(a), the realized transfer function is $H(s) = (1/R_1C_1)/(s + 1/R_1C_1)$

For $Z_1 = 1/sC_2$ and $Z_2 = R_2$, as shown in Figure 2.2(b), $H(s) = s/(s + 1/R_2C_2)$

For $Z_1 = (R_1 + 1/sC_1)$ and $Z_2 = (R_2 + 1/sC_2)$ as shown in Figure 2.2(c), the transfer function becomes:

$$H(s)=\frac{R_2}{R_1+R_2}\frac{(s+1/R_2C_2)}{\left[s+(C_1+C_2)/C_1C_2(R_1+R_2)\right]} \tag{2.4}$$

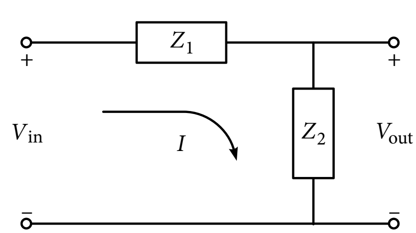

Figure 2.1 First-order bilinear transfer function realization using passive elements.

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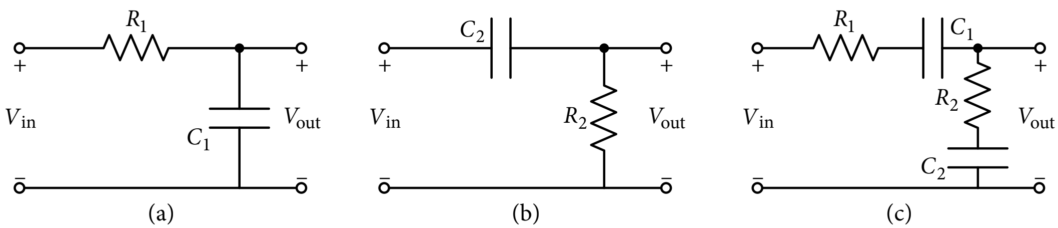

Figure 2.2 Few first-order transfer function realizations using resistors and capacitors only.

The bilinear function of equation (2.4) has a pole p_1 and a zero z_1; their expressions are as follows:

$$p_1 = \frac{(C_1 + C_2)}{C_1C_2(R_1 + R_2)} \text{and } z_1 = \frac{1}{R_2C_2} \tag{2.5a}$$

The gain of the transfer function of equation (2.4) is as follows:

$$\text{DC gain} = \frac{C_1}{(C_1 + C_2)} \text{and gain at high frequencies} = \frac{R_2}{R_1 + R_2} \tag{2.5b}$$

It is important to note that in this passive circuit, the voltage gain at any frequency will never be more than unity.

2.3 Active First-order Filters

In the previous section, it was shown that first-order transfer functions can easily be realized using passive elements. However, there are quite a few limitations in such networks. For example, all the values of pole (s) and zero (s) are not realizable. For example, an ideal integrator is not realizable. An important limitation is a resultant disturbance in the transfer function when the network gets loaded. Significant changes in the response take place because of the loading effect. Hence, it is preferable to realize the elements of the circuit in active form, which will additionally provide gain as well, which is generally needed.

The following sections describe a few simple applications of OAs being used as first-order active filters.

2.3.1 Use of inverting amplifiers

If the elements used in Figure 1.11(a) (lossy inverting integrator) are replaced by impedances Z_1, Z_2 (or admittances Y_1, Y_2), the transfer function of the circuit gets modified as shown in equation (2.6).

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$$H(s) = -(Z_2/Z_1) \text{ or } -(Y_1/Y_2) \tag{2.6}$$

The impedances Z_1, Z_2 (or Y_1, Y_2) can be any series/parallel combination of R_1 and C_1; R_2 and C_2 are as shown in Figure 2.3; the transfer function of the circuit is obtained as follows:

$$H(s) = -(Z_2/Z_1) = -\frac{R_2}{R_1}\frac{s+(1/R_2C_2)}{s+(1/R_1C_1)} = -k\frac{s+z_1}{s+p_1} \tag{2.7}$$

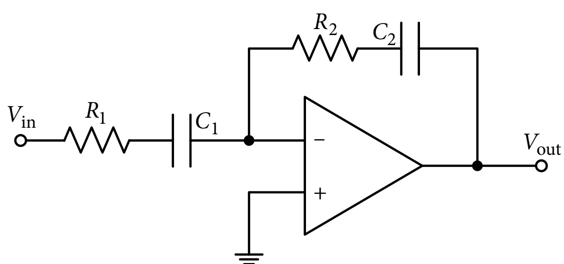

Figure 2.3 Realization of active first-order section using an inverting amplifier.

Comparing this equation with equation (2.3), we can realize a first-order section with the following parameters:

$$k = \frac{R_2}{R_1};\ \text{zero } z_1 = \frac{1}{R_2C_2} \text{and pole } p_1 = \frac{1}{R_1C_1} \tag{2.8}$$

If the designer is aware of the location (value) of the pole and zero and the gain at $s = 0$ (or $s = \infty$), element values can be easily calculated using equation (2.8). It is to be noted that equation (2.7) has three parameters, whereas element values to be found are four. Hence, one element value (or an element ratio) has to be assumed.

Example 2.1: Design a first-order active bilinear function having its zero at 1000 rad/s, pole at 2000 rad/s and gain of 20 dBs at very low frequencies.

Solution: Corresponding to 20 dBs, gain on linear scale is $k = 10^{20/20} = 10$ at very low frequencies (or $s = 0$). Using equation (2.7):

$$H(0) = -(C_1/C_2) = -10$$

If the OA is used in inverting mode, the gain will have a negative value of –10. Corresponding to the given pole and zero values, $1/R_2C_2 = 1000$ and $1/R_1C_1 = 2000$. Selecting $C_1 = 1\ \mu\text{F}$, we get $C_2 = 0.1\ \mu\text{F}$, $R_2 = 10\ \text{k}\Omega$ and $R_1 = 0.5\ \text{k}\Omega$. The desired circuit with element values is shown in Figure 2.4(b). Figure 2.4(c) shows the circuit's PSpice (Simulation Program for

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Integrated Circuits Emphasis: a simulator program used to verify circuit designs and predict their behavior) simulation with an input voltage of 0.1 volt. At low frequencies, the output voltage is approximately 1.0 V with a maximum of 2.0 V or gain of (–20).

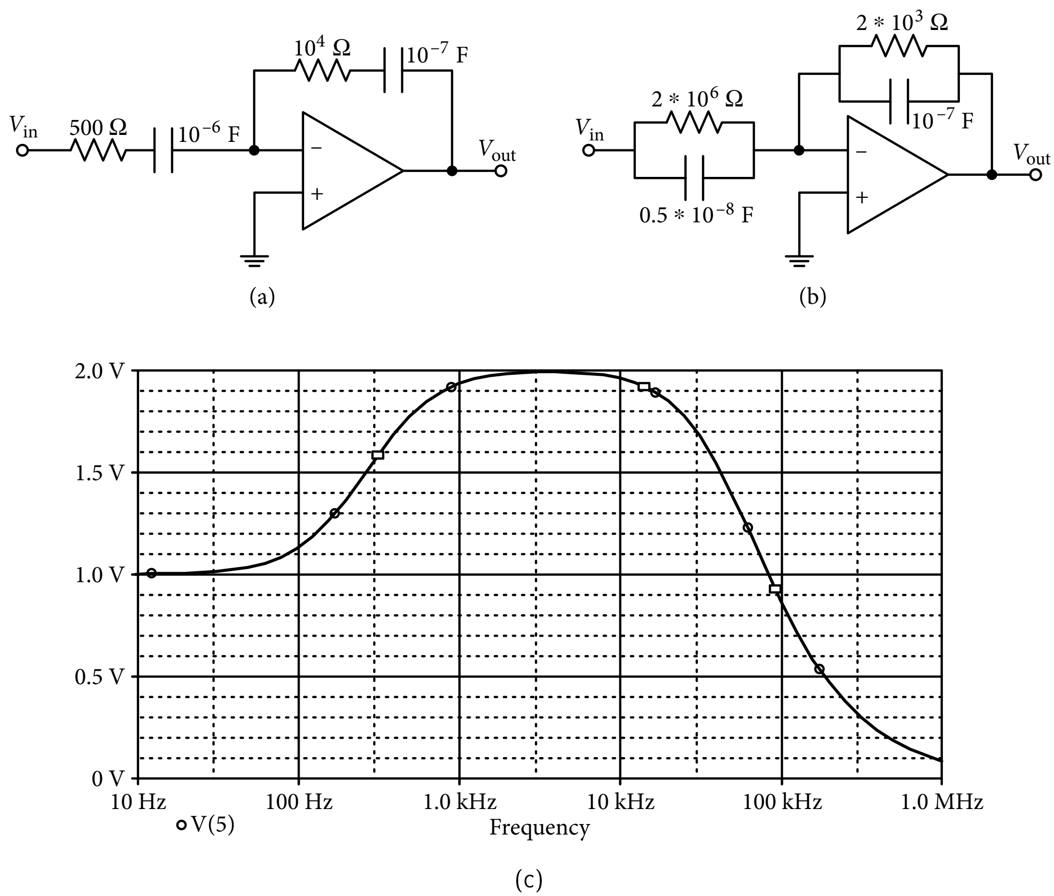

Figure 2.4 (a) Using series form and (b) parallel form of impedances in the circuit for Example 2.1. (c) Magnitude response of the active bilinear circuit for the circuit in Example 2.1.

2.3.2 Bass cut/boost and treble cut/boost filters – application example

In audio systems, low frequencies, which are typically in the range of a few Hz to 100 Hz, are called *bass* notes. Mid-range frequency signals, typically ranging between 100 and 1000 Hz, are called *middle* notes. High frequencies are called *treble* notes; they are typically above 1000 Hz. Low frequencies are responsible for the deep sound of bass guitars and drums. Most instruments create sounds in the mid-range frequency; these include guitars, brass or string instruments, and even the human voice. High frequencies are responsible for the *sparkle* sound of cymbals and clarity of voices. Sound gets muffled if the treble note is missing or weak. All

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the three types of notes are enhanced or *boosted* else weakened or *cut* to improve the quality. If no boost or cut is applied, the response is said to be flat. The following example is a simple practical illustration employing an inverting amplifier for audio systems.

In equation (2.7), $H(s) = -Z_2/Z_1$. Hence, if $Z_1 = R_1$ and Z_2 is a parallel combination of R_2, with resistance R_3 and capacitance C_1 in series, as shown in Figure 2.5(a), we obtain the following relation:

$$H(s) = -\frac{R_2}{R_1}\frac{R_3}{R_2+R_3}\left\{\frac{s+(1/C_1R_3)}{s+1/C_1(R_2+R_3)}\right\} \tag{2.9}$$

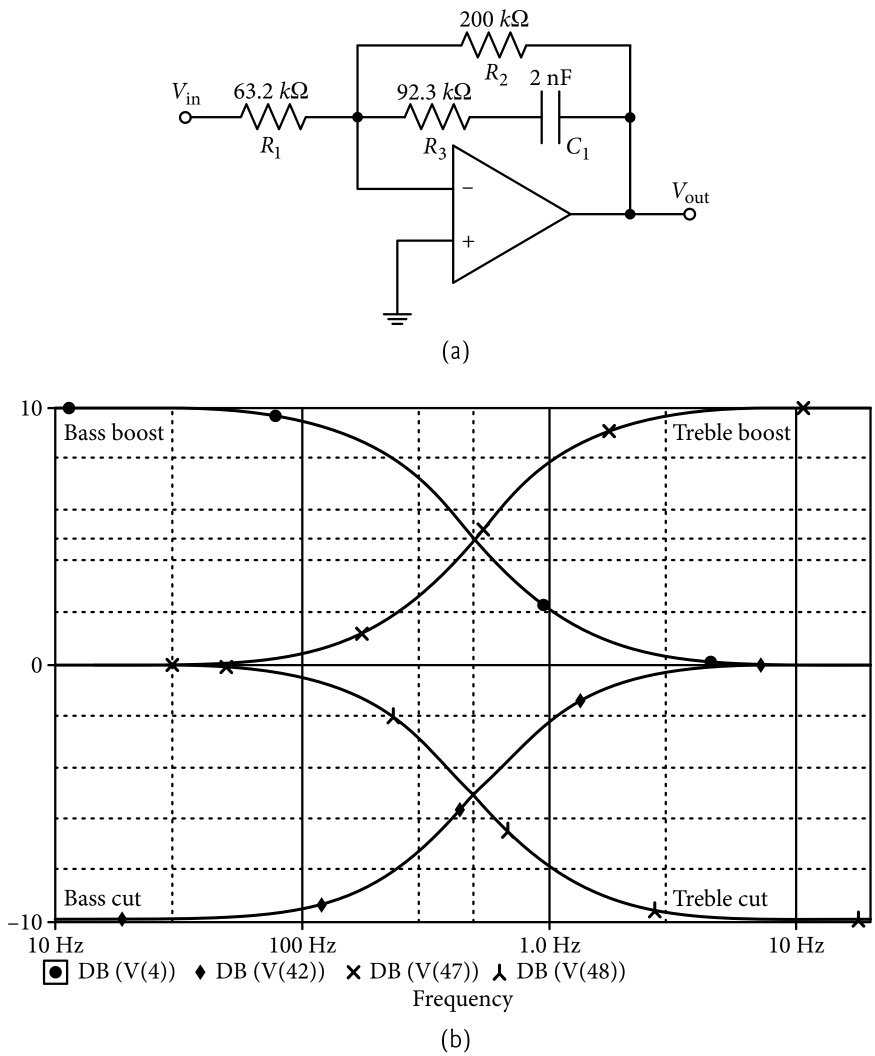

Figure 2.5 (a) A bass boost (and treble cut) circuit. (b) Response of the bass boost/cut and treble boost /cut by 10 dBs.

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The circuit in Figure 2.5(a) realizes a function which performs a bass boost action with the following expressions for gain at high and low frequencies:

$$\text{Gain at dc} = R_2/R_1 \tag{2.10a}$$

$$\text{Gain at high frequencies } = \frac{R_2}{R_1}\frac{R_3}{R_2 + R_3} \tag{2.10b}$$

For getting a boost of 10 dBs, if R_2 is selected as 200 kΩ, we need an R_1 of 63.2 kΩ, and for a high frequency gain of unity, equation (2.10b) gives R_3 = 92.3 kΩ. Pole frequency f_c of the bass boost circuit is decided by the choice of the capacitor C_1, with its expression as follows:

$$f_c = \frac{1}{2\pi C_1 (R_2 + R_3)} \tag{2.11}$$

With C_1 = 2 nF, f_c will be 272 Hz for the selected resistances.

The circuit in Figure 2.5(a) can also be used to function as a *treble cut*. To get a –10 dB treble cut and 0 dB gain at dc with a pole frequency of 272 Hz, equation (2.10) and (2.11) gives the required values of elements as: R_1 = 63.2 kΩ, R_2 = 200 kΩ, R_3 = 92.3 kΩ and C_1 = 2nF. Figure 2.5(b) shows the simulated responses of the designed bass boost and treble cut filter section.

If Z_1 is a series combination of resistance R_4, with resistance R_5 and capacitance C_2 in parallel, and Z_2 = R_6 as shown in Figure 2.6, its transfer function will be as follows:

$$H(s) = \frac{R_6(1 + sC_2R_5)}{(R_4 + R_5) + sC_2R_4R_5} \tag{2.12}$$

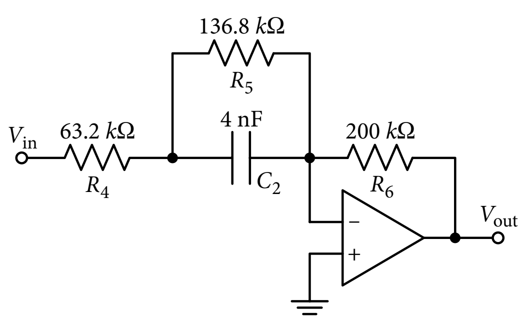

Figure 2.6 A treble boost (and bass cut) circuit.

While realizing a treble boost circuit, expressions of gain at higher frequency and at dc are as follows:

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$$\text{Treble boost gain} = R_6/R_4 \tag{2.13a}$$

$$\text{dc gain} = R_6/(R_4 + R_5) \tag{2.13b}$$

For a treble boost of 10 dB, selecting R_6 = 200 kΩ, we require an R_4 = 63.2 kΩ, for unity dc gain, R_5 = 136.8 kΩ from equation (2.13b). The expression of the pole frequency is:

$$f_c = \frac{(R_4 + R_5)}{2\pi C_2 (R_4 R_5)} \tag{2.14}$$

Hence, with C_2 = 4 nF, f_c will be 837 Hz.

The circuit in Figure 2.6 can also be used for bass cut function. To get a –10 dB bass cut, 0 dB gain at higher frequencies and pole frequency of 837 Hz, equations (2.13) and (2.14) gives element values R_4 = 63.2 kΩ, R_5 = 136.8 kΩ, R_6 = 63.2 kΩ and C_2 = 4 nF. Figure 2.5(b) also shows the simulated responses of the treble boost and bass cut circuits, verifying the design.

2.3.3 Fluorescence spectroscopy: application example

Frequency domain fluorescence measurements in atomic and molecular physics can be modeled in terms of first-order low pass filters (LPFs). Hence, as fluorescence can be mathematically equated to analog filters, a unified treatment of the entire fluorescence chain is possible by cascading their transfer functions [2.2].

Without going into the theoretical background of the representation of fluorescence, let us see the utility of simple LPFs as a useful practical application. It is observed that fluorescence from a three-level system (Figure 2.7(a)) can be represented by a Laplace transform equation [2.2]. This Laplace representation has been realized using two first-order cascaded LPFs. The cascaded filter is simulated for (i) a very fast relaxation from level 2 to 3 in Figure 2.7(b) and (ii) for a slower relaxation from level 2 to 3. The life time of the first-stage LPF for both the cases was set at 1 second. For the second-stage LPF, values of the components were selected by the inspection of the transfer function of a near-practical fluorescence measurement case with a life time of 10^{11} s and 10^3 s. Values of the elements for both the first filter and the two cases of the second filter are obtained from the equation of life time = ($1/2\,\pi RC$):

R_1 = 159 kΩ, C_1 = 1 μF for stage 1 and (a) R_2 = 1.59 kΩ, C_2 = 100 nF for case (i) and (b) R_2 = 0.159Ω, C_2 = 10 pF for case (ii) (2.15)

Simulated response of the two cases is shown in Figure 2.7(c), where curve '-Vp(51)' represents the decay rate of 10^{11} and curve '-Vp(5)' represents the decay rate of 10^3.

The equivalent circuit for the transfer function with two life time components is shown in Figure 2.8(a). The frequencies used for the slow and fast transition rates in the circuit were 100 Hz and 1 MHz. For these frequencies, the design values of the components are as follows:

R_1 = 15.9 kΩ, C_1 = 1 μF R_2 = 15.9 kΩ, C_2 = 100 pF, R_4 = R_5 = 10 kΩ, R_3 varies from 10 kΩ to 11.11 kΩ, 12.5 kΩ, 16.66 kΩ, 25 kΩ, 50 kΩ and 100 kΩ (2.16)

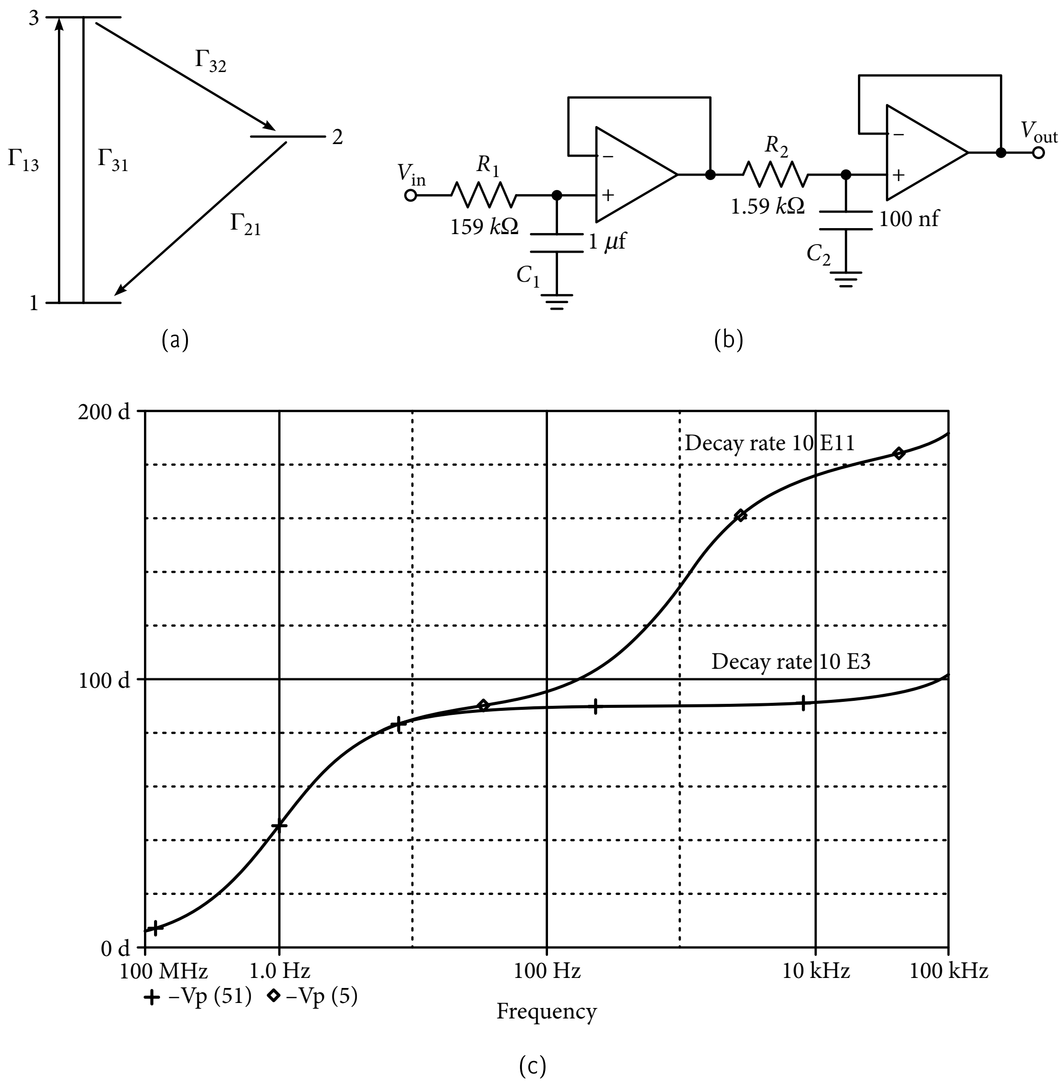

Figure 2.7 (a) Three-level representation of fluorescence. {With permission from R. Trainham et al. [2.2]} (b) Low pass filter realization of the transfer function for the three-level fluorescence. {With permission from R. Trainham et al. [2.2]} (c) Phase shifts corresponding to the two cases of fast and slow RC time constants for the cascaded low pass filter in Figure 2.7(b).

R_3 was varied to change the weight age of the transition rate of the slow component. The signals from the two LPFs were added and the final response is shown for two different ratios of the intensity of the slow component to the intensity of the fast component in Figure 2.8(b). Responses given by the filters of Figures 2.7(b) and 2.8(a) match very well with calculated theoretical responses [2.2].

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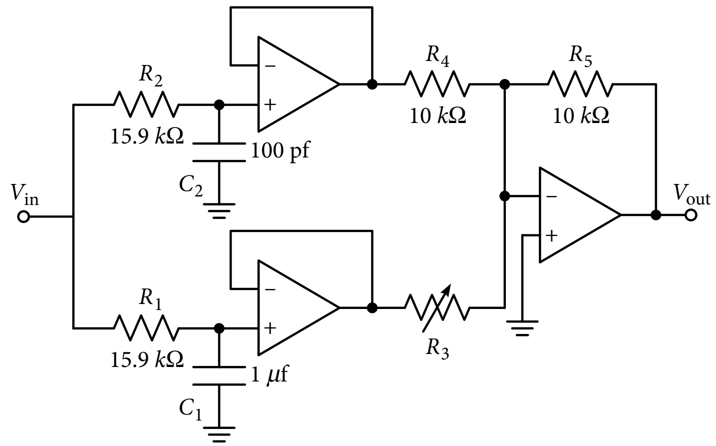
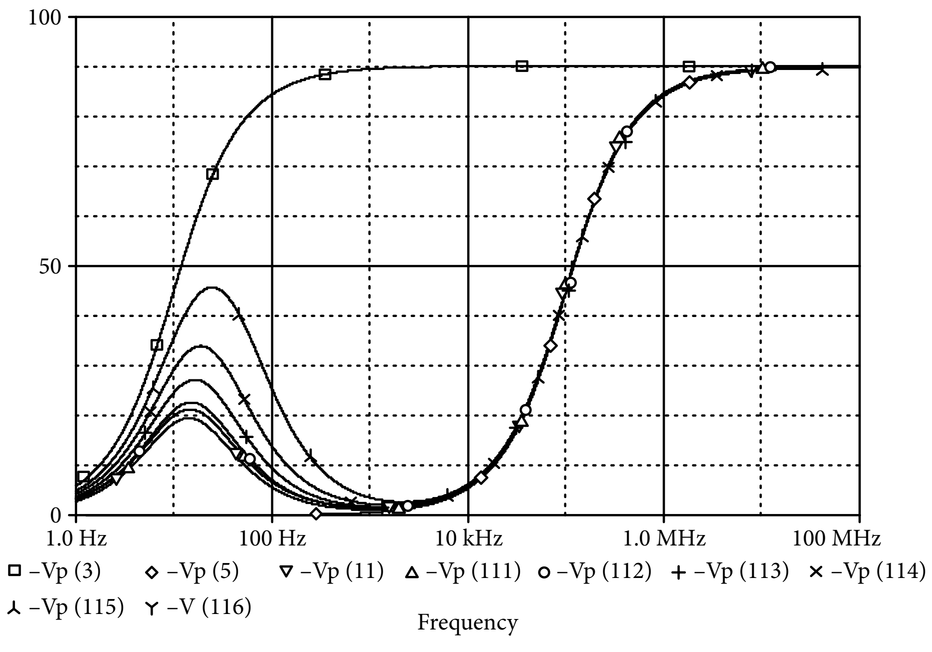


(a)



(b)

Figure 2.8 (a) Realization of the transfer function for the two life time fluorescence. {With permission from R. Trainham et al. [2.2]} (b) The family of curves of phase shifts for different mixtures of two life times separated by four decades.

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2.3.4 Use of non-inverting amplifiers

A non-inverting amplifier using OA can also be used to realize a first-order bilinear function by replacing its resistors with general impedances Z_1, Z_2 (or Y_1, Y_2) as shown in Figure 2.9.

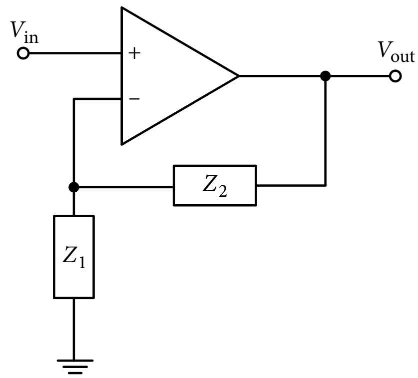

Figure 2.9 Non-inverting amplifier for the realization of a first-order active bilinear transfer function.

The transfer function is as follows:

$$H(s) = 1 + (Z_2/Z_1) = 1 + (Y_1/Y_2) \tag{2.17}$$

Comparison of equation (2.17) with equation (2.3) gives the following relation.

$$\left(Z_2 / Z_1\right) = k\frac{\left(s+z_1\right)}{\left(s+p_1\right)} - 1 = \frac{s(k-1)(kz_1 - p_1)}{(s+p_1)} \tag{2.18}$$

As impedances Z_1, and Z_2 are positive entities, the following constraints are to be met to keep the numerator positive:

$$k \geq 1 \text{ and } (kz_1 - p_1) \ \geq 0 \tag{2.19}$$

The designer needs to be careful about the strict constraint out of the two in equation (2.19). The absolute value is not important as it only adds to the gain which can also be controlled with a cascaded amplifier/ attenuator. Element values depend on the way the impedances Z_1 and Z_2 are realized. For example, if Z_1 is a series combination of R_1 and C_2 and Z_2 is a parallel combination of R_2 and C_2 or vice versa, the circuit will realize a second-order transfer function. With Z_1 and Z_2 both having resistors and capacitors in series, the flow of biasing current in the inverting node of the OA is blocked (this needs to be overcome by connecting a high value resistor in parallel with the input capacitor C_1).

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Example 2.2: Design a first-order bilinear transfer function using a non-inverting amplifier having the pole and zero of Example 2.1 with gain $k = 5$.

Solution: As, $k > 1$, the first constraint in equation (2.19) is valid for selecting Z_1 as a parallel combination of R_1 and C_1 and Z_2 as a parallel combination of R_2 and C_2 in the non-inverting amplifier of Figure 2.9. The obtained transfer function is as follows:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{(C_1 + C_2)}{C_2} \frac{s + (R_1 + R_2)/R_1R_2(C_1 + C_2)}{\left(s + \left(1/R_2C_2\right)\right)} \tag{2.20}$$

Hence, $H(0) = 1 + (R_2/R_1)$, and at high frequencies, say

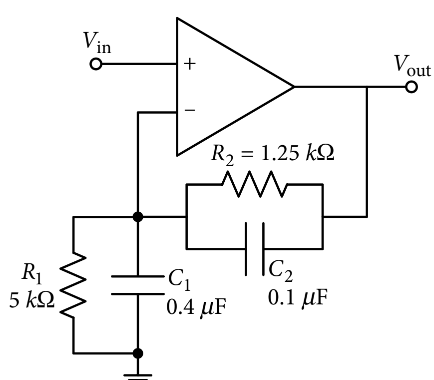
$$H(10\text{kHz}) = 1 + (C_1/C_2) \tag{2.21}$$

Expression of its pole and zero are obtained as follows:

$$p_1 = \frac{1}{R_2C_2} \text{ and } z_1 = \frac{(R_1 + R_2)}{R_1R_2(C_1 + C_2)} \tag{2.22}$$

With $k = H(10\text{kHz}) = 5$, equation (2.21) gives $(C_1/C_2) = 4$; hence, selecting $C_2 = 0.1$ µF, we get $C_1 = 0.4$ µF. For the pole at 2000 rad/s and the zero at 1000 rad/s, use of equation (2.22) gives the values of R_1 and R_2 as 5 kΩ and 1.25 kΩ, respectively.

All the values of the elements used are shown in Figure 2.10(a) and the PSpice simulated response is shown in Figure 2.10(b). For an input voltage of 1.0 V, the circuit has a minimum output voltage of 1.25 V at low frequencies and a maximum of 5.044 V at around 10 kHz. A gain of 5 can be verified at around 10 kHz, the circuit's useful frequency range. The voltage peaks at nearly 81 kHz is due to the effect of the frequency-dependent gain of the OA; the peak voltage is controlled by the supply voltages of the OA. Additionally, if the input voltage is increased beyond nearly 1.0 volts, the output gets distorted due to the effect of the slew rate constraint.



(a)

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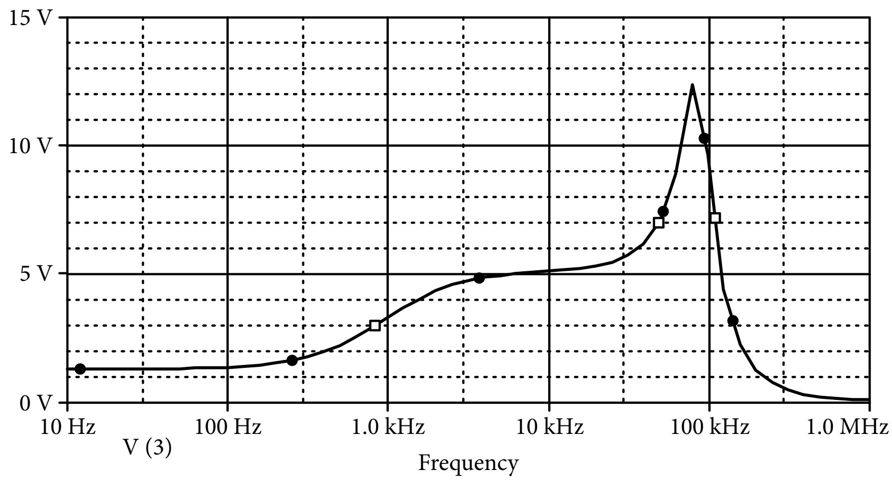

(b)

Figure 2.10 (a) Circuit realizing the transfer function of Example 2.2 using a non-inverting amplifier. (b) Response of the first-order bilinear circuit of Figure 2.10(a).

2.4 Effect of Operational Amplifier's Pole on Integrators

Analysis of the circuits done in the chapter so far assumed OAs as ideal. Finite values of the circuit's input and output resistors do come in the picture but the most significant effect of the resistors is that of the finite and frequency-dependent open-loop gain. This non-idealness is mainly responsible for the use of OA (with the commonly used type, like 741) based circuits being restricted to low frequencies (in the audio frequency range) as was shown in Example 2.2.

This section will look into the performance variation of the first-order filters discussed in Section 2.3 when the OA is represented by its first-pole roll-off model given in equation (1.17). Effect of finite values of R_i and R_o is not considered here for two reasons. First, their values are not far from ideal; hence their effect is minimal and their introduction will only increase the complexity unnecessarily. Second, in critical cases, the effect of the finite values of the resistors can be absorbed in the components employed in the filter realizations.

For the realization of the inverting amplifier of the first-order section shown in Figure 2.3, the ratio of the output to the input voltage is as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{Z_2}{Z_1}\frac{1}{1+(1+Z_2/Z_1)/A(s)} \tag{2.23}$$

With $-\dfrac{Z_2}{Z_1} = -k\dfrac{s+z_1}{s+p_1}$ from equation (2.7) and $k = \dfrac{R_2}{R_1}$ from equation (2.10a), limitation of the use of OA can be observed by using the approximated model given by equation (1.17)

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while neglecting its first-pole frequency of ω_a. The integrator model gives sufficiently accurate results at high frequencies; though at very low frequencies, ω_a cannot be neglected for correct results. Hence, equation (2.23) can be modified to the following:

$$\frac{V_{\text{out}}}{V_{\text{in}}}=-k\frac{(s+z_1)}{(s+p_1)+(s/B)[(s+p_1)+k(s+z_1)]}=-k\frac{(s+z_1)}{s^2\left(\frac{1+k}{B}\right)+s\left(1+\frac{p_1+kz_1}{B}\right)+p_1} \tag{2.24}$$

For an ideal OA, B being infinite, equation (2.24) will reduce to equation (2.7), and when B is finite, larger B or smaller values of z_1 and p_1 will lessen the parasitic effect. Frequency dependence of the OA gain has increased the denominator order by one, resulting in two poles. One of the poles will be near the original pole p_1 as $(p_1 + \Delta p_1)$ and a second pole, p_2 will be far from p_1, but the distance between the two (or the effect of non-ideality) will depend on the value of B (unfortunately, this is not the same for all OAs) and the value of the parameters p_1, z_1, and k.

Example 2.3: Design a first-order circuit using an inverting amplifier which will have a pole at 2×10^5 rad/s, zero at 10^5 rad/s and a gain of 2 at low frequencies. Find the effect of the frequency-dependent open-loop gain of the OA with B as (i) 10^6 rad/s, (ii) 0.5×10^6 rad/s and (iii) 10^5 rad/s. Compare the results with the simulated responses of the circuit.

Solution: In a similar way as in Example 2.1, the structure of the circuit is similar to that in Figure 2.11(a) or 2.11(b) with the following values if OAs were taken as ideal during analysis.

$R_1 = 250\Omega$, $R_2 = 1000\Omega$, $C_1 = 2 \times 10^{-8}\,F$ and $C_2 = 10^{-8}\,F$

$R_1' = 500\Omega, R_2' = 1000\Omega, C_1' = 2\times10^{-8}\,F$ and $C_2' = 0.5\times10^{-8}\,F$

With frequency-dependent gain, $A(s) \cong B/s$, the expression of gain is obtained as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}}=-2\frac{(s+10^5)}{s^2\left(\frac{1+2}{B}\right)+s\left(1+\frac{2\times10^5+2\times10^5}{B}\right)+2\times10^5} \tag{2.25a}$$

For further analysis and in order to determine the effect of the frequency dependence of the gain of the OA, three cases are taken with different values of the gain bandwidth product B.

i. For $B = 10^6$ rad/s, equation (2.25a) can be modified as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}}=-2\frac{(s+10^5)}{3\times10^{-6}s^2+1.4s+2\times10^5} \tag{2.25b}$$

When $B = 10^6$ rad/s, we get conjugate poles $p_{1,2} = (-2.333 \pm j\,1.1055) \times 10^5$ rad/s.

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In this case, the poles are not too far from the design pole value, and the characteristics show small deviation with peak gain having small reduction. The peak value is 3.9735 (against a theoretical value of 4), that too only at a higher frequency of 250.55 kHz, nearly one-fourth of B.

ii. For $B = 0.5 \times 10^6$ rad/s, we get conjugate poles $p_{1,2} = (-1.5 \pm j1.0408) \times 10^5$ rad/s.

In this case, the characteristics gets deviated a bit more, with peak gain going down to 3.95 at 159.3 kHz, still at a reasonably high frequency.

iii. For $B = 10^5$ rad/s, we get two real poles at -1.0×10^5 rad/s and -0.666×10^5 rad/s, and the characteristics get highly deviated with the gain doing down to 3.73, a deviation of 6.75%, at 78.7 kHz, because of the second real pole positions.

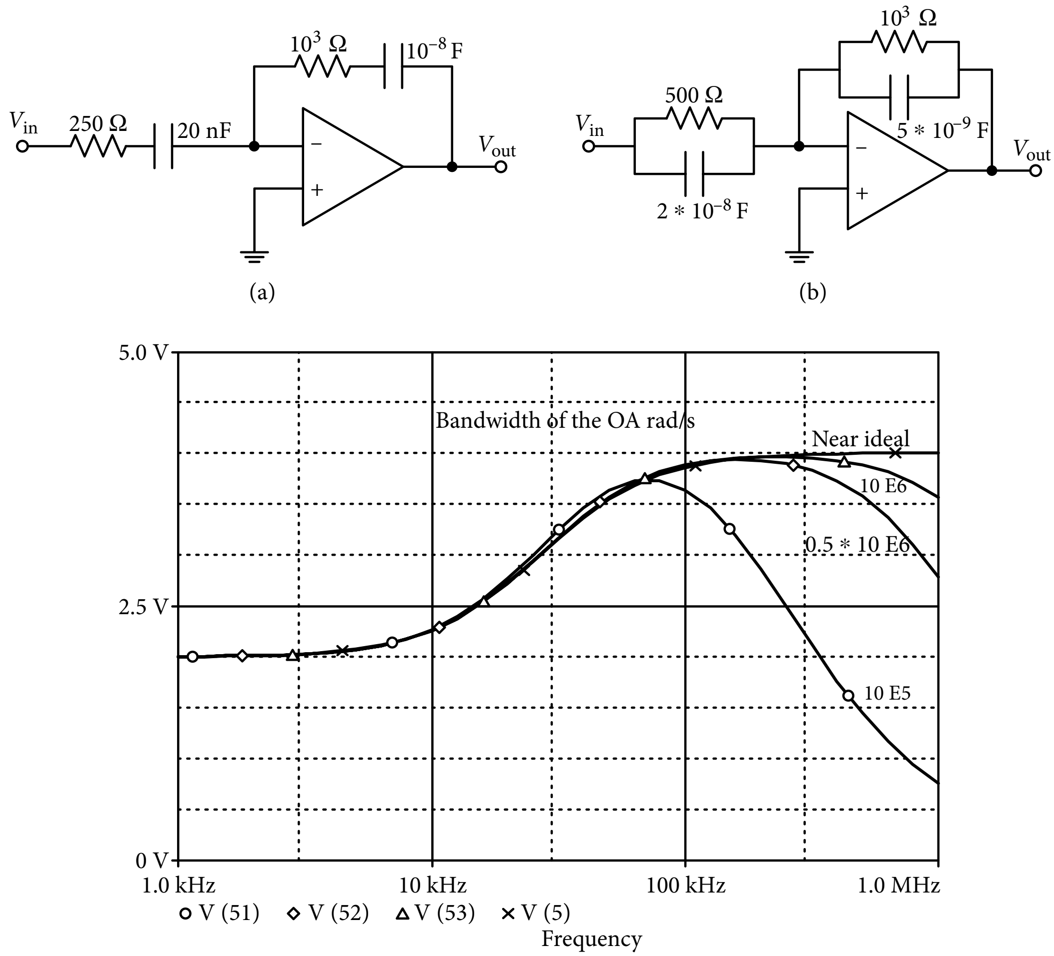

Figure 2.11 (a) Series, and (b) parallel forms of circuits for Example 2.3. (c) Magnitude response of the circuit of Figure 2.11(a) with ideal OA and OA with bandwidth = 10^6, 0.5×10^6 and 10^5 rad/s.

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It is clear from these observations and the response, when OA is ideal in Figure 2.11(c), that the higher the ratio of gain bandwidth product to the working pole frequency, the more the non-ideality effect gets reduced. It may be noted that the value of the dc gain k also plays an important role in the amount of non-ideal effect; the larger the value of k, the more deviation in pole location. In the example provided here, the pole value was intentionally selected high enough and close to B to highlight the non-ideality effect of OAs.

It is observed that when the first-order section is realized using a non-inverting amplifier, the non-ideality of the operational amplifier affects the filter in a similar way. Order of the filter is increased by one for each OA; the original pole deviates and the amount of deviation depends on the ratio of B with the pole value and on the gain value k.

Example 2.4: Design a first-order section using an ideal non-inverting amplifier with the same specifications as in Example 2.3. Evaluate the effect of the non-ideality of the OA through simulation results.

Solution: Using the non-inverting circuit of Figure 2.10 and taking OA as ideal, the obtained expression of the gain is as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{(C_1 + C_2)}{C_2} \frac{s + \dfrac{(R_1 + R_2)}{R_1 R_2 (C_1 + C_2)}}{s + (1/C_2 R_2)} \tag{2.26}$$

Values of the elements for the given specifications $p_1 = 2 \times 10^5$ rad /s, $z_1 = 10^5$ rad/s and dc gain of 2 from equation (2.26) are obtained as follows:

$$R_1 = R_2 = 0.5\ \text{k}\Omega,\ C_1 = 0.03\ \mu\text{F and } C_2 = 0.01\ \mu\text{F}$$

The circuit with the aforementioned element values is shown in Figure 2.12(a) and the simulated response of the ideal case is shown in Figure 2.12(b) having a dc gain of 2 and a high frequency gain of 4.

For the non-inverting amplifier circuit of Figure 2.5, expression of the ratio of the output to input voltage with $A(s) \cong B/s$ is given as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \left(1 + \frac{Z_2}{Z_1}\right) \frac{1}{1 + \dfrac{1}{A(s)}\left(1 + \dfrac{Z_2}{Z_1}\right)} = k \frac{s + z_1}{\dfrac{k}{B} s^2 + \left(1 + \dfrac{k}{B} z_1\right) s + p_1} \tag{2.27}$$

Equation (2.27) results in two poles. Value of the poles will again depend on the value of B, its ratio with p_1 and the value of the gain k.

For $B = 10^6$ rad/s, $p_{1,2} = (-3 \pm j1) \times 10^5$ and peak gain of 3.8 occurs at 126.36 kHz.

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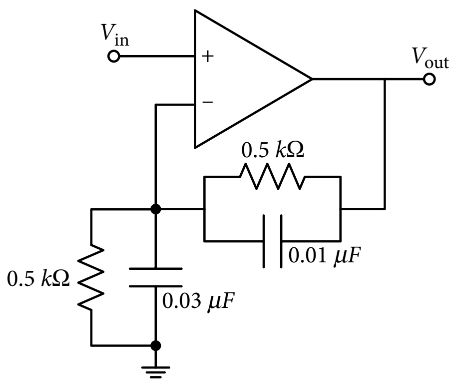
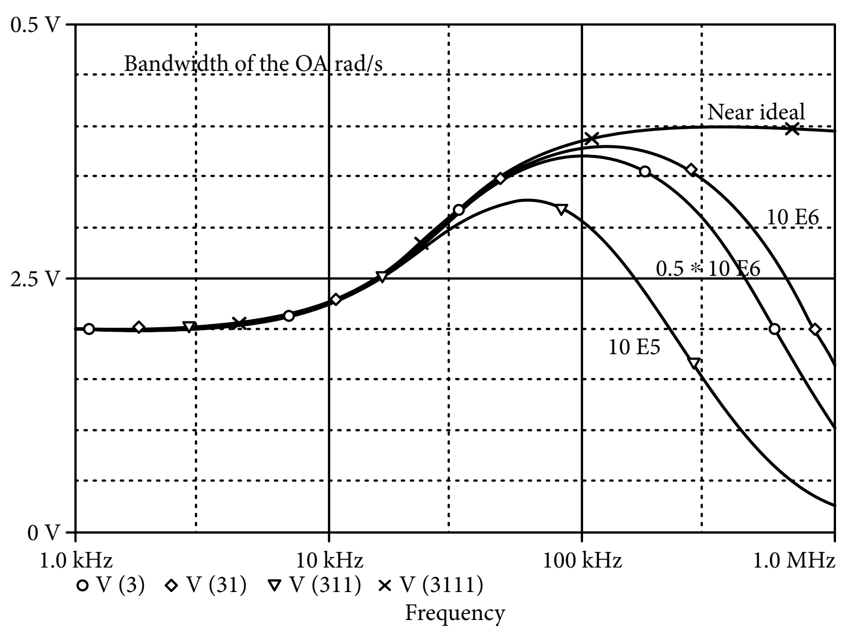
For $B = 0.5 \times 10^6$ rad/s, $p_{1,2} = (-1.75 \pm j1.39) \times 10^5$ and response drops with peak gain, dropping to 3.7 at a frequency of 100 kHz.

For $B = 10^5$ rad/s, $p_{1,2} = (-0.75 \pm j0.661) \times 10^5$, the response further worsens, having a peak gain of only 3.26, and an error of 18.5% at a much lower frequency, 62.28 kHz.

All the three responses, along with the case when OA is almost ideal, are shown in Figure 2.12(b). It is to be noted that distortion in the non-inverting amplifier case is comparatively much larger than the inverting amplifier case.



(a)



(b)

Figure 2.12 (a) Non-inverting amplifier circuit for Example 2.4. (b) Magnitude response of the circuit of Figure 2.12(a) with ideal OA and OA with bandwidth = 10^6 , 0.5×10^6 and 10^5 rps.

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2.5 Biquadratic Section: Parameters ω_o and Q_o

Moving on from first-order sections to second-order ones, transfer functions of second-order sections can be easily obtained by cascading two first-order sections. The overall transfer function is simply the product of the individual transfer function of the first-order sections, provided they satisfy the condition of cascading, that is, they have very high input impedance and very low output impedance. However, such second-order sections realize poles only on the negative real axis (with OAs considered ideal), which can be realized even with only passive elements. This is where the following second-order section, given by equation (2.28), comes in. The equation gives a basic module that is used in different ways to construct various higher-order filter sections, for example, in cascade or multiple feedback form.

$$H(s)=\frac{N(s)}{D(s)}=\frac{a_2s^2+a_1s+a_0}{b_2s^2+b_2s+b_0} \tag{2.28}$$

It is, therefore, useful to concentrate first on such sections expressed in terms of poles and zeroes as follows:

$$H(s)=\frac{a_2}{b_2}\frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)} \tag{2.29}$$

The importance of the aforementioned section, which is commonly known as a *biquadratic section* comes from the condition that the poles are complex conjugate; the zeroes may or may not be complex conjugate. With the condition that the poles are in conjugate pair form, the transfer function can be expressed in terms of real and imaginary parts of the zeros $\mathrm{R_e}(z_1)$ and $\mathrm{I_m}(z_1)$, and the real and imaginary parts of the poles, $\mathrm{R_e}(p_1)$ and $\mathrm{I_m}(p_1)$ as follows:

$$H(s)=k\frac{s^2+\left[2\mathrm{R_e}(z_1)\right]s+\mathrm{R_e^2}(z_1)+\mathrm{I_m^2}(z_1)}{s^2+\left[2\mathrm{R_e}(p_1)\right]s+\mathrm{R_e^2}(p_1)+\mathrm{I_m^2}(p_1)}=k\frac{s^2+(\omega_z/Q_z)s+\omega_z^2}{s^2+(\omega_o/Q_o)s+\omega_o^2} \tag{2.30}$$

Here ω_o is the pole frequency, which is given in terms of its real and imaginary parts as follows:

$$\omega_o^2=\mathrm{R_e^2}(p_1)+\mathrm{I_m^2}(p_1) \tag{2.31a}$$

At frequency ω_o, the gain function becomes approximately the highest. At the zero frequency ω_z, the gain function become approximately the least and its relation is given as follows:

$$\omega_z^2=\mathrm{R_e^2}(z_1)+\mathrm{I_m^2}(z_1) \tag{2.31b}$$

Conjugate zeros and poles, and their real and imaginary components are shown in Figure 2.13.

For the biquadratic function of equation (2.28) or equation (2.30), dc gain is given as follows:

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$$20\log_{10}|H(j0)| = 20\log_{10}\left(k\,\omega_z^2/\omega_o^2\right) \tag{2.32}$$

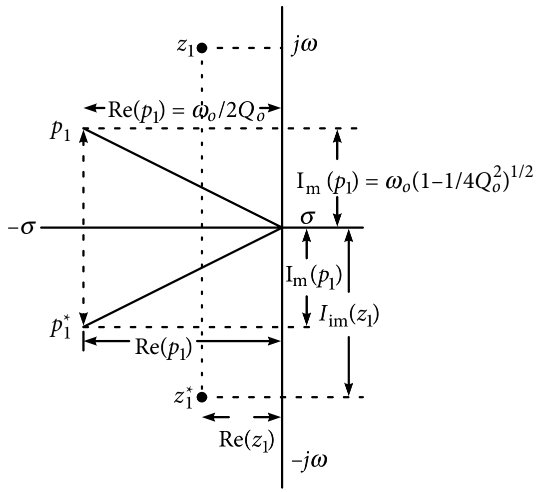

Figure 2.13 Conjugate zeroes and poles of a second-order section, located on the *s* plane and their relation with Q_o.

And the asymptotic gain for ω reaching infinity is as follows:

$$20\log_{10}|H(j\infty)| = 20\log_{10}(k) \tag{2.33}$$

Another important parameter which defines the sharpness of the magnitude response near the maxima, $|H(j\omega_o)|$ is known as pole quality factor Q_o, which is given as follows:

$$Q_o = \frac{\omega_o}{2\mathrm{R_e}(p_1)} = \left[\frac{\mathrm{R_e}(p_1)^2 + \mathrm{I_m}(p_1)^2}{2\mathrm{R_e}(p_1)}\right] \tag{2.34}$$

Whereas the depth of $|H(j\omega_z)|$ is defined by the *zero quality factor* Q_z, given as follows:

$$Q_z = \frac{\omega_z}{2\mathrm{R_e}(z_1)} = \left[\frac{\mathrm{R_e}(z_1)^2 + \mathrm{I_{im}}(z_1)^2}{2\mathrm{R_e}(z_1)}\right] \tag{2.35}$$

In most cases, $\mathrm{R_e}(z_1) = 0$, which means $Q_z \to \infty$ and $\omega_z = \mathrm{I_{im}}(z_1)$. This results in infinite (ideally) attenuation at ω_z. From equation (2.34), $\mathrm{R_e}(p_1)$ can be expressed as follows, which is shown in Figure 2.13.

$$\mathrm{R_e}(p_1) = \omega_o/2Q_o \tag{2.36}$$

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Combining equations (2.34) and (2.31a), we get the following relation.

$$I_m(p_1) = \omega_o \left(1 - 1/4Q_o^2\right)^{1/2} \tag{2.37}$$

This is also shown in Figure 2.13, and from equations (2.36) and (2.37), we get:

$$\left\{R_e^2(p_1) + I_m^2(p_1)\right\}^{1/2} = \omega_o \tag{2.38}$$

which means that for all values of Q_o, the location of pole p_1 will lie on a circle with radius ω_o. For $Q_o = 0.5$, the poles became real; whereas for high Q_o, the poles are close to the imaginary axis.

2.6 Responses of Second-order Filter Sections

It is important to note that the zero ω_z can be anywhere on the s plane while deciding the nature of the filter, namely, low pass (LP), high pass (HP), and band pass (BP), and pole frequency ω_o and pole quality factor Q_o are the main design parameters. It is the value of ω_o which differentiates between the pass band and stop band of the LP and HP filters or decides the center frequency of the BP or BR (band reject) filters. The value of Q_o does have an effect on the gain response of the LP and HP sections at ω_o, but it is most significant in deciding the quality of BP or BR filters. Significance of ω_o and Q_o will be illustrated in detail in the next sections.

2.7 Second-order Low Pass Response

When $a_1 = a_2 = 0$, in equation (2.28), the expressions of the transfer function $H(s)$ in equations (2.28)–(2.30) will change to that of a second-order LP transfer function. Since the constant k is only a magnitude multiplier and does not affect the frequency response, it can be scaled, and the LP transfer function can be written as follows:

$$H_{LP}(s) = \frac{k\omega_o^2}{s^2 + (\omega_o / Q_o)s + \omega_o^2} \tag{2.39}$$

The network analysis is done assuming the input to be sinusoidal (or to be a combination of sinusoidal signals), $s \rightarrow j\omega$. Hence, the magnitude and phase of the LP transfer function shall be as follows:

$$\left|H_{LP}(j\omega)\right| = \frac{k\omega_o^2}{\left[\left(\omega_o^2 - \omega^2\right) + (\omega_o / Q_o)^2 \omega^2\right]^{1/2}} \tag{2.40}$$

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$$= \frac{k}{\left[\left(1-\omega_n^2\right)+(\omega_n / Q_o)^2\right]^{1/2}} \tag{2.41}$$

$$\varphi_{\text{LP}} = -\tan^{-1}\frac{(\omega / \omega_o Q_o)}{(1-\omega^2 / \omega_o^2)} \tag{2.42}$$

$$= -\tan^{-1}\frac{(\omega_n / Q_o)}{(1-\omega_n^2)} \tag{2.43}$$

In equations (2.41) and (2.43), $\omega_n = (\omega/\omega_o)$ is termed as a *normalized frequency*.

Magnitude and phase function of the LP are shown in Figure 2.14(a) and (b), respectively, where the magnitude function is as follows:

$$|H_{\text{LP}}(j0)| = k,\ |H_{\text{LP}}(j\omega_o)| = kQ_o \text{ and } |H_{\text{LP}}(j\infty)| \to 0 \tag{2.44}$$

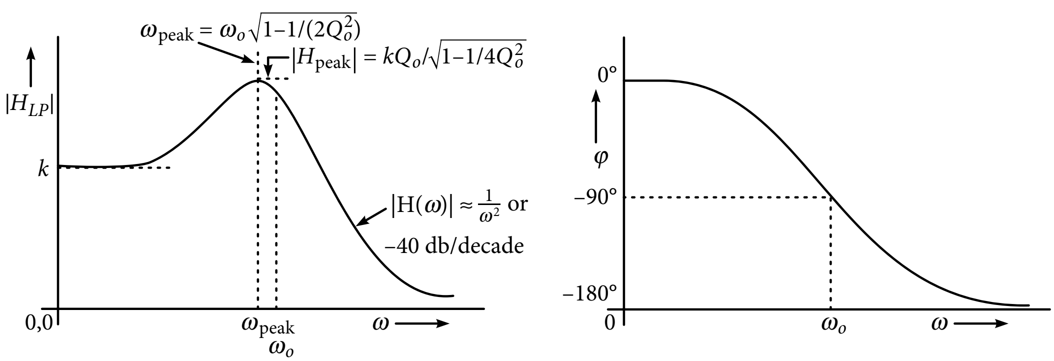

Figure 2.14 (a) Gain variation of a second-order low pass section, and (b) phase variation of the second-order low pass section for all values of Q_o.

These are shown in Figure 2.14(a), where the peak value is obtained by differentiating the magnitude function. Peak value of the transfer function H_{peak}, and the frequency at which it occurs ω_{peak}, are respectively given as follows:

$$H_{\text{peak}} = kQ_o / \left\{1-1/4Q_o^2\right\}^{1/2} \cong kQ_o \tag{2.45}$$

$$\omega_{\text{peak}} = \omega_o\left\{1-1/2Q_o^2\right\}^{1/2} \cong \omega_o \tag{2.46}$$

Approximation in equations (2.45) and (2.46) are satisfactory only with large values of Q_o. For $\omega \gg \omega_o$, rate of drop of the magnitude function is proportional to $(1/\omega^2)$or –20 dB/dec. The

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rate of drop for a second-order section is also known as *two-pole roll-off*, as compared to *single-pole roll-off* for a first-order transfer function having only one pole.

For the LP filter, sometimes it is desirable not to have a significant peak in the pass band. However, it is shown in Figure 2.14(a) that relative to $|H_{\text{LP}}(j0)|$, H_{peak} is larger by Q_o times, which implies that for avoiding significant peak, Q_o should have a low value (say < 0.9) for the LP filters. For $Q_o = 0.707$, equation (2.46) gives that the peak of the magnitude function shall occur at $\omega = 0$.

It is important to note that while designing LP filters, the usual specifications are given in terms of the *half-power frequency* ω_c, where $|H_{\text{LP}}(j\omega_c)|$ is 0.707 times its value at dc $|H_{\text{LP}}(j0)|$ (for $Q_o < 0.9$). Since the gain (v_o/v_{in}) falls by a factor of 0.707 at the *half-power frequency* ω_c, it is also called –3 dB frequency. Another required specification for the LP filter design is $|H_{\text{LP}}(j0)|$, which decides the gain required by the filter at dc.

For the phase function shown in Figure 2.14(b), value of phase change with ω is as follows:

$$\varphi(0) = 0,\ \varphi(\omega_o) = -90°, \text{ and } \varphi(\omega \to \infty) \to -180° \tag{2.47}$$

Example 2.5: Show that the circuit in Figure 2.15 behaves as a second-order LP function. Design it for $\omega_o = 10$ krad/s and $Q_o = 1/\sqrt{2}$ and $\sqrt{2}$.

Solution: Taking OA as ideal, nodal equations at nodes 2 and 3, respectively, are as follows:

$$V_1(G_1 + G_2 + G_3 + sC_2) - V_{\text{out}}G_2 - V_{\text{in}}G_1 = 0 \tag{2.48}$$

$$V_1G_3 + V_{\text{out}}sC_1 = 0 \tag{2.49}$$

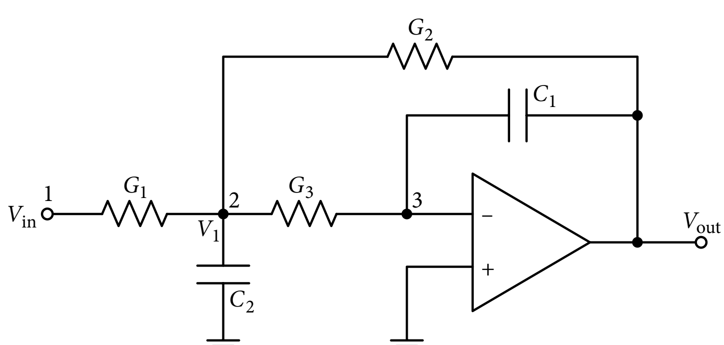

Figure 2.15 Second-order low pass filter section for Example 2.5.

Combining equations (2.48) and (2.49), the transfer function is obtained as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{(G_1G_3/C_1C_2)}{s^2 + s\left\{(G_1+G_2+G_3)/C_2\right\} + (G_2G_3/C_1C_2)} \tag{2.50a}$$

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It gives the expressions for ω_o and Q_o as:

$$\omega_o^2 = \frac{G_2 G_3}{C_1 C_2},\ Q_o = \frac{C_2}{G_1 + G_2 + G_3}\left(\frac{G_2 G_3}{C_1 C_2}\right)^{1/2} \text{ and } k = \left(\frac{G_1 G_3}{G_2 G_3}\right) \tag{2.50b}$$

Selecting $R_2 = R_3 = 5$ kΩ, with $\omega_o = 10$ krad/s, we get the following from equation (2.50b):

$$C_1 C_2 = 0.04 \times 10^{-14} \tag{2.51a}$$

Corresponding to $Q_o = \sqrt{2}$, selecting $R_1 = 1$ kΩ gives a dc gain of $k = 5$. Required values of the capacitors are obtained from equations (2.50b) and (2.51a) as follows:

$$C_1 = 2.0206 \text{ nF and } C_2 = 0.1974\ \mu\text{F} \tag{2.51b}$$

Figure 2.16 shows the magnitude responses of the PSpice simulation of the LP filter having Q_o value as $1/\sqrt{2}$ and $\sqrt{2}$. Magnitude response for $Q_o = 1/\sqrt{2}$ does not show any peak and its 3 dB frequency is 1.582 kHz (9.944 krad/s) with a dc gain of 5. However, the response for the corresponding LP filter with $Q_o = \sqrt{2}$, for which, with the same resistance values, the capacitances required are $C_1 = 4.04$ nF and $C_2 = 0.09899\ \mu F$, shows a peak gain of 7.526 at a frequency of 1.378 kHz (8.661 krad/s) in conformity with equations (2.46) and (2.45). Figure 2.16 also shows the corresponding phase responses for the two cases. Though the rate of variation in phase differs, in both the cases, a phase shift of 90° occurs at 1.592 kHz (10.0068 krad/s).

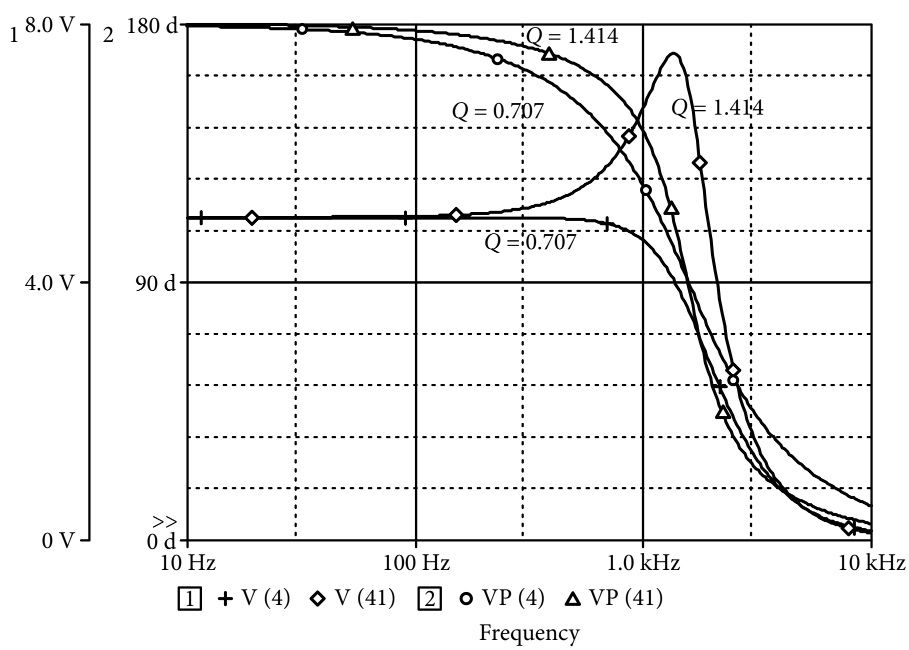

Figure 2.16 Magnitude and phase response of the low pass filter of Figure 2.15 with $Q = \sqrt{2}$ and $1/\sqrt{2}$.

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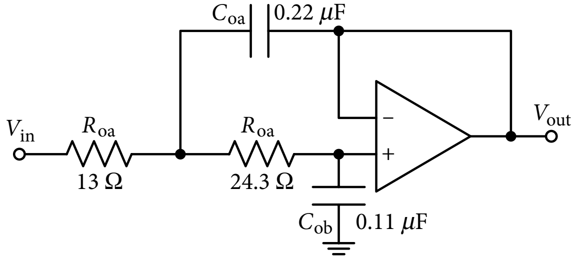
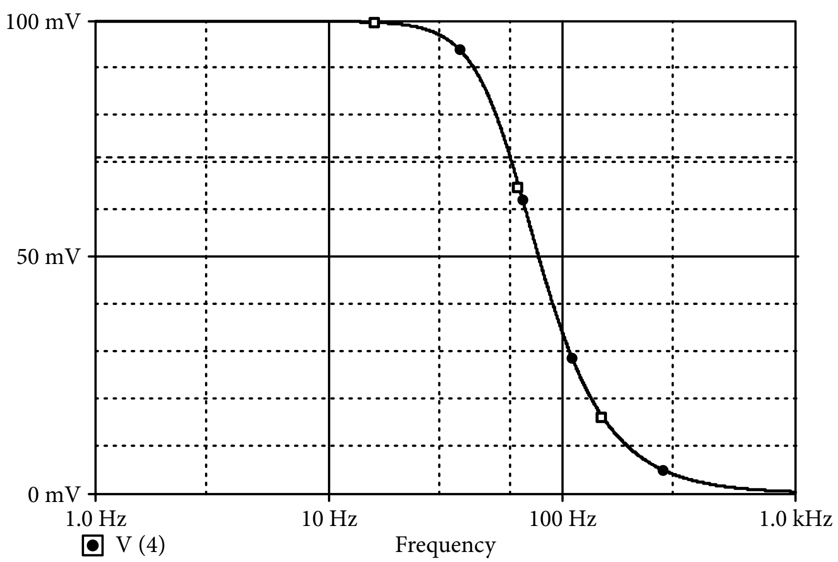
2.7.1 Earthworm seismic data acquisition: application example

The Earthworm System is a seismic network data acquisition and processing system developed by the US Geological Survey in the 1990s [2.3]. The system contained a number of real time electronic seismic wave forms (may be more than 16) that were fed to a multichannel digitizer (consisting of one, two or four 64 channel multiplexer boards).

Like any other data acquisition system, this system also faced the problem of picking up noise. In the beginning, passive filters were used to eliminate/reduce noises. However, passive filters introduced a 24 kΩ impedance between the source and the input. To overcome this limitation, a two-pole LP active filter using a single non-inverting OA as shown in Figure 2.17(a) was employed [2.4]. Quad OA Tl064 provided low impedance while consuming less power. Consumption of less power was an important parameter as a large number of such filters were used in the system.



(a)



(b)

Figure 2.17 (a) Low pass filter used in the Earthworm System [2.3]. (b) Simulated response of the low pass filter of Figure 2.17(a).

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Figure 2.17(b) shows the simulated response with the 3 dB frequency being 60.17 Hz. It may be noted that even a simple filter can be utilized for major projects.

2.8 Second-order High Pass Response

A biquadratic function can be converted to an HP response when coefficients $a_0 = a_1 = 0$. Equation (2.30) is modified as follows:

$$H_{\mathrm{HP}}(s) = \frac{ks^2}{s^2 + (\omega_o / Q_o)s + \omega_o^2} \tag{2.52}$$

Here k is the high frequency gain. The gain function of the HP section is shown in Figure 2.18, which is very similar in nature with the LP response,

$$\omega_{\mathrm{peak}} = \omega_o / \left\{1 - 1/2Q_o^2\right\}^{1/2} \cong \omega_o \tag{2.53a}$$

$$H_{\mathrm{peak}} = kQ_o / \left\{1 - 1/4Q_0^2\right\}^{1/2} \cong kQ_o \tag{2.53b}$$

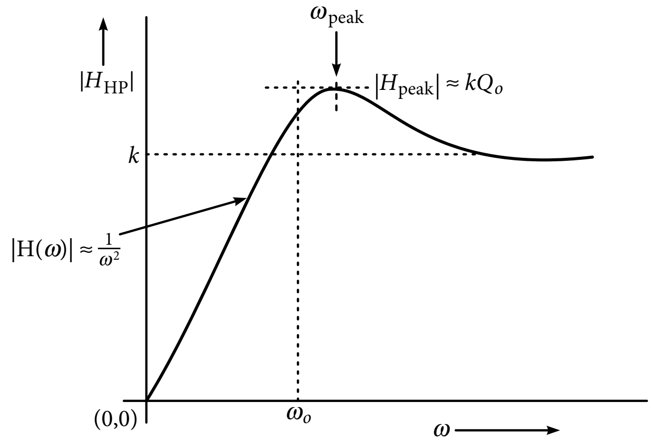


Figure 2.18 Gain variation of a second-order high pass section.

Approximation in equations (2.53) and (2.46) are satisfactory with large values of Q_o. Once again, its gain drops at a rate of –40 dB/dec in the pass band for $\omega \ll \omega_c$, and the gain is 3 dB below the high frequency gain, of $|H_{\mathrm{HP}}(\omega \to \infty)| = k$ at the half-power frequency ω_c (for Q_o < 0.9). The rate of gain drop is also known as *two-pole roll-off*, as compared to the *single-pole roll-off* for a transfer function having one pole only.

For the HP filter, it is desirable not to have a significant peak in the pass band. However, it is shown in Figure 2.18 that relative to $|H_{\mathrm{HP}}(j\infty)|$, H_{peak} is larger by nearly Q_o times, which implies that to avoid significant peak, Q_o should have a low value (say < 0.9). For Q_o = 0.707, equation (2.53b) gives that the peak of the magnitude function shall occur at $\omega = \infty$.

It is important to note that while designing an HP filter, the usual specifications are given in terms of the *half-power frequency* ω_c, where the $|H_{\text{HP}}(j\omega)|$ is 0.707 times its value at infinity, $|H_{\text{HP}}(j\infty)|$ (for $Q_o < 0.9$). Since the gain (v_o/v_{in}) falls by a factor of 0.707 at the *half-power frequency* ω_c, it is called –3 dB frequency. Another required specification for the HP filter design is $|H_{\text{HP}}(j\infty)|$, which decides the gain required by the filter at very high frequencies.

Example 2.6: Show that the circuit in Figure 2.19(a) behaves as a second-order HP function. Design it for ω_o = 10krad/s and $Q_o = 1/\sqrt{2}$ and $\sqrt{2}$.

Solution: Taking OA as ideal, the nodal equations at nodes 2 and 3, respectively, are as follows:

$$V_1(sC_1 + G_2 + sC_3 + sC_4) - V_{\text{out}}\,sC_4 - V_{\text{in}}\,sC_1 = 0 \tag{2.54a}$$

$$V_1\,sC_3 + V_{\text{out}}\,G_5 = 0 \tag{2.54b}$$

Combining equations (2.54a) and (2.54b), the transfer function is obtained as follows:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{s^2(C_1/C_4)}{s^2 + s\left\{G_5\left(C_1 + C_3 + C_4\right)/C_3C_4\right\} + (G_2G_5/C_3C_4)} \tag{2.55a}$$

This gives the expressions for ω_o and Q_o as

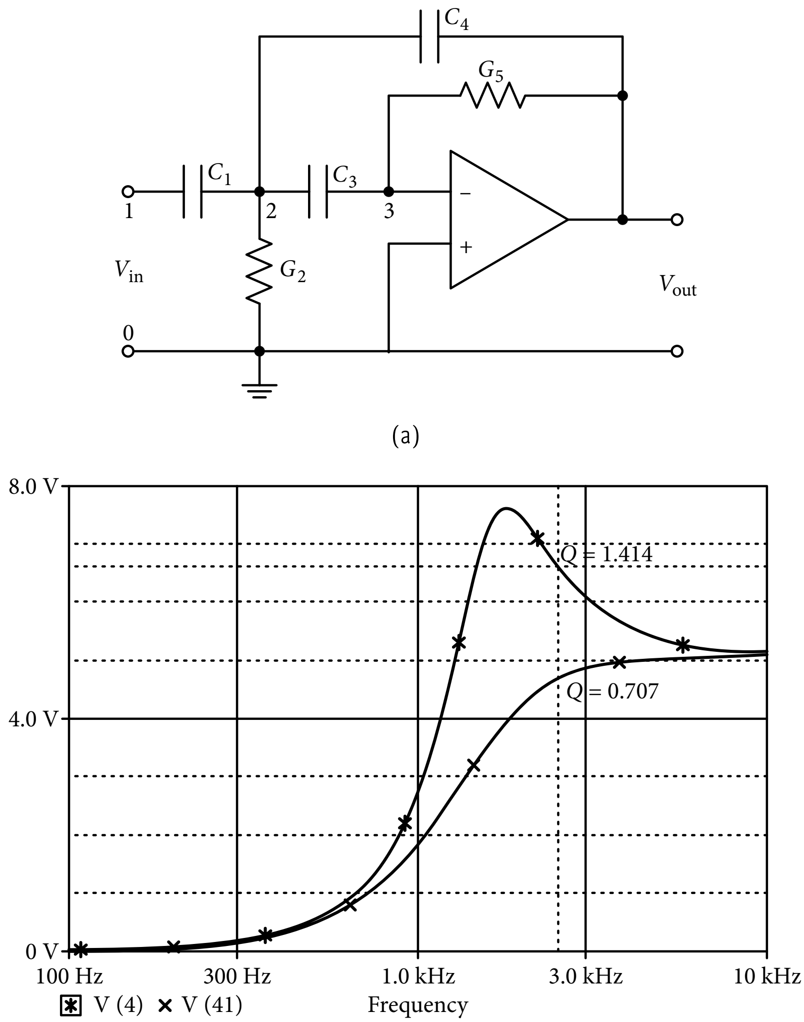
$$\omega_o^2 = G_2G_5/C_3C_4,\ Q_o = \frac{C_3C_4}{G_5(C_1 + C_3 + C_4)}\left(\frac{G_2G_5}{C_3C_4}\right)^{1/2} \text{ and } k = (C_1/C_4) \tag{2.55b}$$

Selecting $C_3 = C_4$ = 100 nF, we get the value of C_1 = 500 nF for k = 5, and the following relation from equation (2.55b) with ω_o = 10 krad/s:

$$G_2G_5 = 10^{-6} \tag{2.56}$$

Corresponding to $Q_o = \sqrt{2}$, using equation (2.55b), and selecting R_5 = 10 kΩ, the required values of the resistance R_2 = 100 Ω is obtained from equation (2.56).

Figure 2.19(b) shows the magnitude responses of the PSpice simulation of the HP filter having $Q_o = 1/\sqrt{2}$ and $\sqrt{2}$. Magnitude response for $Q_o = 1/\sqrt{2}$ does not show any peak and its 3 dB frequency is 1.602 kHz (10.069 krad/s) with a dc gain of 5.09. However, the response for the corresponding HP filter with $Q_o = \sqrt{2}$, for which, with the same capacitance values, the required resistances are R_2 = 202 Ω and R_5 = 4.949 kΩ, shows a peak gain of 7.598 at a frequency of 1.8078 kHz (11.363 krad/s) in conformity with equations (2.53a) and (2.53b).



(a)

(b)

Figure 2.19 (a) Second-order high pass filter circuit for Example 2.6. (b) Magnitude response of the high pass filter of Figure 2.19(a) with $Q = \sqrt{2}$ and $1/\sqrt{2}$.

2.9 Second-order Band Pass Response

When $a_0 = a_2 = 0$, a biquadratic section become a BP section, whose transfer function is given as follows:

$$H_{BP}(s) = \frac{k(\omega_0 / Q_o)s}{s^2(\omega_o / Q_o)s + \omega_o^2} \tag{2.57}$$

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$$\text{Hence, } \left|H_{\text{BP}}(j\omega)\right| = \frac{k s\omega(\omega_o / Q_o)}{\left[(\omega_o^2 - \omega^2)^2 + (\omega_o / Q_o)^2 \omega^2\right]^{1/2}} \tag{2.58}$$

$$\text{and} \quad \varphi(j\omega) = 90^\circ - \tan^{-1}\frac{(\omega\omega_o / Q_o)}{(\omega_o^2 - \omega^2)} \tag{2.59}$$

Equations (2.58) and (2.59) are the magnitude and phase function of the BP section, which are sketched in Figure 2.20(a) and (b), respectively. Since $H_{\text{BP}}(s)$ has a zero at 0 and ∞, the magnitude function reduces to zero at dc and at infinite frequency; the peak occurs at $\omega = \omega_o$.

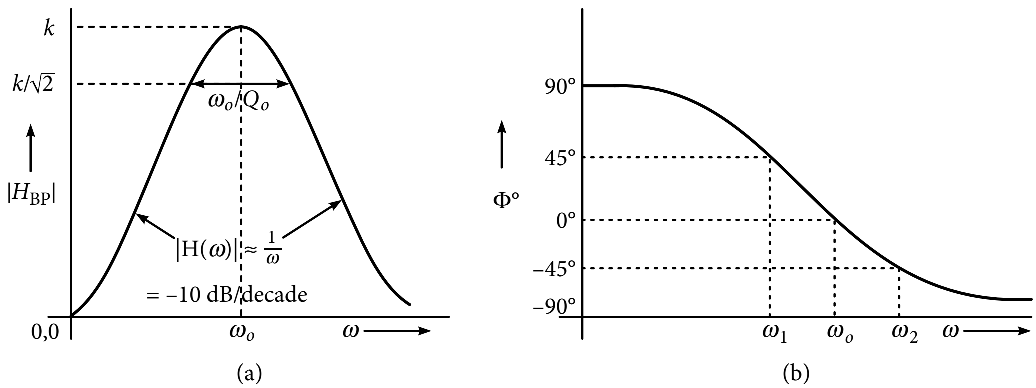

Figure 2.20 (a) Magnitude and (b) phase variation of a second-order band pass section.

The magnitude function drops from the peaks on both sides at a rate of 10 dB/dec with its value becoming 3 dB less than the peak value of k at the half-power frequencies ω_1 and ω_2.

For designing a BP section, important specifications include the bandwidth (BW), the distance between ω_1 and ω_2, or the range of frequencies for which the power output remains more than half of the peak power. The BW, ω_1 and ω_2 are found by putting the square of the magnitude function $|H_{\text{BP}}(j\omega)|^2 = (1/2)$. It gives

$$\omega_1, \omega_2 = \omega_o\left[\left\{1 + \left(1/2Q_o\right)^2\right\}^{1/2} \pm \left(1/2Q_o\right)\right] \tag{2.60}$$

The product and difference of the two frequencies are as follows:

$$\omega_1 \times \omega_2 = \omega_o^2 \text{ and} \left(\omega_2 - \omega_1\right) = \left(\omega_o / Q_o\right) = \text{BW} \tag{2.61}$$

which means that ω_o is the geometric mean of ω_1 and ω_2 and the BW is inversely proportional to the pole $Q(Q_o)$. Figure 2.21 shows the effect of the value of Q_o on the BP response, which becomes thinner/ sharper as Q increases.

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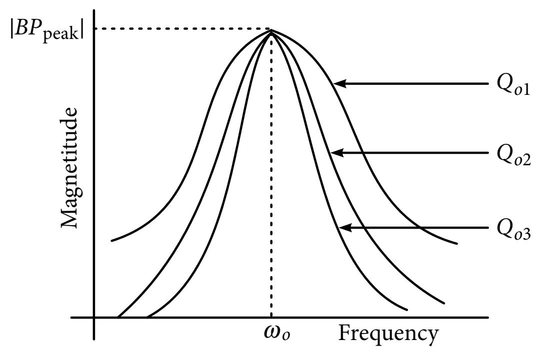

Figure 2.21 Typical response of a band pass filter with varying Q_o ($Q_{o1}< Q_{o2}<Q_{o3}$).

Regarding the phase-function of equation (2.59), it is observed that it is similar to that for the LP case except with the addition of 90° at dc, which means that it asymptotes at –90° for $\omega \to \infty$. Moreover, the values for both the frequencies ω_1 and ω_2 are 45° and –45° from equation (2.59).

Example 2.7: Figure 2.22 shows a single OA based BP filter. Derive its transfer function and compare the response for pole Q value of 2, 5 and 10 at a center frequency of 10 krad/s.

Solution: Considering OA as ideal, with its inverting terminal at virtual ground, the nodal equations at terminal 2 and 3, respectively, are obtained as follows:

$$V_1 (G_1 + G_2 + sC_1 + sC_2) - V_{\text{out}}\, sC_2 - V_{\text{in}}\, G_1 = 0 \tag{2.62}$$

$$V_1 sC_1 + V_{\text{out}}\, G_3 = 0 \tag{2.63}$$

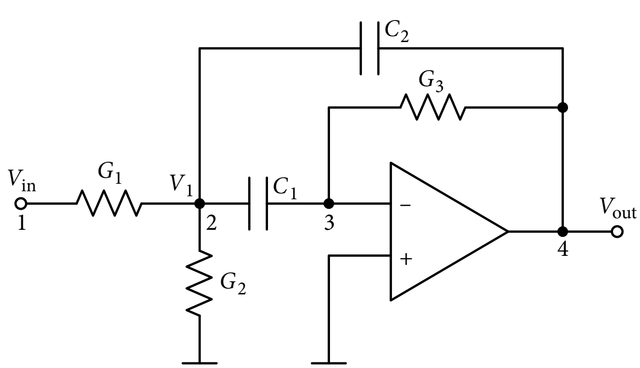

Figure 2.22 A second-order band pass filter section for Example 2.7.

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Combining equations (2.62) and (2.63), the transfer function is obtained as:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{s(G_1 / C_2)}{s^2 + sG_3\{(C_1 + C_2)/C_1C_2\} + \{G_3(G_1 + G_2)/C_1C_2\}} \tag{2.64}$$

It gives the expressions for ω_o and Q_o as

$$\omega_o^2 = \frac{G_3(G_1 + G_2)}{C_1C_2} \text{ and } Q_o = \frac{\left[\left(\frac{C_1C_2}{G_3}\right)(G_1 + G_2)\right]^{1/2}}{(C_1 + C_2)} \tag{2.65}$$

Selecting equal values for capacitors $C_1 = C_2 = 0.005\ \mu F$, for ω_o = 10krad/s, equation (2.65) provides the following element values.

For $Q_o = 2$, $R_1 = R_2 = 10$ kΩ and $R_3 = 80$ kΩ.

For $Q_o = 5$, $R_1 = R_2 = 4$ kΩ and $R_3 = 200$ kΩ.

For $Q_o = 10$, $R_1 = R_2 = 2$ kΩ and $R_3 = 400$ kΩ.

Figures 2.23 and 2.24 show the magnitude and phase response for the aforementioned three cases; the respective center frequencies, bandwidth, and quality factor obtained through PSpice simulation is as follows:

f_o = 1.587 kHz, bandwidth BW = 790 Hz, resulting in Q_o = 2.008.

f_o = 1.578 kHz, bandwidth BW = 313.7 Hz, resulting in Q_o = 5.033.

f_o = 1.567 kHz, bandwidth BW = 162.9 Hz, resulting in Q_o = 10.1.

As the responses and the resulting parameters show, the circuit works very well at this frequency range.

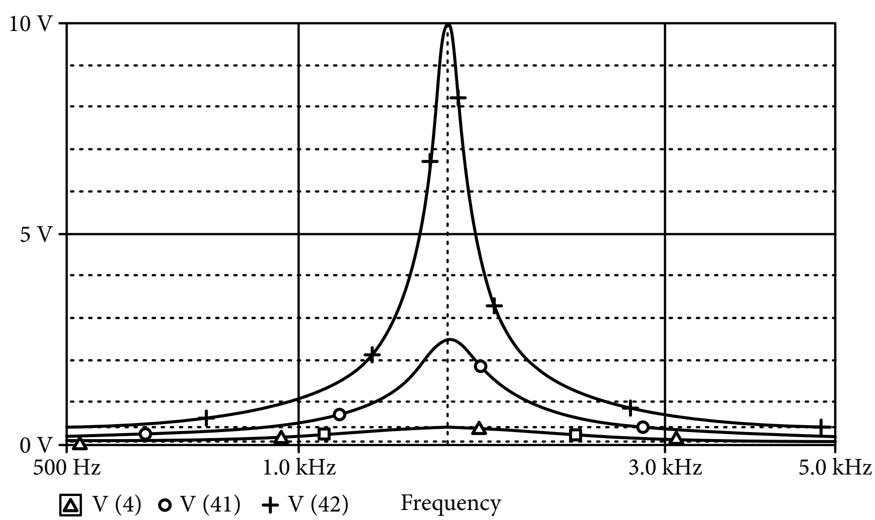

Figure 2.23 Magnitude response of the band pass filter of Figure 2.22 with Q = 2, 5 and 10.

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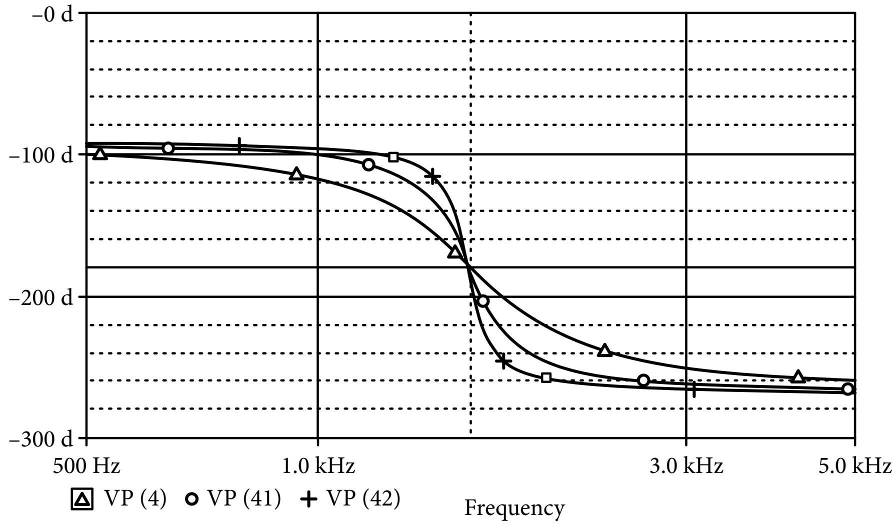

Figure 2.24 Phase response of the band pass filter of Figure 2.22 with $Q = 2$, 5 and 10.

2.10 Band Reject (BR) Response

A BR response, which passes all signals except those falling in certain band of frequencies, is obtained by putting $a_1 = 0$ in the biquadratic function of equation (2.28). It results in the following:

$$H_{BR}(s) = \frac{a_2 s^2 + a_0}{s^2 + (\omega_o / Q_o)s + \omega_o^2} \tag{2.66}$$

$$= \frac{K(s^2 + \omega_z^2)}{s^2 + (\omega_o / Q_o)s + \omega_o^2} \tag{2.67}$$

Here, $K = |H_{BR}(j\omega)|$ is the gain as $\omega \to \infty$ and the rejection band of frequencies is centered at $\omega = \omega_z$ as the numerator has a zero at ω_z. It is the value of Q_o which determines the rate of change of the BR response beyond ω_z, as well as the amount of bump in the response.

The BR filter is also known as *notch filter* because of the shape of the magnitude characteristics. However, depending on the relative value of ω_z in comparison to ω_o, notch filter is called a *symmetrical notch*, *high pass notch* (HPN) or a *low pass notch* (LPN) for $\omega_o = \omega_z$, $\omega_o > \omega_z$ and $\omega_o < \omega_z$, respectively. The three types of notch responses are shown in Figure 2.25(a), (b), and (c). Here the bump in HPN or LPN occurs at $\omega = \omega_{peak}$. Expressions for the frequency ω_{peak} and the maxima of the transfer function for the LPN and HPN, which occurs at ω_{peak} are given as:

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$$\omega_{\text{peak}} = \omega_o \sqrt{\left[1 + \frac{1}{\left|\left\{1-(\omega_o \omega_z)^2\right\}\right| 2Q_o^2}\right]} \tag{2.68}$$

$$|H_{\text{BR}}(j\omega)|_{\max} = KQ_o|\{1 - (\omega_z/\omega_o)^2\}| \tag{2.69}$$

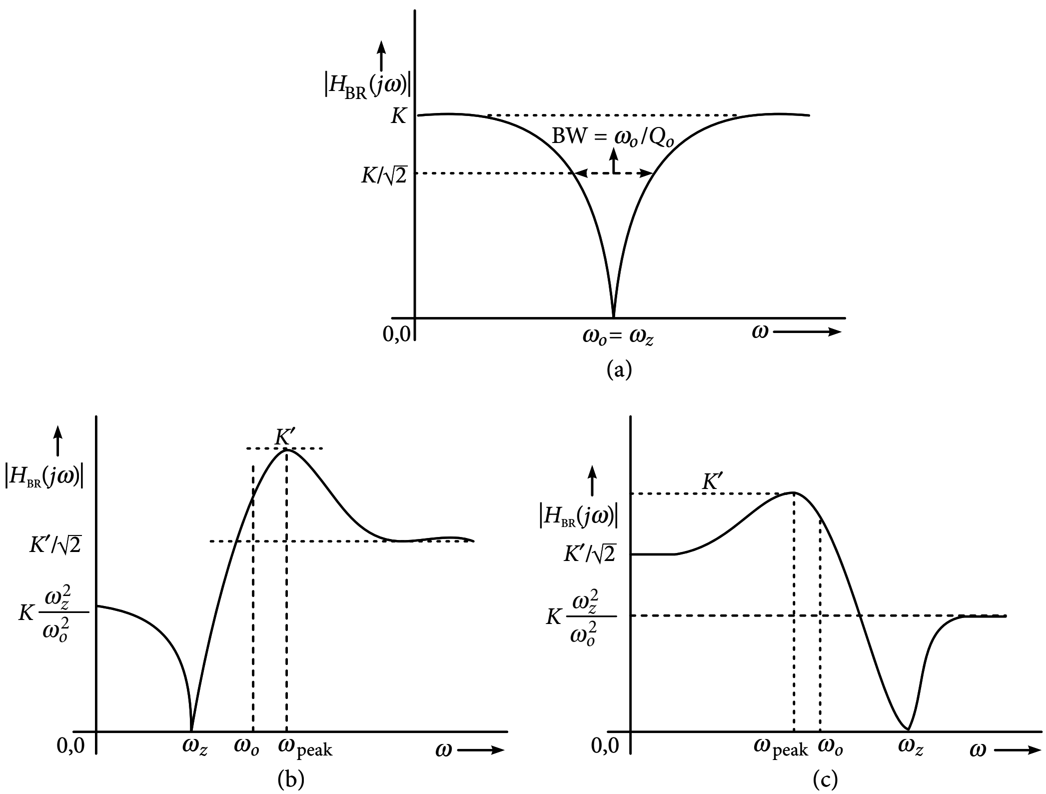

Figure 2.25 (a) Gain response of a symmetrical notch, (b) high pass notch, and (c) low pass notch with $K' = |H_{\text{BR}}(j\omega)|_{\max}$.

Bandwidth of the BR filter is same as that for the BP filter:

$$(\text{BW})_{\text{BR}} = \omega_o/Q_o \tag{2.70}$$

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2.11 Second-order All Pass Response

An all pass (AP) filter has constant magnitude response for all frequencies. For this type of filter to be realized, coefficients of the biquadratic section are selected in such a way that the transfer function becomes:

$$H_{\text{AP}}(s) = K\frac{s^2 - (\omega_o/Q_o)s + \omega_o^2}{s^2 + (\omega_o/Q_o)s + \omega_o^2} \tag{2.71}$$

Hence, for sinusoidal input

$$H_{\text{AP}}(j\omega) = K\frac{(\omega_o^2 - \omega^2) - j\omega(\omega_o/Q_o)}{(\omega_o^2 - \omega^2) + j\omega(\omega_o/Q_o)} \tag{2.72}$$

Here, $H_{\text{AP}}(j\omega)$ has to remain constant for all frequencies and the phase and delay of the AP filter are obtained as follows

$$\varphi_{\text{AP}}(\omega_o) = -2\tan^{-1}\frac{\omega(\omega_o/Q_o)}{(\omega_o^2 - \omega^2)} \tag{2.73}$$

$$D_{\text{AP}}(\omega_o) = 2\left(\frac{\omega_o}{Q_o}\right)\frac{(\omega_o^2 + \omega^2)}{\left\{(\omega_o^2 - \omega^2)^2 + \omega^2\left(\frac{\omega_o}{Q_o}\right)^2\right\}} \tag{2.74}$$

Figure 2.26 shows the variation of phase of the AP filter for a certain value of Q_o along with its magnitude response and Figure 2.27 shows the variation of one-half delay for a few values of Q_o. It is observed that for $Q_o = 1/\sqrt{3}$, the delay become *maximally flat*, whereas for $Q_o > 1/\sqrt{3}$, the delay variations have a peak.

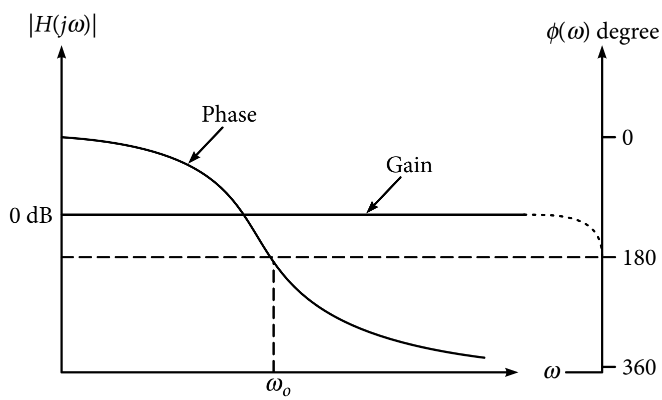

Figure 2.26 Variation of phase and gain response of a second-order all pass filter for a certain value of Q_o.

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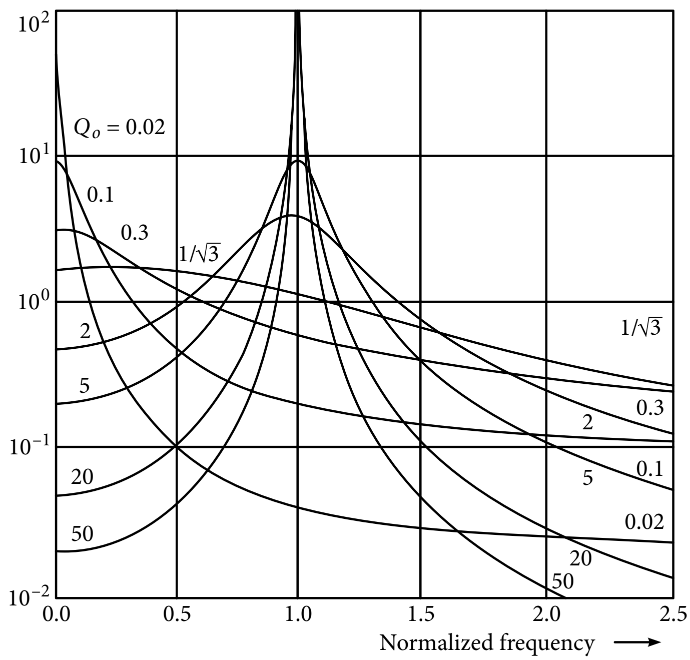

Figure 2.27 One-half delay of second-order all pass filter as a function of Q_o.

In the second-order BR as well as AP filters, finite zeroes are to be realized for which realization methods are a bit different than LP, HP, and BP types. It is for this reason that simulation examples for notch and AP filters shall be taken up at a later stage.

2.12 Effect of Operational Amplifier's Pole on Biquads

Finite frequency dependent gain of the OA, represented by the single-pole roll-off model of equation (1.17) introduces one extra pole for the first-order filter section. In fact, it introduces as many extra poles as the order of the filter section. It affects the filter characteristics by changing all its important parameters like gain, cut-off/ pole frequency and rate of fall of the signal in the stop band; each with varying degree. Amount of variation in the parameter depends on the filter specifications (values of the required gain, pole frequency and pole Q), finite value of the gain–bandwidth product of the OA, and on the method (structure) used for the realization of the filter, like generating biquads using the two-integrator loop method, or any single amplifier generating biquad method, and then cascading them using these biquads in a multiple feedback structure or using direct forms of realizations for higher-order filters. Hence, the effect of OA's poles shall be taken up in later chapters along with different methods of filter realization and the corrective steps applied to overcome the deviations occurring in the filter parameters.

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[2.4] Jenson, E. G. 2000. 'A Filter Circuit Board for the Earthworm Seismic Data Acquisition System.' Open-file Report 00-379. California: US Geological Survey.

Practice Problems

2-1 Find the transfer function for the circuit shown in Figure P2-1. Calculate and verify the frequency at which its gain changes by 3 dBs from dc level using PSpice, with $R_1 = R_2 = 10$ kΩ and $C = 1$ nF. Consider the OA as near ideal.

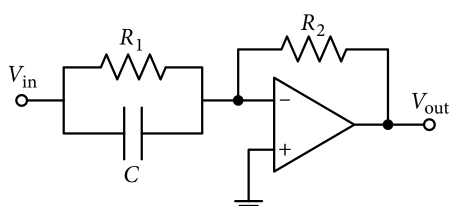

Figure P2.1 Figure for Problem 2-1 and 2-3.

2-2 Find the transfer function for the circuit shown in Figure P2-2 Calculate 3 dB frequency and test the circuit using PSpice with $R_1 = R_2 = 10$ kΩ and $C = 1$ nF. Consider the OA as near ideal.

2-3 Repeat the problem 2-1 with the bandwidth of the OA as (a) 100 krad/s, (b) 50 krad/s, and (c) 25 krad/s. Find percentage error in the gain at the 3-dB frequency level for the three cases and compare with the case when OA was considered near ideal in the problem 2-1.

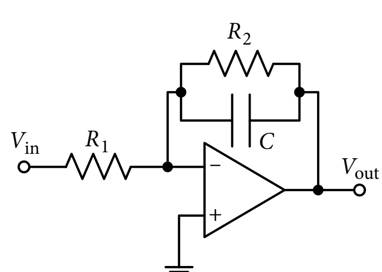

Figure P2.2 Figure for Problem 2-2 and 2-4.

2-4 Repeat the problem 2-2 with the bandwidth of the OA as (a) 100 krad/s, (b) 50 krad/s, and (c) 25 krad/s. Compare the gain at the 3-dB frequency for the three cases and find percentage error in it with the case when OA was considered near ideal in the problem 2-2.

2-5 Find the transfer function for the circuit shown in Figure P2-3 Calculate the peak magnitude and find the frequency at which it occurs using PSpice with $R_1 = R_2 = 10$ kΩ and $C_1 = C_2 = 2$ nF. Consider the OA as near ideal.

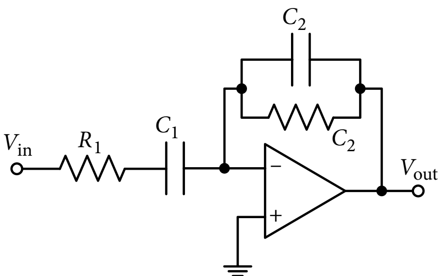

Figure P2.3 Figure for Problem 2-5 and 2-6.

2-6 Repeat the problem 2-5 with the bandwidth of the OA as (a) 100 krad/s, (b) 50 krad/s, and (c) 25 krad/s. Compare the frequency at which peak gain occurs and obtain percentage error in the result for the three cases with the case when OA was considered near ideal in the problem 2-5.

2-7 Find the transfer function for the circuit shown in Figure P2-4. Test the circuit using PSpice with $R_1 = R_2 = 10$ kΩ and $C = 0.5$ nF and find the frequency at which gain drops by 3 dBs. Consider the OA as near ideal.

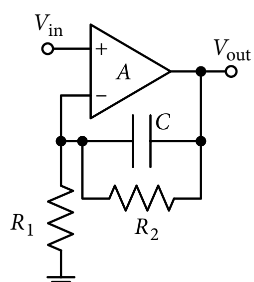

Figure P2.4 Figure for Problem 2-6 and 2-7.

2-8 Repeat the problem 2-7 with the bandwidth of the OA as (a) 100 krad/s, (b) 50 krad/s and (c) 25 krad/s. Compare the result for the three cases with the case when OA was considered ideal in the problem 2-7 while finding error in the frequency at which gain falls by 3 dBs.

2-9 Figure P2.5 shows a second-order passive RLC filter. (a) Derive its transfer function and mention the type of response given by the filter section. (b) Find the values of poles and zeroes when $R = 500$ **Ω**, $L = 10$ mH and $C = 0.04$ μF. (c) Calculate the parameters ω_o, Q_o and dc gain. (d) If the magnitude response has a peak, then what is the value of the voltage gain and at which frequency does it occur?

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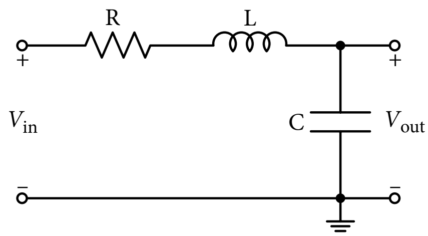

Figure P2.5 Figure for Problem 2-9.

2-10 For the circuit of Figure P2.5, calculate the value of Q_o in each case if R changes from 500 Ω to 250 Ω, 100 Ω and 50 Ω. Find the location of poles on the complex frequency variable plane and show that the poles lie on a semi-circle. What is the radius of the semi-circle?

2-11 Check whether peak in the magnitude response of the circuit in Problem 2-9 occurs at ω_o or not. Justify the location of the peak.

2-12 Repeat Problem 2-9 if location of inductor and the capacitor are interchanged.

2-13 Use the circuit of Figure 2.15 to design a second-order LP filter with the following specifications: cut-off frequency f_o= 15.9 kHz, Q = 2.5 and dc gain of zero dB. (b) Test the magnitude and phase with PSpice/EWB while using 741 type OA.

2-14 Calculate magnitude and phase of the LP filter with following specifications: f_o = 1.59 kHz and Q = 2.5. for frequencies $0.25 \times f_o$, $0.5 \times f_o$, f_o, $1.5 \times f_o$, $2 \times f_o$ and compare it with the simulated response.

2-15 Verify equations (2.45) and (2.46) for the LP filter of Problem 2-13 by comparing the parameters by obtaining theoretically and from the simulated response.

2-16 Derive the voltage ratio transfer function for the circuit shown in Figure P2.6. What kind of response is available from it?

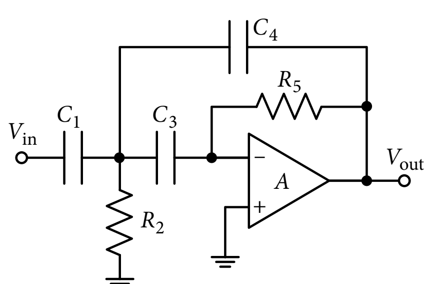

Figure P2.6 Figure for Problem 2-16.

2-17 Design the circuit of Figure P2.6 for critical frequency of 7.95 kHz and Q_o = 2.5 and test the magnitude and phase response.

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2-18 Verify equations (2.55a) and (2.55b) for the filter section in Problem 2-17 theoretically and from the practical/simulated test results.

2-19 Repeat Problem 2-14 for the filter of Figure P2.6.

2-20 (a) Determine suitable element values for the realization of the BP filter shown in Figure 2.22 for realizing pole $Q_o = 5$ and a complex pole pair lying on a circle of radius = 50 krad/s.

(b) Determine the peak gain.

(c) Determine the spread in element values.

(d) Determine error in complex pole radius and Q_o when OA has $B = 500$ krad/s.

2-21 Redesign the circuit in Figure 2.22 for $\omega_o = 40$ krad/s and $Q_o = 10$.

(a) Calculate and verify the simulated value of the filter bandwidth while using ideal OA.

(b) Repeat (a) for OA with $B = 400$ krad/s.

(c) Calculate and verify the phase shift of the filter at 3 dB frequencies.

2-22 Derive equations (2.73) and (2.74).

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