3

Alternating current circuits

3.1 Introduction

The class of circuits described as 'alternating current circuits' (abbreviated to a.c. circuits) comprises networks of linear lumped elements that may include capacitance and inductance as well as resistance. It has become a common and convenient practice to use the abbreviation, 'a.c.' as a qualifying adjective. Thus, we speak of an 'a.c. voltage', an 'a.c. current', and so on. In such circuits the sources of excitation produce time-varying voltages and currents described by sinusoidal functions of the form:

$$v = V_{\mathrm{m}}\sin\omega t \qquad \text{or} \qquad i = I_{\mathrm{m}}\sin\omega t \tag{3.1}$$

We may regard the above expressions as functions of time t or functions of angle ωt, the latter often being the more convenient. Waveforms corresponding to (3.1) are shown in fig. 3.1, as functions of both time and angle, and the various relevant parameters are defined.

An a.c. circuit is, by definition, one in which steady-state conditions obtain; that is, any transient conditions arising in the circuit at the time of switching will have died away leaving the circuit in an equilibrium state in which the amplitudes of all currents and voltages are constant. The time origin in the above equations is therefore of no consequence so that the alternating voltages and currents in an a.c. circuit can be described equally by the cosine functions:

$$V = V_{\mathrm{m}}\cos\omega t \qquad \text{or} \qquad i = I_{\mathrm{m}}\cos\omega t \tag{3.2}$$

For this reason the term *cisoid* is sometimes used to describe in a general way the waveforms encountered in a.c. circuits.

Although the choice of time origin is arbitrary the relative time (or angular) displacement between waveforms, which we call *phase*, is of vital importance in describing the electrical behaviour of an a.c. circuit. In fig. 3.2 we illustrate voltage and current waveforms displaced by phase angle ϕ. Either of the two waveforms may be regarded as the reference with respect to which the phase of the other is measured. If we select the voltage waveform as reference, then, in terms of the sine function the two waveforms are described by

$$v = V_m \sin\omega t \text{ (phase reference)}$$

and

$$i = I_m \sin(\omega t - \phi)$$

In this case we say that the current waveform *lags* the voltage waveform by angle ϕ since, as may be seen from fig. 3.2, the current waveform passes through zero in a positive going direction (point B) at an instant of time

Fig. 3.1. Sinusoidal voltage and current waveforms.

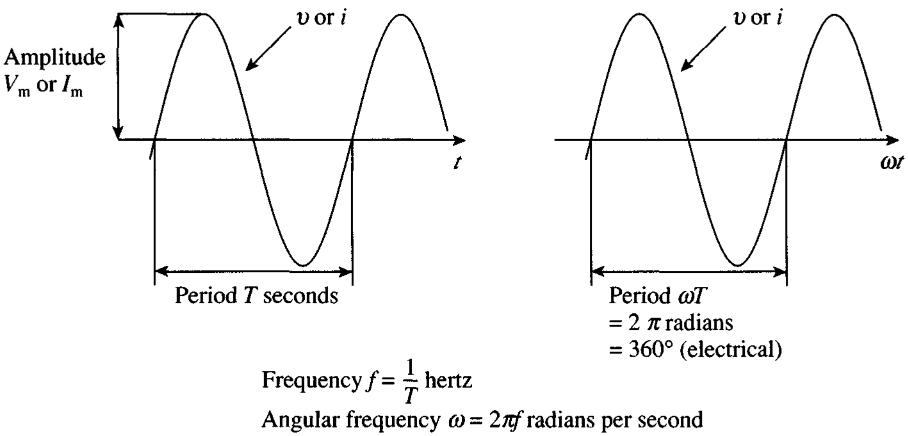
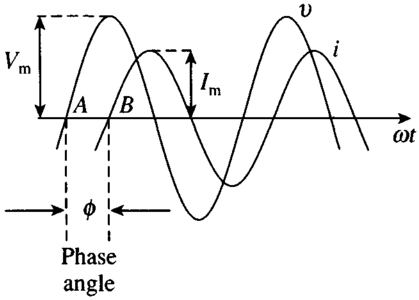


Fig. 3.2. Phase displacement between two sinusoidal waveforms.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

later than the corresponding point (point A) of the voltage waveform. If, on the other hand, we choose to take the current waveform as the reference, then the two waveforms are described by the functions:

$$i = I_m \sin\omega t \text{ (phase reference)}$$

and

$$v = V_m \sin(\omega t + \phi)$$

and we say that the voltage waveform *leads* the current waveform by angle ϕ. In all of the above we could have used the cosine function instead of the sine function; providing either one or the other is used consistently the method of describing phase is the same.

The following points concerning the meaning and use of phase should be noted: (1) Its definition in any circuit depends upon the choice of the phase reference waveform. (2) It is independent of waveform amplitude. (3) It has meaning only when referred to waveforms of the same frequency. (4) A negative sign attached to the phase angle signifies that the associated waveform is lagging, conversely, a positive sign signifies a leading waveform. (5) A waveform leading by a phase angle β, say, (measured in degrees) may also be described as lagging by an angle $\gamma = 360 - \beta$. It is conventional practice to choose whichever of the two possible angles is numerically less than 180°.

An important part of a.c. circuit analysis is concerned with the calculation of power. In d.c. circuits the calculation of power in resistance is effected using the expressions I^2R or V^2/R (equations (1.20) and (1.21)); in a.c. circuits these same expressions can be conveniently used by working in terms of *effective* values of current and voltage rather than with their amplitudes. The relationship between the effective value of an alternating current and its amplitude may be derived by considering the amplitude of the alternating current required to produce the same mean energy dissipation in a resistance R as that produced by a d.c. current of I amperes.

The instantaneous power in a resistance R carrying a current i is, from (1.20), i^2R; therefore, the energy dissipation over one complete period T is

$$\int_0^T i^2 R \, dt$$

If I is the effective value of the current waveform, then over a period T the energy dissipation must be I^2RT hence,

$$I^2RT = \int_0^T i^2 R \, dt$$

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

or

$$\text{Effective value } I = \left[\frac{1}{T}\int_0^T i^2\,\mathrm{d}t\right]^{\frac{1}{2}} \tag{3.3}$$

This expression allows the effective value of any periodic current waveform to be evaluated. Because of the mathematical form of (3.3) the effective value is also known as the *root mean square* (r.m.s.) value or magnitude.

For the particular case of a sinusoidal waveform we have

$$I = \left[\frac{1}{T}\int_0^T (I_{\mathrm{m}}\sin\omega t)^2\,\mathrm{d}t\right]^{\frac{1}{2}} = \frac{I_{\mathrm{m}}}{\surd 2} \tag{3.4}$$

In a similar way it can be shown that the effective value of a sinusoidal alternating voltage is $V_{\mathrm{m}}/\surd 2$.

Finally, we mention a parameter which is useful for purposes of comparison between various types of periodic waveform; this is the *form factor* defined as the ratio of the r.m.s. value to the half-cycle average value. (The average value of a sine-wave over a complete cycle is, of course, zero.)

It can be shown that the half-cycle average of a sine-wave is equal to $2V_{\mathrm{m}}/\pi$, hence the form factor is $(V_{\mathrm{m}}/\surd 2)/(2V_{\mathrm{m}}/\pi) = 1.11$. This may be compared with a form factor of unity for a square-wave and 1.155 for a triangular wave. The form factor indicates the extent to which a wave form exhibits a 'peaked' characteristic. (See ref. 11, for a more complete discussion.)

3.2 A.C. voltage–current relationships for the linear circuit elements

We now consider the form of the voltage developed across each of the three circuit elements when a sinusoidal current passes through them (fig. 3.3). In the following analysis we choose to describe the current by the cosine function because it is mathematically slightly more convenient. Essentially the same results would be obtained using the sine function.

(a) *Resistance.* The instantaneous voltage across the resistance R is

$$\begin{aligned} v_{\mathrm{R}} &= Ri \\ &= RI_{\mathrm{m}}\cos\omega t \end{aligned}$$

or

$$v_R = V_{R\mathrm{m}}\cos\omega t \tag{3.5}$$

where

$$V_{R\mathrm{m}} = RI_{\mathrm{m}} \tag{3.6}$$

Converting to r.m.s. magnitudes by dividing both sides of (3.6) by $\sqrt{2}$ we obtain

$$V_R = RI \tag{3.7}$$

Thus, an Ohm's law type of relationship exists between alternating current and voltage magnitudes for a resistive element. From (3.5) we see that voltage and current are in phase.

(b) *Inductance.* Using (1.40) the voltage across the inductance is given by

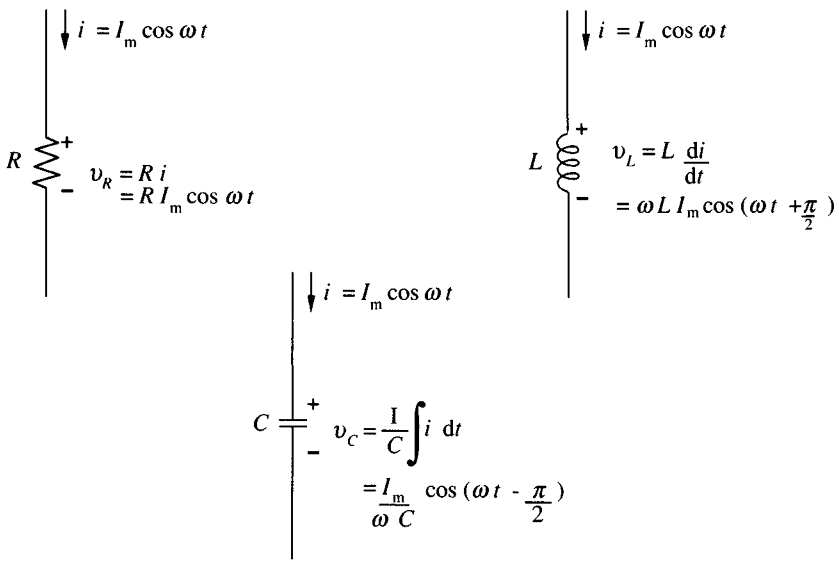
$$\begin{aligned} v_L &= L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}}{\mathrm{d}t}(I_\mathrm{m}\cos\omega t) \\ &= -\omega L I_\mathrm{m}\sin\omega t \\ &= \omega L I_\mathrm{m}\cos\left(\omega t + \frac{\pi}{2}\right) \end{aligned}$$

or

$$v_L = V_{L\mathrm{m}}\cos\left(\omega t + \frac{\pi}{2}\right) \tag{3.8}$$

where $V_{L\mathrm{m}} = \omega L I_\mathrm{m}$.

Fig. 3.3. A.C. voltage–current relationships, in terms of instantaneous values, for the basic circuit elements.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

Converting to r.m.s. magnitudes we obtain

$$V_L = \omega L I \tag{3.9}$$

The quantity ωL is known as the *inductive reactance*, and since it is the ratio of a voltage to a current, it has dimensions of ohms. It is usually denoted by the symbol X_L hence (3.9) may be written

$$V_L = X_L I \tag{3.10}$$

Equation (3.8) shows that for an inductive element the voltage waveform *leads* the current waveform by phase angle $\pi/2$ radians.

(c) *Capacitance.* Using (1.31) the voltage across the capacitance is given by

$$v_C = \frac{1}{C}\int_0^t i\,\mathrm{d}t + v_0 = \frac{1}{C}\int_0^t I_\mathrm{m}\cos\omega t\,\mathrm{d}t + v_0$$

Since we are dealing with circuits in which voltages are purely sinusoidal, there is no initial voltage across the capacitance so that we may put $v_0 = 0$. On integrating we obtain:

$$v_C = \frac{I_\mathrm{m}}{\omega C}\sin\omega t$$

$$= \frac{I_\mathrm{m}}{\omega C}\cos\left(\omega t - \frac{\pi}{2}\right)$$

or

$$v_C = V_{C\mathrm{m}}\cos\left(\omega t - \frac{\pi}{2}\right) \tag{3.11}$$

where $V_{C\mathrm{m}} = \dfrac{I_\mathrm{m}}{\omega C}$

In terms of r.m.s. magnitudes we have

$$V_C = \frac{I}{\omega C} = X_C I \tag{3.12}$$

The quantity $X_C = 1/\omega C$ is called the *capacitive reactance*, measured in ohms, and we see from (3.11) that in this case the voltage *lags* the current by $\pi/2$ radians.

We conclude that, for each of the three elements, voltage is proportional to current in terms of either amplitudes or r.m.s. magnitudes. For the inductive and capacitive elements both voltage and current are sinusoidal but suffer a phase displacement of $\pi/2$ radians or 90 electrical degrees. We say that, for these elements, voltage and current are 'in quadrature'.

It is evident from the above theory that in a.c. circuit analysis the application of Kirchhoff's laws requires the addition of voltages or currents that will differ in phase if the circuit contains two or more elements of different kinds. Because of this we cannot combine the magnitudes of the voltages given by (3.7), (3.10) and (3.12) using direct algebraic addition; however, we could proceed by adding the instantaneous values given by the trigonometric functions (3.5), (3.8) and (3.11). For instance, if the total voltage across a series combination of just two elements, say resistance and inductance, were required, we could proceed as follows:

$$
\begin{aligned}
\text{Total voltage } v &= v_R + v_L \\
&= V_{Rm}\cos\omega t + V_{Lm}\cos\left(\omega t + \frac{\pi}{2}\right) \\
&= RI_m\cos\omega t + X_L I_m\cos\left(\omega t + \frac{\pi}{2}\right) \\
&= RI_m\cos\omega t + X_L I_m\cos\omega t\cos\frac{\pi}{2} - X_L I_m\sin\omega t\sin\frac{\pi}{2} \\
&= RI_m\cos\omega t - X_L I_m\sin\omega t \\
v &= \sqrt{(R^2 + X_L^2)}I_m\cos(\omega t + \alpha)
\end{aligned}
$$

where

$$\alpha = \tan^{-1}\frac{X_L}{R} \quad \left(\text{also written as } \alpha = \arctan\frac{X_L}{R}\right)$$

The result is a voltage of amplitude $\sqrt{(R^2 + X_L^2)}I_m$, and phase angle α with respect to the original current flowing through the two elements. The quantity $\sqrt{(R^2 + X_L^2)}$ has dimensions of ohms and is called the *impedance* of the series-connected elements. In principle the total voltage across any number of series-connected elements could be derived by repeated application of the above trigonometrical procedure taking voltages two at a time. We should then find that the general result was of the form:

$$\begin{matrix}\text{Instantaneous} \\ \text{voltage}\end{matrix} = [\text{Impedance}]\begin{bmatrix}\text{Amplitude} \\ \text{of} \\ \text{current}\end{bmatrix}\begin{bmatrix}\text{Phase displaced} \\ \text{sine or cosine} \\ \text{function}\end{bmatrix} \qquad (3.13)$$

Both the impedance and the phase displacement are functions of the resistances and reactances in the circuit under consideration, and their determination by the above trigonometrical procedure for each particular circuit would be tedious in the extreme. Alternative approaches are therefore adopted based either (a) on a geometrical and graphical interpretation of the trigonometrical equations presented above or (b) on the use of the complex exponential and complex algebra. The latter is the most convenient and flexible approach, and is presented in the following sections.

3.3 Representation of a.c. voltage and current by the complex exponential: Phasors

From the discussion and results contained in the previous sections it will be clear that in a.c. circuit analysis it is only the magnitudes and relative phases of voltages and currents that are of interest. The use of the complex exponential to represent a.c. voltages and currents allows the analysis of circuits to be effected in terms of magnitude and phase only; it provides also a direct and simple means for depicting graphically the relationships among a.c. quantities. We assume in the following that the reader is familiar with the meaning of complex number and with the elements of complex algebra.

The basis of the method is provided by the Euler relation:

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta \tag{3.14}$$

where $j = \sqrt{-1}$ and A and θ are respectively the modulus (or amplitude) and argument (or angle) of the complex exponential.

The relation (3.14) defines a complex number $\boldsymbol{A}$* whose real and imaginary parts are respectively $A\cos\theta$ and $A\sin\theta$, thus,

$$\boldsymbol{A} = a + jb \tag{3.15}$$

where

$$a = A\cos\theta, \; b = A\sin\theta$$

Therefore, we may write

$$a^2 + b^2 = A^2\cos^2\theta + A^2 sin^2\theta$$

that is

$$A = \sqrt{(a^2 + b^2)}$$

and

$$\frac{b}{a} = \frac{\sin\theta}{\cos\theta}$$

* Complex quantities will be signified in this text by the use of bold italic type.

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

that is

$$\theta = \tan^{-1}\frac{b}{a}$$

We may depict the relationship (3.14) graphically by means of the Argand diagram (fig. 3.4) in which the complex exponential defines a point P in the plane with polar coordinates (A, θ). The right-hand side of (3.14) defines the same point in terms of the Cartesian coordinates $(A\cos\theta,\ A\sin\theta)$.

From a slightly different point of view we may regard the complex exponential $e^{j\theta}$ as an operator. With this interpretation multiplication of a real scalar quantity A by $e^{j\theta}$ simply causes A to rotate in the Argand diagram by an amount θ without change of amplitude. An important special case occurs when $\theta = \frac{\pi}{2}$ radians (or 90°); then (3.14) becomes:

$$Ae^{j\pi/2} = A\cos\frac{\pi}{2} + jA\sin\frac{\pi}{2}$$

that is,

$$Ae^{j\pi/2} = A(0) + jA(1)$$

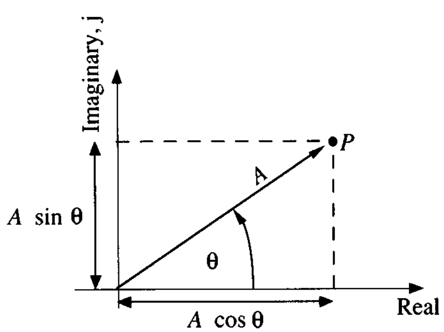
or

$$e^{j\pi/2} = j$$

Similarly,

$$e^{j(-\frac{\pi}{2})} = -j$$

Fig. 3.4. Representation of the complex exponential $Ae^{j\theta}$ on the Argand diagram.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

Thus multiplication of a real quantity by $+j$ causes a rotation of 90° in a positive (counter-clockwise) sense, while multiplication by $-j$ causes a rotation of 90° in a negative sense. It also follows that multiplication by j^2 is equivalent to a rotation of 180°. This interpretation of the complex exponential must be borne in mind throughout the following theory.

A saving of labour, particularly in numerical work, is achieved by writing the complex exponential as

$$Ae^{j\theta} \equiv A\underline{/\theta} \tag{3.16}$$

Now consider the argument θ in (3.14) to be a function of time such that $\theta = (\omega t + \phi)$, and let A be identically equal to the amplitude V_m of an alternating voltage wave form; (3.14) then becomes

$$V_m e^{j(\omega t + \phi)} = V_m \cos(\omega t + \phi) + jV_m \sin(\omega t + \phi) \tag{3.17}$$

We see that the complex exponential on the left-hand side of this expression can be used to represent mathematically either the co-sinusoidal or sinusoidal forms of the alternating voltage; the particular form being expressed by specifying either the real part (Re) or imaginary part (Im) as required, viz.

$$v = V_m \cos(\omega t + \phi) = \mathrm{Re}\, V_m e^{j(\omega t + \phi)} \tag{3.18}$$

or

$$v = V_m \sin(\omega t + \phi) = \mathrm{Im}\, V_m e^{j(\omega t + \phi)} \tag{3.19}$$

The interpretation of (3.17) on the Argand diagram is shown in fig. 3.5(*a*). The line OP, representing the amplitude V_m of the alternating voltage, rotates with angular speed ω and we refer to this line as a *rotating phasor*. The projection of the point P on to the real and imaginary axes defines time-varying coordinates proportional respectively to $V_m \cos(\omega t + \phi)$ and $V_m \sin(\omega t + \phi)$.

If the complex exponential is not specified by (3.18) or (3.19), then it is understood that either form is applicable and we can describe in general terms the instantaneous value of any alternating voltage by

$$v = V_m e^{j(\omega t + \phi)} \tag{3.20}$$

This expression may be rewritten as

$$v = V_m e^{j\phi} e^{j\omega t} = \mathbf{V}_m e^{j\omega t} \tag{3.21}$$

where

$$\mathbf{V}_m = V_m e^{j\phi} \tag{3.22}$$

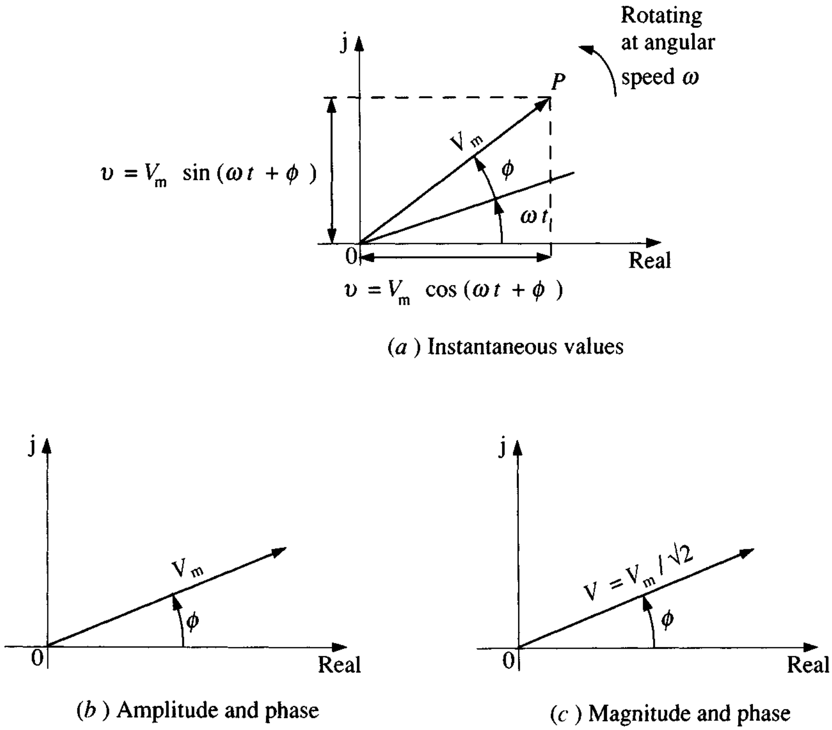
The expression (3.22) defines a *complex voltage* that is *independent* of time and which contains only the amplitude and phase information concerning the alternating voltage. We interpret the complex voltage on the Argand diagram as shown in fig. 3.5(*b*). Since the complex voltage is time-invariant the line OP does not now rotate and it is consequently termed a *stationary phasor*. The complex voltage function defined by (3.22) is also referred to as a stationary phasor or, more simply, as a phasor.

In practical circuit analysis it is, for the reasons given in section (3.1), better to work in terms of r.m.s. values in which case the complex voltage is written

$$\boldsymbol{V} = V e^{j\phi} \equiv V\underline{/\phi} \tag{3.23}$$

where $V = V_m/\sqrt{2}$, and the notation of (3.16) has been used. The Argand diagram is modified accordingly as shown in fig. 3.5(*c*).

Fig. 3.5. Representation of an alternating voltage in complex exponential form by means of the Argand diagram. (*a*) Rotating phasors. (*b*) and (*c*) Stationary phasors.



The above treatment is, of course, applicable to the representation of currents and we may write

$$\boldsymbol{I} = I\mathrm{e}^{\mathrm{j}\beta} \equiv I\underline{/\beta} \tag{3.24}$$

3.4 Voltage–current relationships for the general network branch: Impedance

The results of section (3.3) will now be used to derive the voltage–current relationship for the series-connected elements shown in fig. 3.6. This arrangement is called the *general network branch* because it is completely representative of any series-connected combination of lumped passive elements. Once the voltage–current relationship is determined for this circuit then it becomes possible to solve, at least in principle, any a.c. linear lumped network.

To make the following treatment completely general we assume that the current i passing through the branch has amplitude I_{m} and possesses a phase angle ϕ measured with respect to some other voltage or current elsewhere in the circuit of which the branch shown in fig. 3.6 forms part. We wish to determine the amplitude and phase angle of the total branch voltage v.

The branch voltage v is, by Kirchhoff's voltage law,

$$v = v_R + v_L + v_C$$

which may be written, using the instantaneous voltage–current relationships for the individual circuit elements derived in chapter 1 (equations (1.16), (1.31) and (1.40)):

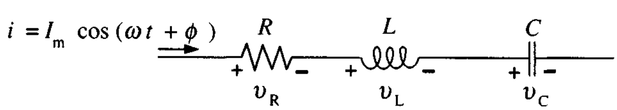
$$v = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i\,\mathrm{d}t \tag{3.25}$$

Now the instantaneous current may be represented in complex exponential form (see equation (3.18)) by

$$\begin{aligned} i &= I_{\mathrm{m}}\cos(\omega t + \phi) = \mathrm{Re}\, I_{\mathrm{m}}\mathrm{e}^{\mathrm{j}(\omega t + \phi)} \\ &= \mathrm{Re}\, \boldsymbol{I}_{\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t} \end{aligned}$$

therefore (3.25) becomes (taking the real part as understood)

Fig. 3.6. The general network branch.



$$v = RI_m e^{j\omega t} + LI_m \frac{d}{dt} e^{j\omega t} + \frac{I_m}{C} \int e^{j\omega t}$$

$$= RI_m e^{j\omega t} + j\omega LI_m e^{j\omega t} + \frac{I_m}{j\omega C} e^{j\omega t} + (\text{const.})$$

The constant of integration is zero since the voltages and currents in an a.c. circuit are purely sinusoidal. (A finite value for this constant would imply that a direct voltage existed across the terminals of the capacitor.) We may therefore write

$$v = \left(R + j\omega L + \frac{1}{j\omega C} \right) \mathbf{I}_m e^{j\omega t} \tag{3.26}$$

The quantity in brackets in the above equation is called the *complex impedance* and is denoted by the symbol $\boldsymbol{Z}$, thus

$$\text{Complex impedance } \boldsymbol{Z} = R + j\omega L + \frac{1}{j\omega C} \tag{3.27}$$

$$= R + j\left(\omega L - \frac{1}{\omega C} \right)$$

or

$$\boldsymbol{Z} = R + jX \tag{3.28}$$

where

$$X = \left(\omega L - \frac{1}{\omega C} \right) \tag{3.29}$$

The quantity X, called the *reactance* of the branch, is the difference between the inductive and capacitive reactances.

Since $\boldsymbol{Z}$ is a complex number it may be converted from the Cartesian form (3.28) to polar form:

$$\boldsymbol{Z} = R + jX = Ze^{j\theta} = Z \underline{/\theta} \tag{3.30}$$

where

$$Z = \sqrt{(R^2 + X^2)} \tag{3.31}$$

and

$$\theta = \tan^{-1} \frac{X}{R} \tag{3.32}$$

The modulus Z of the complex impedance, measured in units of ohms, is

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

often referred to simply as the impedance. ($\boldsymbol{Z}$, the complex impedance, is also often called the impedance and we have to understand from the context of the theory or argument in question which is meant.) The argument θ in (3.30) is called the *angle* of the complex impedance.

Substituting (3.27) into (3.26), the expression for the instantaneous branch voltage becomes

$$v = \boldsymbol{Z}\boldsymbol{I}_m e^{j\omega t} \tag{3.33}$$

Now, using (3.30) and recalling that $\boldsymbol{I}_m = I_m e^{j\phi}$, this may be written

$$v = Ze^{j\theta} I_m e^{j\phi} e^{j\omega t}$$

or

$$v = ZI_m e^{j(\omega t + \phi + \theta)} \tag{3.34}$$

In all of the above expressions for v, the real part has been understood, therefore, from (3.18) the branch voltage is

$$v = ZI_m \cos(\omega t + \phi + \theta) \tag{3.35}$$

We see from this expression that the amplitude V_m of the branch voltage is given by

$$V_m = ZI_m \tag{3.36}$$

and the phase angle with respect to the current is θ, the angle of the complex impedance.

We have thus established, with the aid of complex exponential theory, the required voltage–current relationship in terms of instantaneous values. The reader should now compare (3.35) with (3.13). (The same result could have been obtained, rather more laboriously, by using the trigonometrical methods and results of section 3.2.) Of rather greater practical significance, however, is the voltage–current relationship for the general branch in terms of complex voltages and currents; this is derived as follows. In terms of the complex exponential the instantaneous branch voltage may be written

$$v = V_m e^{j(\omega t + \phi + \theta)} = \boldsymbol{V}_m e^{j\omega t} \tag{3.37}$$

where

$$\boldsymbol{V}_m = V_m e^{j(\phi + \theta)} \text{ is the complex voltage.}$$

Combining (3.37) and (3.33) gives

$$\boldsymbol{V}_m e^{j\omega t} = \boldsymbol{Z}\boldsymbol{I}_m e^{j\omega t}$$

or

$$V_m = ZI_m \tag{3.38}$$

Note that eliminating $e^{j\omega t}$ from both sides of the above equation has the effect of converting from a rotating phasor system to a stationary phasor system.

In terms of r.m.s. magnitudes (3.38) becomes

$$V = ZI \tag{3.39}$$

In words:

Complex voltage = complex impedance × complex current

It will be recognized that (3.39) is of a similar mathematical form to the Ohm's law encountered in d.c. circuit theory, consequently it is often referred to as Ohm's law for a.c.

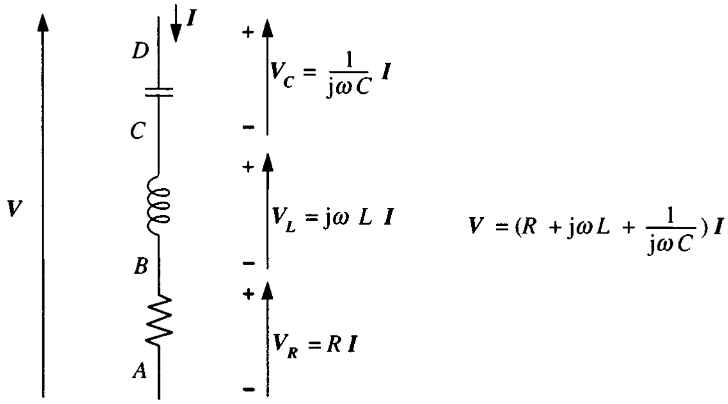
This equation allows us to write down the voltage drops across the elements of the general branch in terms of the complex impedance. From (3.27) the complex impedance of the general branch is $Z = R + j\omega L + 1/j\omega C$ therefore

$$V = \left(R + j\omega L + \frac{1}{j\omega C}\right)I$$

or

$$V = RI + j\omega LI + \frac{1}{j\omega C}I \tag{3.40}$$

Fig. 3.7. Voltage drops across the elements of the general branch and the complex voltage–current relationship.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

The terms in this equation represent the voltage drops across each of the elements in the general branch, as shown in fig. 3.7. Practical a.c. circuit analysis is carried out in terms of complex voltage drops and currents using relationships of the form (3.40), not in terms of instantaneous quantities. It will be noted that arrows, as well as (+) and (−) signs, have been used in fig. 3.7 to indicate the polarity, or reference direction, of the a.c. voltage drops across the circuit elements. This practice has been widely adopted in British textbooks, although not in American textbooks.

3.5 Phasor and impedance diagrams

The representation of complex voltages and currents on the Argand diagram provides a valuable pictorial aid to the interpretation of the algebraic procedures used in a.c. circuit analysis, and it greatly facilitates the understanding of circuit operation; indeed, it is customary to illustrate the operation of a.c. circuits by this means, often without explicit reference to the complex exponential notation. Such diagrams are referred to as *phasor diagrams* since they illustrate the relationships between the various sinusoidal voltages and currents in a circuit interpreted in their phasor form. (The concept of phasors and phasor diagrams can also be developed on the basis of a geometrical interpretation of sinusoidal wave forms. See for example reference 1.)

The phasor diagram for the general network branch provides a basis for the construction of all other such diagrams. To derive this we use the relationship (3.40). Recalling that $1/\mathrm{j} = -\mathrm{j}$, this may be written as

$$\boldsymbol{V} = R\boldsymbol{I} + \mathrm{j}\omega L\boldsymbol{I} - \mathrm{j}\frac{1}{\omega C}\boldsymbol{I} \tag{3.41}$$

Now let us in the first instance suppose that we have chosen the current as the reference waveform in the circuit, that is, its phase angle is chosen to be zero. Then we can write $\boldsymbol{I} = I\mathrm{e}^{\mathrm{j}0} = I$, and (3.41) becomes

$$\boldsymbol{V} = RI + \mathrm{j}\omega LI - \mathrm{j}\frac{1}{\omega C}I \tag{3.42}$$

This equation is represented diagrammatically in fig. 3.8. Interpreted strictly from the point of view of the Argand diagram we have the situation shown in fig. 3.8(a). The voltage drop across the resistance is $V_R = RI$, and this is represented by a phasor along the real or reference axis which is, of course, the direction of the current phasor. The voltage drop across the inductance is of magnitude $V_L = \omega LI$, and this is represented by a phasor along the positive imaginary axis. Similarly, the voltage drop across the

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

capacitance is $V_C = I/\omega C$; represented by a phasor along the negative imaginary axis.

An alternative interpretation of (3.42) is shown in fig. 3.8(*b*). In this diagram the phasors have been drawn head-to-tail (in a manner analogous to the vector polygon used in force diagrams in the fields of mechanics and structures), and the polygon is closed by a *resultant* voltage equal to ZI, the magnitude of the total branch voltage. The angle α, which the resultant voltage makes with the reference direction, is given by (3.32), namely, $\alpha = \tan^{-1}(\omega L - 1/\omega C)/R$. Notice that there is a topological similarity between the circuit diagram of fig. 3.7 and the phasor diagram of fig. 3.8(*b*): the order of the phasors in the phasor diagram corresponds to the order of the voltage drops in the circuit diagram. Corresponding points have been indicated in the two diagrams following the order $ABCD$.

In developing fig. 3.8 we chose to take the current as reference; if more generally we take the branch current to have some phase angle ϕ measured with respect to another voltage or current variable elsewhere in the circuit, then (3.40) becomes

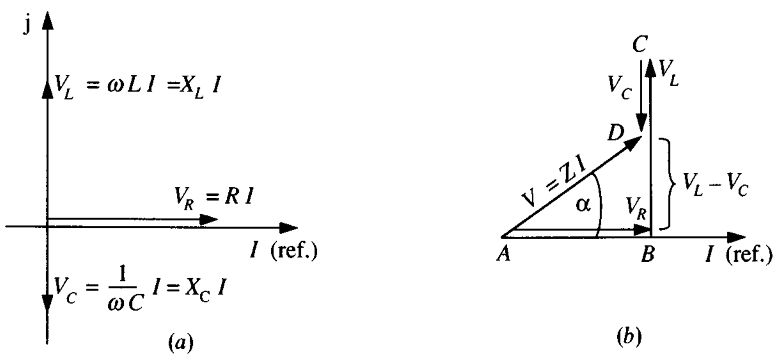
$$V = RIe^{j\phi} + j\omega LIe^{j\phi} - j\frac{1}{\omega C}Ie^{j\phi}$$

$$= \left(RI + j\omega LI - j\frac{I}{\omega C}\right)e^{j\phi} \qquad (3.43)$$

We see that (3.43) is simply (3.42) multiplied by $e^{j\phi}$, which means that in the phasor diagram all phasors are rotated bodily through an angle ϕ as shown in fig. 3.9.

It is important to appreciate that the phasor diagram shows phasor voltages and currents in a fixed relationship to one another, consequently, although it is customary to draw the Argand diagram with real and

Fig. 3.8. Phasor diagrams for the general branch.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

imaginary axes respectively horizontal and vertical, it is not mandatory to draw the phasor diagram with the reference phasor horizontal. Electrical power engineers, for instance, often take the system voltage as reference and traditionally draw this vertically on their phasor diagrams. Phasor diagrams may therefore be constructed in any of the forms typified by figs. 3.8 or 3.9, and with considerable freedom as to choice of reference phasor and reference direction.

Finally, we mention one other diagram (related to the phasor diagram) that shows the relationships between resistance, reactance and impedance in a circuit or part of a circuit; this is the so-called *impedance diagram*. The impedance diagram for the general network branch is shown in fig. 3.10, and it is obtained by dividing each of the voltage phasors in fig. 3.8(*b*) by the magnitude of the current I. The impedance diagram may be oriented in any convenient direction to suit the problem in hand.

3.6 Linear circuit theorems and techniques in a.c. circuit analysis

In chapter 2 a number of analytical techniques and circuit theorems were developed based essentially on the linear properties of the direct current circuits considered. This property of linearity depended upon the constancy of the ratio of voltage to current for each resistive element of the circuit, that is, upon Ohm's law. The same approach may be used in the case of a.c. circuits since, as we have seen in the immediately preceding

Fig. 3.9. Illustrating the effect on the phasor diagrams in fig. 3.8 of shifting the phase of the current by angle ϕ.

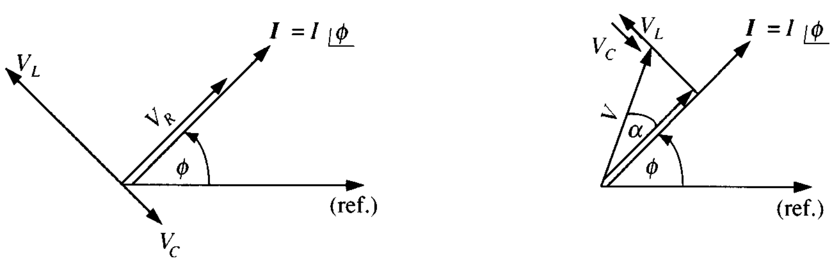
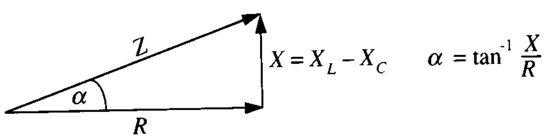


Fig. 3.10. The impedance diagram.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

Table 3.1

D.C. Formulation	Equation No.	A.C. Formulation
Ohm's law		
$V = RI$	(1.16)	$V = \boldsymbol{Z}I$
$V = \dfrac{1}{G}I$	(1.17)	$V = \dfrac{1}{Y}I$
Elements in series		
$R = R_1 + R_2 + \ldots + R_n$	(1.23)	$\boldsymbol{Z} = \boldsymbol{Z}_1 + \boldsymbol{Z}_2 + \ldots + \boldsymbol{Z}_n$
Elements in parallel		
$\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \ldots + \dfrac{1}{R_n}$	(1.27)	$\dfrac{1}{\boldsymbol{Z}} = \dfrac{1}{\boldsymbol{Z}_1} + \dfrac{1}{\boldsymbol{Z}_2} + \ldots + \dfrac{1}{\boldsymbol{Z}_n}$
$G = G_1 + G_2 + \ldots + G_n$	(1.28)	$Y = Y_1 + Y_2 + \ldots + \dfrac{1}{Y_n}$
$R = \dfrac{R_1 R_2}{R_1 + R_2}$	(1.26)	$\boldsymbol{Z} = \dfrac{\boldsymbol{Z}_1 \boldsymbol{Z}_2}{\boldsymbol{Z}_1 + \boldsymbol{Z}_2}$
Voltage divider		
$V_2 = \dfrac{R_2}{R_1 + R_2} V_1$	(2.6)	$V_2 = \dfrac{\boldsymbol{Z}_2}{\boldsymbol{Z}_1 + \boldsymbol{Z}_2} V_1$
$V_2 = \dfrac{G_1}{G_2 + G_1} V_1$	(2.7)	$V_2 = \dfrac{Y_1}{Y_2 + Y_1} V_1$

sections, the ratio of complex voltage to complex current is constant for linear resistive, inductive and capacitive elements. Commencing, therefore, with the statement of Ohm's law in its a.c. form, (3.39), all the theorems and techniques developed for d.c. circuits can equally well be developed for a.c. circuits; the only difference being the replacement of the symbol R by the symbol $\boldsymbol{Z}$ in the linear equations. We may, therefore, immediately adopt the theory of chapter 2 in its entirety by the simple expedient of working in terms of complex voltages and currents and writing $\boldsymbol{Z}$ for R. Some analogous expressions and theorems for d.c. and a.c. circuits are presented in table 3.1. In this table the symbol Y, called the *admittance*, denotes the reciprocal of the impedance $\boldsymbol{Z}$ (see section 3.8).

As an example of the procedure we consider the Thévenin equivalent circuit for an a.c. network. This comprises an ideal a.c. voltage source in series with a complex impedance as shown in fig. 3.11(*a*). The Thévenin–Norton transformation is carried out in a way exactly analogous to that shown in fig. 2.16; with the result as shown in fig. 3.11(*b*).

The graphical symbol used in this book to indicate a complex impedance is also shown in fig. 3.11. It will be observed that the ideal a.c. voltage and current sources are distinguished by having a sine-wave symbol enclosed within the circle; this is a common but not universal practice.

The application of the phasor method will now be illustrated with reference to the circuit shown in fig. 3.12, which is analogous to the single-mesh d.c. circuit of fig. 2.5. Analysis of the circuit proceeds, as in the d.c. case, with the assignment of a current $\boldsymbol{I}$, and this is followed by the application of Kirchhoff's voltage law to set up the circuit equation. Exactly the same conventions are applied to determine the signs of the terms in the circuit equation, and the working rules discussed in section 2.3 may be used with advantage. Although not strictly necessary for the purposes of analysis, the voltage drop across the resistance is also specified in fig. 3.12 so that the relationship between all three voltages in the mesh may be clearly seen on the phasor diagram (fig. 3.13).

Applying KVL to this circuit:

$$\boldsymbol{I}R = \boldsymbol{V}_1 - \boldsymbol{V}_2$$

Fig. 3.11. The Thévenin–Norton transformation for an a.c. network.

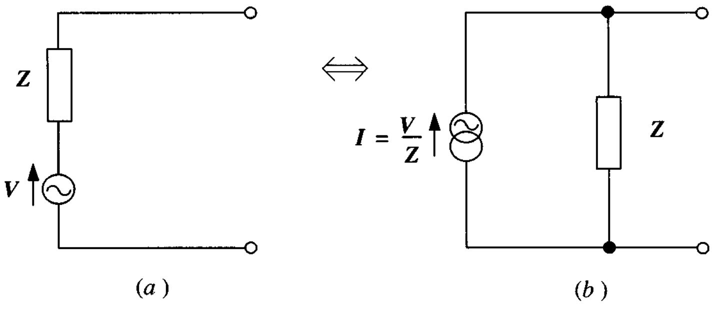
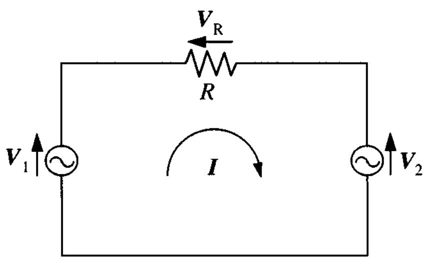


Fig. 3.12. Single-mesh a.c. circuit.



or

$$I = \frac{V_1 - V_2}{R} = \frac{V_R}{R} \tag{3.44}$$

Let us now suppose that the two sources have equal magnitude V and that the phase angle of one can be varied with respect to the other, that is, $V_1 = V\underline{/0}$ (phase reference), and $V_2 = V\underline{/\phi}$, where ϕ is variable. With both sources equal in magnitude and with $\phi = 0$ no current will flow; the phasor diagram for this situation is shown in fig. 3.13(*a*). If ϕ is now increased, say to 60°, (3.44) becomes

$$I = \frac{V\underline{/0} - V\underline{/60}}{R}$$

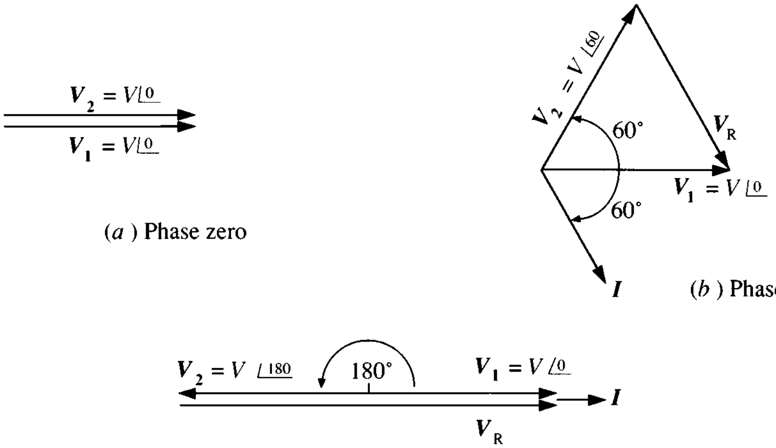
$$= \frac{V}{R}[(\cos 0 + j\sin 0) - (\cos 60 + j\sin 60)]$$

$$= \frac{V}{R}(0.5 - j0.866)$$

$$= \frac{V}{R}\underline{/-60}$$

This result is interpreted on the phasor diagram as shown in fig. 3.13(*b*). The directions of the phasors should be carefully observed; in particular, it

Fig. 3.13. Phasor diagrams for the circuit of fig. 3.12. Illustrating the effect of changing the phase of V_2 while keeping the magnitudes of V_1 and V_2 constant.



(*a*) Phase zero

(*b*) Phase 60°

(*c*) Phase 180°

should be noted that the direction of V_R is from the tip of V_2 towards the tip of V_1. This accords with the usual convention for vector addition, representing the relations $V_R = V_1 - V_2$ or $V_R + V_2 = V_1$. The phasor V_R lies parallel to the current phasor; a result which is to be expected since voltage and current must be in phase for a purely resistive element.

If now the phase of V_2 is further advanced until $\phi = 180°$, the phasor diagram becomes as shown in fig. 3.13(*c*). The current has a magnitude of $2V/R$ and a phase angle zero. We see that the effect of changing the phase of V_2 by 180° is the same as reversing the reference direction of V_2 in the circuit diagram, that is, changing the terminal polarity of the source.

3.7 Worked example

The circuit of fig. 3.14(*a*) is to be operated at mains power frequency (50 Hz).

(a) Find the complex impedance between the terminals AB and between the nodes D and B. Sketch an impedance diagram for the circuit.

(b) The circuit is connected to a mains power supply of 240 V; find the currents in each branch of the circuit and the voltage of node D with respect to B.

(c) Check the value for the node voltage obtained in (b) by using the method of nodal analysis.

(d) Sketch a phasor diagram for the complete circuit.

Solution

(a) First the reactances of the two inductances and the capacitance are found:

$$\begin{aligned}\omega L_1 &= 2\pi \times 50 \times 15.9 \times 10^{-3} = 5\\ \omega L_2 &= 2\pi \times 50 \times 66.8 \times 10^{-3} = 21\\ 1/\omega C &= 1/(2\pi \times 50 \times 398 \times 10^{-6}) = 8\end{aligned}$$

These values are entered on the diagram (in brackets) as shown. Units of ohms are understood throughout the problem.

Let the complex impedances of the left- and right-hand branches between D and B be $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ respectively, then

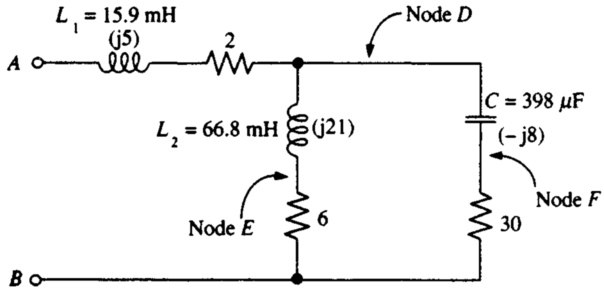
$$\boldsymbol{Z}_1 = 6 + \mathrm{j}21; \qquad \boldsymbol{Z}_2 = 30 - \mathrm{j}8$$

Expressed in polar form, using the notation of (3.16), these impedances become

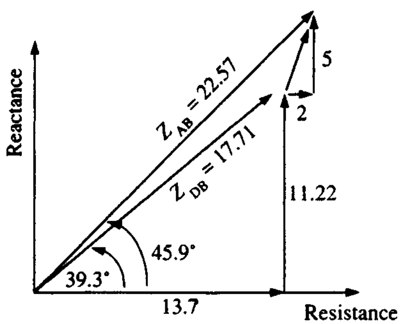
$$\boldsymbol{Z}_1 = 21.84\,\underline{/74.1}; \qquad \boldsymbol{Z}_2 = 31.04\underline{/-14.9}$$

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

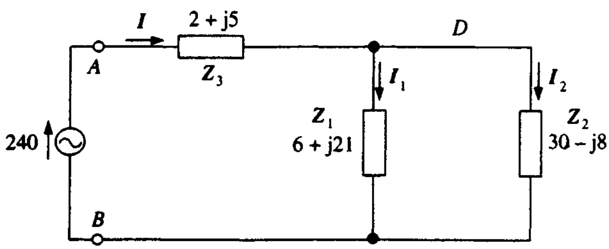
Fig. 3.14. Diagrams for the worked example of section 3.7.



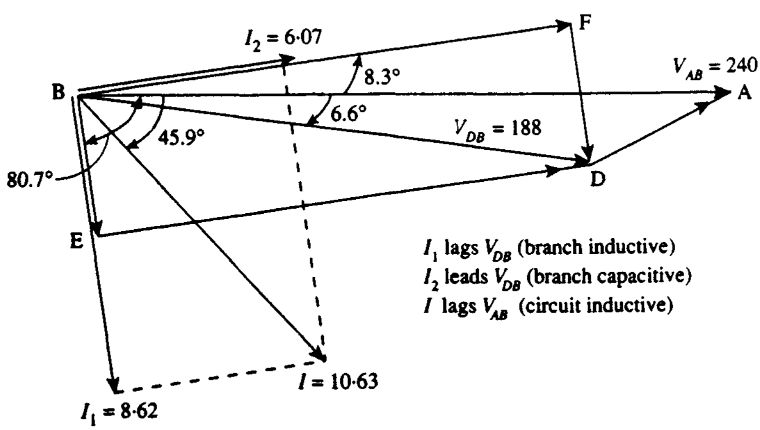
(*a*)



(*b*)



(*c*)



(*d*)

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

(Note: in numerical work it is customary to express the argument of a complex quantity in degrees rather than radians.)

The resultant impedance between DB is

$$\boldsymbol{Z}_{DB} = \boldsymbol{Z}_1 // \boldsymbol{Z}_2 = \frac{\boldsymbol{Z}_1 \boldsymbol{Z}_2}{\boldsymbol{Z}_1 + \boldsymbol{Z}_2}$$

To evaluate this impedance the reader will recall that addition of complex quantities is effected by expressing them in Cartesian form, while multiplication or division is effected by expressing quantities in polar form. Thus

$$\boldsymbol{Z}_1 + \boldsymbol{Z}_2 = 6 + \mathrm{j}21 + 30 - \mathrm{j}8 = 36 + \mathrm{j}13 = 38.27\underline{/19.9}$$

and

$$\boldsymbol{Z}_{DB} = \frac{21.84\underline{/74.1} \times 31.04\underline{/-14.9}}{38.27\underline{/19.9}}$$

$$= \frac{21.84 \times 31.04}{38.27}\underline{/74.1 - 14.9 - 19.9}$$

$$\boldsymbol{Z}_{DB} = 17.71\underline{/39.3} = 13.7 + \mathrm{j}11.2$$

(Note that the procedure known as rationalization is generally to be avoided in numerical evaluation of expressions of the above form.)

The impedance between AB is then

$$\boldsymbol{Z}_{AB} = 13.7 + \mathrm{j}11.22 + 2 + \mathrm{j}5 = 15.7 + \mathrm{j}16.22$$

or

$$\boldsymbol{Z}_{AB} = 22.57\underline{/45.9}$$

The impedance diagram is illustrated in Fig. 3.14(*b*).

(b) Unless stated otherwise voltages are always expressed in terms of their r.m.s. magnitudes. Fig. 3.14(*c*) shows the 240 V supply, assumed to be an ideal source, connected to AB. The supply voltage is chosen as the reference phasor, that is, it takes zero phase angle. Since it is the branch currents which have to be evaluated, we assign currents to each branch as shown. (Assignment of mesh currents would entail an extra step in the calculation.)

The current drawn from the supply is found from the a.c. form of Ohm's law:

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}_{AB}} = \frac{240\underline{/0}}{22.57\underline{/45.9}} = 10.63\underline{/-45.9}\text{ A}$$

This result is interpreted as a current of magnitude 10.63 A, lagging the supply voltage by angle 45.9°.

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

The voltage of node D with respect to B is given by

$$\boldsymbol{V}_{DB}=\boldsymbol{I}\boldsymbol{Z}_{DB}=10.63\underline{/-45.9}\times 17.71\underline{/39.3}$$

or

$$\boldsymbol{V}_{DB}=188.3\underline{/-6.6}\text{ V}$$

The currents I_1 and I_2 are found by dividing this voltage by the respective branch impedances:

$$\boldsymbol{I}_1=\frac{\boldsymbol{V}_{DB}}{\boldsymbol{Z}_1}=\frac{188.3\underline{/-6.6}}{21.84\underline{/74.1}}=8.62\underline{/-80.7}\text{ A}$$

and

$$\boldsymbol{I}_2=\frac{\boldsymbol{V}_{DB}}{\boldsymbol{Z}_2}=\frac{188.3\underline{/-6.6}}{31.04\underline{/-14.9}}=6.07\underline{/8.3}\text{ A}$$

Alternatively, for this part of the problem, the branch currents could be evaluated by means of the current divider principle discussed in section 2.2. In this case

$$\boldsymbol{I}_1=\frac{\boldsymbol{Z}_2\boldsymbol{I}}{\boldsymbol{Z}_1+\boldsymbol{Z}_2}=\frac{31.04\underline{/-14.9}\times 10.63\underline{/-45.9}}{38.27\underline{/19.9}}=8.62\underline{/-80.7}\text{ A}$$

and

$$\boldsymbol{I}_2=\frac{\boldsymbol{Z}_1\boldsymbol{I}}{\boldsymbol{Z}_1+\boldsymbol{Z}_2}=\frac{21.84\underline{/74.1}\times 10.63\underline{/-45.9}}{38.27\underline{/19.9}}=6.07\underline{/8.3\text{A}}$$

(c) The nodel equation at D is

$$\frac{\boldsymbol{V}_{DB}-240\underline{/0}}{\boldsymbol{Z}_{AD}}+\frac{\boldsymbol{V}_{DB}}{\boldsymbol{Z}_1}+\frac{\boldsymbol{V}_{DB}}{\boldsymbol{Z}_2}=0$$

But

$$\boldsymbol{Z}_{AD}=2+\text{j}5=5.38\underline{/68.2}$$

therefore

$$\frac{\boldsymbol{V}_{DB}-240\underline{/0}}{5.38\underline{/68.2}}+\frac{\boldsymbol{V}_{DB}}{21.84\underline{/74.1}}+\frac{\boldsymbol{V}_{DB}}{31.04\underline{/-14.9}}=0$$

Rearranging this equation gives

$$\boldsymbol{V}_{DB}\left(\frac{1}{5.38\underline{/68.2}}+\frac{1}{21.84\underline{/74.1}}+\frac{1}{31.04\underline{/-14.9}}\right)=\frac{240\underline{/0}}{5.38\underline{/68.2}}$$

$$V_{DB}(6.9-j17.26+1.254-j4.4+3.11+j0.828)\times10^{-2}$$
$$=44.61\underline{/-68.2}$$

$$V_{DB}(11.27-j20.83)\times10^{-2}=44.61\underline{/-68.2}$$

$$V_{DB}=\frac{44.61\underline{/-68.2}}{0.237\underline{/-61.6}}=188.4\underline{/-6.6}\text{ V}$$

(d) The phasor diagram for the complete circuit is shown in Fig. 3.14(*d*). Note that the current $\boldsymbol{I}$ is the phasor resultant of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ (indicated by the dash lines). The voltage drop from *A* to *D* and the voltage drop from *B* to *D* are together equal to the source voltage. The voltage drops across the 6 Ω and 30 Ω resistances in each branch are in phase, respectively, with the currents, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ (lines *BE* and *BF*); the voltage drops (lines *ED* and *FD*) are in quadrature.

As an additional exercise the reader may care to evaluate the voltage $\boldsymbol{V}_{EF}$. (Answer $188.43\underline{/-155.77}$)

The reader should note that in working through problems of this type in manuscript, complex quantities (here indicated by bold italic type) may, if so desired, be indicated by a bar above or below the quantity concerned; thus, $\boldsymbol{V}_{DB}$ may be written $\underline{V_{DB}}$ or $\overline{V_{DB}}$. Normally, however, this is unnecessary since, as will be appreciated from the above calculations, there is in numerical work little possibility of confusion between complex quantities and their moduli.

3.8 Admittance

It was found to be convenient in d.c. circuit analysis to define a quantity called conductance, the reciprocal of resistance. The formal equations of nodal analysis were expressed in terms of conductance in section 2.10. In a similar way we find it convenient in a.c. circuit analysis to define a quantity called the complex admittance, the reciprocal of complex impedance:

$$\boldsymbol{Y}\equiv\frac{1}{\boldsymbol{Z}} \tag{3.45}$$

If α is the angle of the impedance and β the angle of the admittance this becomes, in polar form

$$Y\underline{/\beta}=\frac{1}{Z\underline{/\alpha}}$$

where $Y=\dfrac{1}{Z}$, and $\beta=-\alpha$.

Thus, there is a reciprocal relationship between the magnitudes of the admittance and the impedance.

Expressed in Cartesian form the complex admittance may be written

$$Y = G + \mathrm{j}B \tag{3.46}$$

where G is the conductance and B the susceptance. Both G and B are expressed in units of siemens.

Some care is required in the interpretation of G and B, as defined in (3.46), since these quantities do not always bear a simple reciprocal relationship to resistance and reactance. This will be clear if we consider the complex admittance of the general network branch shown in Fig. 3.15(a). At an angular frequency ω we have

$$Y = \frac{1}{\mathbf{Z}} = \frac{1}{R + \mathrm{j}\omega L + \dfrac{1}{\mathrm{j}\omega C}}$$

or

$$Y = \frac{1}{R + \mathrm{j}X}$$

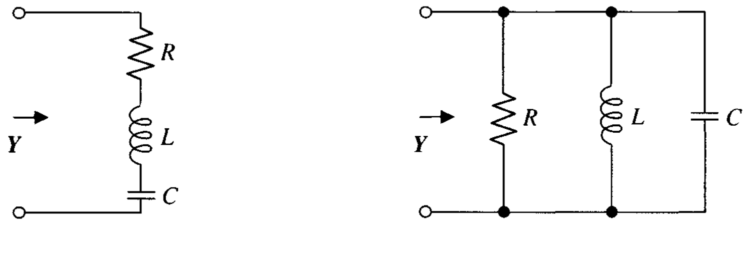
where

$$X = \left(\omega L - \frac{1}{\omega C}\right)$$

The process of rationalization (multiplying numerator and denominator by the complex conjugate of $R + \mathrm{j}X$), allows us to deduce the conductance and susceptance.

$$Y = \frac{1}{R + \mathrm{j}X}\,\frac{R - \mathrm{j}X}{R - \mathrm{j}X} = \frac{R - \mathrm{j}X}{R^2 + X^2}$$

Fig. 3.15. Complex admittance of series and parallel circuits.



(a) Series (b) Parallel

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

or

$$Y = \frac{R}{R^2 + X^2} + \mathrm{j}\frac{-X}{R^2 + X^2} \tag{3.47}$$

Comparing (3.47) with (3.46) we see that for the general series-connected network branch

$$\text{Conductance} = \frac{R}{R^2 + X^2}$$

$$\text{Susceptance} = \frac{-X}{R^2 + X^2}$$

Thus, conductance and susceptance for a series circuit are functions of both resistance and reactance.

The practical use of admittance lies mainly in situations where there are a number of elements connected in parallel. For instance, in the circuit of fig. 3.15(*b*) we have, by an expression analogous to (1.27)

$$Y = \frac{1}{Z} = \frac{1}{R} + \frac{1}{\mathrm{j}\omega L} + \mathrm{j}\omega C$$

or

$$Y = \frac{1}{R} + \mathrm{j}\left(\omega C - \frac{1}{\omega L}\right)$$

In this case the conductance is identically equal to the reciprocal of the circuit resistance. It should also be noted that if the inductive reactance ωL is smaller than the capacitive reactance $1/\omega C$, the susceptance is negative, the converse of the series-connected case.

The formal equations of nodal analysis for a.c. circuits expressed in terms of admittance, yield expressions analogous to those expressed in terms of conductance for the d.c. case. For example, the a.c. formulation for a two-node circuit (corresponding to (2.28)) would be:

$$Y_{11}V_1 + Y_{12}V_2 = I_{11}$$

and (3.48)

$$Y_{21}V_1 + Y_{22}V_2 = I_{22}$$

Here Y_{11} and Y_{22} are the self admittances at nodes (1) and (2) respectively, and $Y_{12} = Y_{21}$ is the mutual admittance. The currents I_{11} and I_{22} represent the net current injected into nodes (1) and (2) respectively from current sources attached to those nodes.

3.9 Frequency response: transfer function

The concept of a two-port network has already been mentioned in section 2.2, and the theory of such networks is considered in more detail in chapter 8. Here we introduce the concept of the transfer function for an a.c. two-port network. The basic circuit is shown in fig. 3.16. An alternating voltage V_1, with angular frequency ω, impressed at the input port (1) gives rise to a voltage V_2 at the output port (2). In the context of electronics and communications systems these voltages would be referred to as *signals.*

The input and output voltages will be related by some linear function dependent upon the arrangement of elements in the network; in general this will be a function of frequency. The network can therefore be characterized by a *transfer function* (or *frequence response function*) defined by

$$V_2 = H(\mathrm{j}\omega)V_1$$

or

$$H(\mathrm{j}\omega) = \frac{V_2}{V_1} = \frac{\text{Output voltage}}{\text{Input voltage}} \tag{3.49}$$

The transfer function defined in this way, as a complex function of frequency (jω), refers to the steady-state conditions only. Other more general definitions are given in section 6.9.6.

The concept of the transfer function is not necessarily restricted to voltage ratios in a network; the relationship between any two network parameters (the ratio of output current to input voltage for example) may be expressed in a similar fashion.

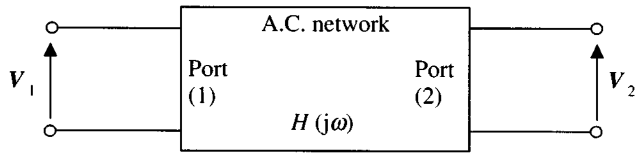
In practice the transfer function is most useful when expressed in its polar form:

$$H(\mathrm{j}\omega) = |H(\mathrm{j}\omega)|\,\mathrm{e}^{\mathrm{j}\arg H(\mathrm{j}\omega)} \tag{3.50}$$

Now $|H(\mathrm{j}\omega)|$ (the modulus) and $\arg H(\mathrm{j}\omega)$ (the angle or phase) are real functions of ω so that (3.50) may be written

$$H(\mathrm{j}\omega) = H(\omega)\mathrm{e}^{\mathrm{j}\phi(\omega)} = H(\omega)\underline{/\phi(\omega)} \tag{3.51}$$

Fig. 3.16. Two-port a.c. network with transfer function $H(\mathrm{j}\omega)$.



where

$$H(\omega)=|H(\mathrm{j}\omega)| \text{ and } \underline{/\phi(\omega)}=\arg H(\mathrm{j}\omega)$$

It may be noted that in conformity with our usual notation for complex quantities (see section 3.3) $H(\mathrm{j}\omega)$ could also be written simply as $\boldsymbol{H}$, but because other definitions of the transfer function exist we have chosen to use the explicit form here.

To demonstrate the analytical procedures used to find the transfer function of a network, we consider the simple capacitance–resistance CR network shown in fig. 3.17. This circuit is employed extensively as an inter-stage coupling network in electronic amplifiers, its function being to prevent or block the passage of signals of zero frequency (d.c.) while allowing the passage of signals of higher frequency.

To find the transfer function of this network we may use the voltage divider principle (table 3.1: a.c. analogue of (2.6)). Assuming that no current is drawn from the output port, we may write

$$V_2=\frac{R}{R+\dfrac{1}{\mathrm{j}\omega C}}V_1$$

Hence

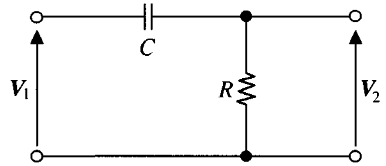
$$H(\mathrm{j}\omega)=\frac{V_2}{V_1}=\frac{R}{R+\dfrac{1}{\mathrm{j}\omega C}}$$

It is usually most convenient to express the transfer function in polar form; we therefore rewrite the above equation as

$$H(\mathrm{j}\omega)=\frac{1}{1+\dfrac{1}{\mathrm{j}\omega CR}} \tag{3.52}$$

The denominator can now be expressed as

Fig. 3.17. CR coupling network.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$1+\frac{1}{\mathrm{j}\omega CR}=1-\mathrm{j}\left(\frac{1}{\omega CR}\right)=\sqrt{\left[1+\left(\frac{1}{\omega CR}\right)^2\right]}\,\underline{/\alpha}$$

where

$$\alpha=\tan^{-1}\left(-\frac{1}{\omega CR}\right)$$

The modulus of $H(\mathrm{j}\omega)$ is then

$$H(\omega)=\frac{1}{\sqrt{\left[1+\left(\frac{1}{\omega CR}\right)^2\right]}} \tag{3.53}$$

and the angle is given by

$$\phi(\omega)=-\alpha=-\tan^{-1}\left(\frac{-1}{\omega CR}\right) \tag{3.54}$$

(Note that it is not necessary to rationalize (3.52) in order to express $H(\mathrm{j}\omega)$ in polar form.)

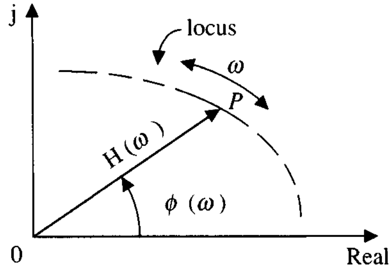
If we examine these expressions for the magnitude and angle of the transfer function, we see that

for $\omega\to 0$: $H(\omega)\to 0$; $\phi(\omega)\to 90°$
for $\omega\to\infty$: $H(\omega)\to 1$; $\phi(\omega)\to 0$

This result is to be expected since as far as direct voltages are concerned the capacitance acts as an open circuit; as far as very high frequencies are concerned the capacitance is effectively a short circuit.

Two methods of illustrating graphically the way in which the magnitude and phase angle of the transfer function vary with frequency are commonly used. The first method, based on the Argand diagram, is shown in fig. 3.18. In this diagram the modulus and angle are plotted in polar coordinates as a function of ω. The locus of the tip of the vector OP, traced out as ω varies,

Fig. 3.18. Locus diagram: plot of $H(\mathrm{j}\omega)=H(\omega)\,\underline{/\phi(\omega)}$ on the Argand diagram.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

defines a line characteristic of the circuit transfer function. Diagrams of this type are called variously *locus*, *polar* or *Nyquist* diagrams. (Similar diagrams can also be drawn to show the effect of varying any one of the circuit parameters at a fixed frequency). It is important to appreciate that ω is not a function of time and, although based on the Argand diagram, the locus diagram is not to be confused with the rotating phasor diagram.

The locus diagram corresponding to (3.52) is shown in fig. 3.19; for the simple CR network the diagram takes the form of a semi-circle. Many other circuits exhibit a similar behaviour and give rise to locus diagrams of circular form. This graphical approach has been elaborated and extended to include many different electrical devices, particularly rotating electrical machines, and it has come to be known as the *circle diagram* method.

The second method whereby the transfer function may be depicted graphically is shown in fig. 3.20. This figure is drawn for the particular CR network of fig. 3.17. Here the modulus and phase of $H(j\omega)$ are plotted

Fig. 3.19. Locus diagram for the CR network of fig. 3.17.

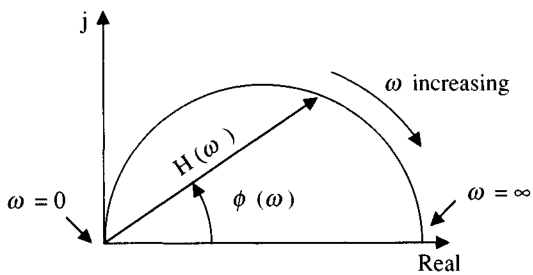
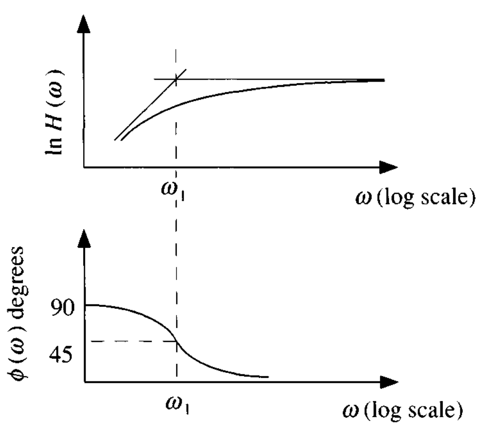


Fig. 3.20. Bode diagrams for the CR network.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

separately as functions of ω on a logarithmic basis. Graphs of this type are known as *Bode diagrams* or *Bode plots* after their originator.

Taking natural logarithms of (3.51) we obtain

$$\ln H(j\omega)=\ln H(\omega)+\ln e^{j\phi(\omega)}=\ln H(\omega)+j\phi(\omega) \tag{3.55}$$

In fig. 3.20(a) the real part of this expression, namely $\ln H(\omega)$, is plotted on a *linear* scale against ω which is plotted on a *logarithmic* scale. Likewise the imaginary part $\phi(\omega)$ is plotted in fig. 3.20(b) also using linear-log scales. The logarithmic basis of the Bode diagram allows large changes in the values of the parameters to be accommodated, and it provides additional practical advantages which are discussed below.

Frequently the modulus of the transfer function is expressed in logarithmic units of the decibel (abbreviation dB) (for an explanation of the decibel, see for example reference 5) and the ordinate in the Bode plot is scaled in these units in preference to the natural logarithmic scale used in fig. 3.20(a). The decibel is defined strictly in terms of the ratio of two power levels P_1 and P_2, that is,

$$\text{Power ratio (dB)}=10\log_{10}\frac{P_2}{P_1} \tag{3.56}$$

If we are concerned with two voltages V_1 and V_2 established across *identical* resistances, the power ratio is proportional to $(V_2/V_1)^2$ and (3.56) becomes

$$\text{Power ratio (dB)}=10\log_{10}\left(\frac{V_2}{V_1}\right)^2=20\log_{10}\frac{V_2}{V_1} \tag{3.57}$$

In general the resistances at the input and output ports of a network are not the same, nevertheless, by convention (3.57) is applied to the modulus of the transfer function, defined by (3.49), without regard to the resistances associated with the input and output ports. According to this convention we therefore write

$$|H(j\omega)|\,(\text{dB})=20\log_{10}H(\omega) \tag{3.58}$$

Fig. 3.21 shows the Bode diagram for the CR network (modulus only) with the ordinate scaled in decibels. The dB values are all negative since $V_2=V_1$ corresponds to 0 dB, and V_2 can never be greater than V_1.

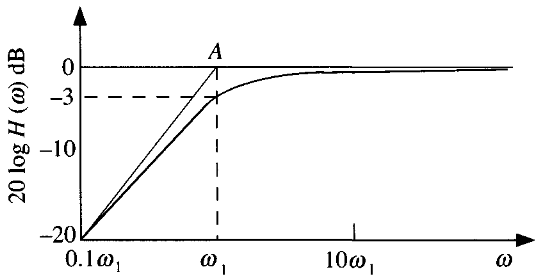
It will be observed that in fig. 3.21 (or fig. 3.20(a)) the curve is asymptotic to two straight line segments. The reason for this will be appreciated if we consider the modulus of $H(j\omega)$ for very low and very high frequencies.

At sufficiently low frequencies, such that $\omega CR \ll 1$, (3.52) becomes $H(\omega)=\omega CR$. $H(\omega)$ is therefore proportional to ω which means that the log-modulus versus ω relationship is of straight-line form provided ω is plotted on a logarithmic scale. Using (3.57) we have $|H(j\omega)|$ (dB) $= 20\log_{10}\omega CR$. For each decade of frequency therefore the modulus changes by $20\log_{10}(10/1) = 20$ dB. This determines the asymptote for low frequencies.

At sufficiently high frequencies ($\omega CR \gg 1$) we have $H(\omega) = 1$, and $20\log_{10}H(\omega) = 0$. Thus, the curve is asymptotic to a horizontal straight line at high frequencies. The two asymptotes meet (point A in Fig. 3.21) at a frequency ω_1 such that $\omega_1 CR = 1$ or $\omega_1 = 1/CR$. At this frequency the *actual* value of $H(\omega)$ is, from (3.53), $1/\sqrt{2}$ or 0.707, in other words the true value of $H(\omega)$ is about 30% lower than the value represented by the intersection of the asymptotes. Expressed in decibels this is equal to -3.03 dB. At this frequency the phase angle is 45° (fig. 3.20(*b*). The frequency ω_1 is called variously the *turnover frequency*, the *corner frequency*, the *break frequency*, or the '3 *dB point*'. Since the maximum departure of the true curve from the approximate curve formed by the two asymptotes is only 3 dB it is sufficient for many practical purposes to represent the modulus of the transfer function by its straight-line approximation. The principles of construction of the asymptotes in the Bode diagram, here discussed in relation to the *CR* network, may be readily extended to networks of greater complexity.

The chief advantage of the Bode diagram is that it affords a ready means of finding graphically the overall transfer function of several networks connected in cascade. Suppose we wish to find the overall transfer function $H_0(j\omega)$ of two cascaded networks with individual transfer functions $H_1(j\omega)$ and $H_2(j\omega)$. From the definition of the transfer function given in (3.49) it will be readily apparent that $H_0(j\omega) = H_1(j\omega)H_2(j\omega)$. It is a somewhat tedious procedure to multiply the moduli of the two transfer functions together over a wide frequency range, especially if one or both have been determined by experiment. However, if the results are presented on a Bode plot, advantage can be taken of the properties of logarithms, and the

Fig. 3.21. Bode diagram for the CR network: modulus of the transfer function expressed in units of the decibel.



ordinates can be simply added since

$$\begin{aligned}\ln H_0(j\omega) &= \ln [H_1(j\omega)H_2(j\omega)] \\ &= \ln H_1(\omega) + \ln H_2(\omega) + j[\phi_1(\omega) + \phi_2(\omega)]\end{aligned}$$

This result applies equally well of course when moduli are expressed in decibels. The ordinate addition is a particularly simple operation when the straight-line approximation is used in the Bode diagram.

In the above discussion it has been assumed that individual transfer functions will not be changed by connection of the networks one to another; often in practice this will not be the case. Each network will be affected by connection to both preceding and succeeding networks to some extent, and due allowance must be made for this when determining the individual transfer functions either from theory or experiment.

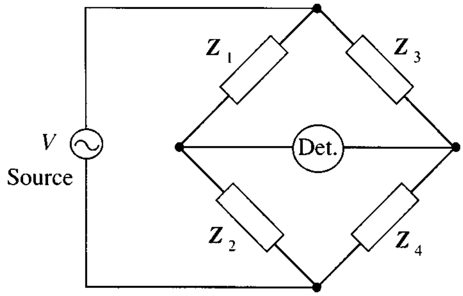
3.10 A.C. bridges

A.C. bridge networks form a large and important class of circuits used for measurement purposes, for filtering (separating wanted signals from unwanted signals), and for use in certain types of oscillator. The essential characteristic common to all of the various circuits falling under this heading is that for a finite input or source signal they produce zero output signal under one particular set of conditions: the so-called *balance* conditions.

The basic circuit configuration of one common form of a.c. bridge, used mainly for measurement purposes, is shown in fig. 3.22; this will be recognized as the a.c. analogue of the Wheatstone bridge discussed in section 2.8.

The balance conditions may be derived using methods essentially similar to those of section 2.8 (equation 2.21); they occur when the impedances in the arms of the bridge are such that

Fig. 3.22. A.C. bridge network.



$$Z_1 Z_4 = Z_2 Z_3$$

or

$$Z_1 = \frac{Z_2 Z_3}{Z_4} \tag{3.59}$$

By arranging Z_1 to be the component whose value is to be measured, and by arranging Z_2, Z_3 and Z_4 to be components of accurately known value, the value of Z_1 can be determined. We can arrange either to make Z_2 and Z_3 fixed and to vary Z_4 to achieve balance, in which case the bridge is known as a *product* bridge, or we can make Z_2 and Z_4 fixed and vary Z_3, in which case the bridge is known as a *quotient* bridge. In all bridges of this type the balance conditions are independent of the magnitude of the source voltage although the sensitivity with which the balance can be detected will suffer if the voltage is too low. The balance conditions may or may not depend on frequency.

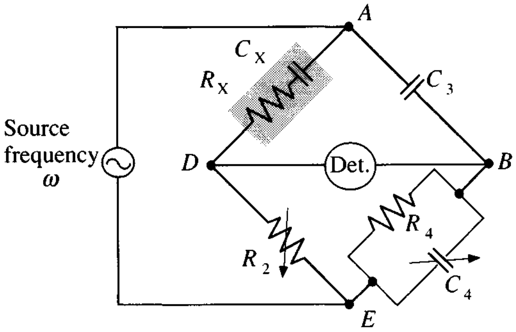
For any given arrangement of elements forming a bridge circuit the source and detector may be interchanged without in any way affecting the signal registered by the detector. This follows directly from the reciprocity theorem. The new bridge circuit resulting from such an interchange is called the *conjugate* of the original and it has the same balance conditions.

Two examples of bridges conforming to the basic configuration shown in fig. 3.22 will serve to illustrate the techniques of analysis and some of the important characteristics of this type of circuit.

3.10.1 The Schering bridge

The circuit of this bridge is shown in fig. 3.23. Its main use is for the measurement of capacitance, particularly the capacitance of high voltage cables and insulators under working conditions. The unknown capacitance

Fig. 3.23. The Schering bridge.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

is C_x, and R_x is a series resistance representing the losses associated with this capacitance (see section 3.13.1 for an account of losses in capacitors). C_3 and C_4 are respectively fixed and variable low-loss capacitors of accurately known values. R_2 is an accurately calibrated variable resistance and R_4 is a fixed standard resistance. (When this bridge is used in high voltage applications, the node E is earthed, and the node A is at high potential.)

From (3.59) the balance conditions are given by

$$\boldsymbol{Z}_1 = \frac{\boldsymbol{Z}_2 \boldsymbol{Z}_3}{\boldsymbol{Z}_4} = \boldsymbol{Z}_2 \boldsymbol{Z}_3 \boldsymbol{Y}_4$$

where $\boldsymbol{Y}_4$ is the admittance of the arm BE. Therefore,

$$R_x + \frac{1}{\mathrm{j}\omega C_x} = R_2 \frac{1}{\mathrm{j}\omega C_3}\left(\frac{1}{R_4} + \mathrm{j}\omega C_4\right) = \frac{R_2}{\mathrm{j}\omega C_3 R_4} + \frac{R_2 C_4}{C_3}$$

Equating real parts on the two sides of this equation gives

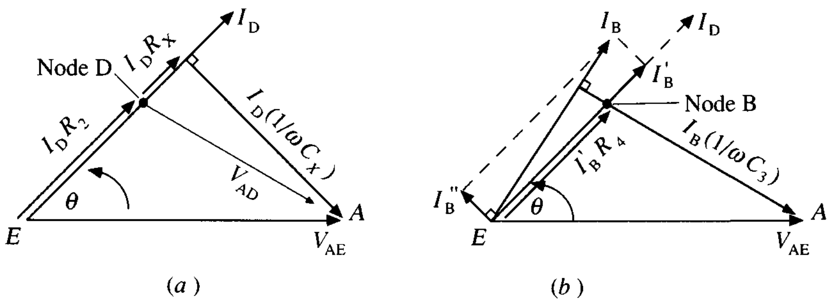
$$R_x = \frac{R_2 C_4}{C_3}$$

and equating imaginary parts gives

$$C_x = \frac{C_3 R_4}{R_2}$$

Two balance conditions are thus obtained reflecting the fact that the potentials of D and B at balance must be the same in both magnitude and phase. It will be noticed that the balance conditions are independent of frequency so that a highly stable source is not required. By adjusting R_2 and C_4 alternately the bridge can be made to converge to a balance indicated by

Fig. 3.24. Phasor diagrams for the balanced Schering bridge of fig. 3.23. (*a*) Left-hand arm *ADE*; (*b*) right-hand arm *ABE*. Currents I_B' and I_B'' are the components of I_B through R_4 and C_4 respectively.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

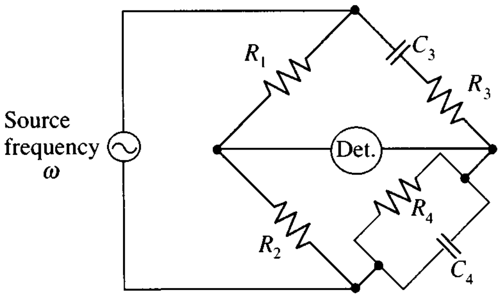
a null detector reading. The detector would normally consist of an oscilloscope or a sensitive integrated circuit amplifier with its output connected to some form of indicator. For very accurate measurements the detector may be tuned so that it is sensitive only to the bridge source frequency; by this means the effects of spurious signals arising, for example, from the power supply mains (hum pick-up) may be reduced.

Phasor diagrams for the balanced Schering bridge are shown in fig. 3.24. For the sake of clarity the diagrams for the left and right-hand arms are shown separately. Referring first to the left-hand arm fig. 3.24(*a*), we may understand how this diagram is constructed by considering the current I_D flowing from A to E via D (remembering that no current flows in the detector branch at balance) and its relationship to the source voltage V_{AE}, which is conveniently taken as reference. The branch ADE contains both capacitance and resistance so that the current I_D must lead V_{AE} by some angle less than 90°. The arm DE is purely resistive so that the voltage $V_{DE}(=I_D R_2)$ must be in phase with the current I_D, also the voltage drop across R_x must be in phase with I_D. These voltage phasors therefore coincide with the current phasor I_D. The voltage across $C_x(=I_D/(\omega C_x))$, lags I by 90° and this voltage phasor is therefore perpendicular to the current phasor. The phasor diagram for the right-hand arm of the bridge is similarly constructed and is shown in fig. 3.24(*b*). When the two diagrams are superimposed, the points marked D and B must coincide since there is no potential difference across the detector at balance.

3.10.2 The Wien bridge

This bridge (fig. 3.25) is an example of the type for which the balance conditions depend upon frequency. It finds application as a frequency determining network in certain types of oscillator (see reference 5).

Fig. 3.25. The Wien bridge.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

From (3.59) we have

$$Z_1 = \frac{Z_2 Z_3}{Z_4} = Z_2 Z_3 Y_4$$

that is

$$R_1 = R_2\left(R_3 + \frac{1}{j\omega C_3}\right)\left(\frac{1}{R_4} + j\omega C_4\right)$$

$$= \frac{R_2 R_3}{R_4} + \frac{R_2}{j\omega C_3 R_4} + j\omega C_4 R_2 R_3 + \frac{R_2 C_4}{C_3}$$

Hence equating real and imaginary parts:

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} + \frac{C_4}{C_3}$$

and

$$\omega^2 = \frac{1}{R_3 C_3 R_4 C_4}$$

This second balance condition gives, for any fixed set of component values, the frequency at which there will be zero detector signal. At this frequency the phase shift between source and detector signals will be zero, and it is this property of the circuit which is important when it is incorporated in so-called phase-shift oscillator circuits.

3.11 Worked example

The output at the detector terminals of a sensitive bridge circuit using a high impedance detector contains an unwanted sinusoidal signal component at an angular frequency ω_0.

(a) Show that by interposing a twin-T filter, of the type shown in fig. 3.26, between the bridge detector terminals and the detector, the unwanted signal may be eliminated.

(b) If the unwanted signal is caused by inductively coupled pick-up from mains power lines operating at 50 Hz, suggest suitable values for R and C in fig. 3.26.

(c) Sketch the general form of the modulus of the transfer function for the twin-T filter as a function of frequency.

Solution

The twin-T filter is itself a form of bridge circuit in which the balance conditions are frequency dependent. At one particular frequency, therefore, the output V_2 falls to zero.

Several of the methods discussed in Chapter 2 may be used for solving part (a) of this example; from the arguments presented in section 2.11 it will be apparent that mesh analysis is the most cumbersome. Two alternative methods are presented below: one employing nodal analysis, the other the T-π transformation.

(a) *Method 1: nodal analysis*

We may assume that no current is drawn from the output port of the filter since the detector has a high input impedance. The input voltage V_1 supplied by the bridge may, for the purposes of this problem, be regarded as a source voltage of fixed magnitude.

Choosing the reference node as shown, there remain three nodes whose voltages are undetermined. One of these, the node O, is the output node with assigned voltage V_2. The other nodes are P and Q with assigned voltages V_P and V_Q.

At O the nodal equation is

$$\frac{V_2 - V_Q}{R} + \frac{V_2 - V_P}{1/\mathrm{j}\omega C} = 0$$

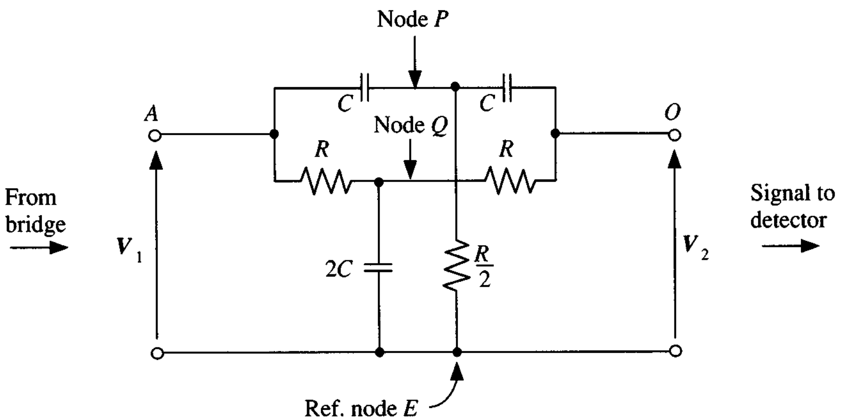
which reduces to

$$\left(\frac{1}{R} + \mathrm{j}\omega C\right) V_2 - \mathrm{j}\omega C\, V_P - \frac{1}{R}\, V_Q = 0 \tag{3.60}$$

The nodal equations at P and Q are

$$\frac{V_P - V_1}{1/\mathrm{j}\omega C} + \frac{V_P - V_2}{1/\mathrm{j}\omega C} + \frac{V_P}{R/2} = 0$$

Fig. 3.26. Circuit for worked example: twin-T filter.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

and

$$\frac{V_Q-V_1}{R}+\frac{V_Q-V_2}{R}+\frac{V_Q}{1/\mathrm{j}\omega 2C}=0$$

which reduce to

$$-\mathrm{j}\omega C V_2+\left(\frac{2}{R}+2\mathrm{j}\omega C\right)V_P+(0)V_Q=\mathrm{j}\omega C V_1 \tag{3.61}$$

and

$$-\frac{1}{R}V_2-(0)V_P+\left(\frac{2}{R}+2\mathrm{j}\omega C\right)V_Q=\frac{V_1}{R} \tag{3.62}$$

Note that the coefficients of V_Q in (3.61) and of V_P in (3.62) are both zero. This is because no mutual element exists between the nodes P and Q. Solving for V_2 using the method of determinants (section 2.5) we obtain

$$V_2=\frac{1}{\Delta}\begin{vmatrix} 0 & -\mathrm{j}\omega C & -\dfrac{1}{R} \\ \mathrm{j}\omega C V_1 & \left(\dfrac{2}{R}+2\mathrm{j}\omega C\right) & 0 \\ \dfrac{V_1}{R} & 0 & \left(\dfrac{2}{R}+2\mathrm{j}\omega C\right) \end{vmatrix} \tag{3.63}$$

where Δ is the determinant formed by the array of coefficients in (3.60), (3.61) and (3.62).

Now at the frequency ω_0 of the unwanted signal $V_2=0$, therefore, expanding the numerator of (3.63) and equating to zero we obtain

$$-\mathrm{j}\omega_0 C V_1\left[-\mathrm{j}\omega_0 C\left(\frac{2}{R}+2\mathrm{j}\omega_0 C\right)\right]+\frac{V_1}{R}\left[-\left(\frac{2}{R}+2\mathrm{j}\omega_0 C\right)\left(-\frac{1}{R}\right)\right]$$
$$=0$$

Equating real (or imaginary) parts of this expression to zero gives, finally,

$$\omega_0=\frac{1}{CR}$$

(a) *Method 2: T-π transformation*

Using relationships analogous to (2.23), each of the two T-connected sections of the filter may be transformed separately to give the corresponding π-configuration as shown in fig. 3.27. These two π-sections may then be combined to form the π-equivalent of the original network. When this is done we see that impedance $(Z_1//Z_4)$ spans the input port and, since the voltage there is constant and equal to V_1, it cannot affect the output in any way. The two remaining arms of the π-section constitute a voltage divider, hence we may derive a general expression from which V_2 may be found at any frequency. However, in this problem we wish to find only the frequency at which V_2 becomes zero. This will occur when $Z_2//Z_5 = Z_2Z_5/(Z_2+Z_5)$ becomes infinite, that is when (Z_2+Z_5) tends to zero.

From (2.23) and fig. 2.17

$$Z_2 = \frac{1}{j\omega C} + \frac{1}{j\omega C} + \frac{(1/j\omega C)(1/j\omega C)}{(R/2)} = \frac{2}{j\omega C} - \frac{2}{\omega^2 C^2 R}$$

$$Z_5 = R + R + \frac{(R)(R)}{(1/2j\omega C)} = 2R + 2j\omega CR^2$$

Therefore V_2 is zero at the frequency of the unwanted signal ω_0 when

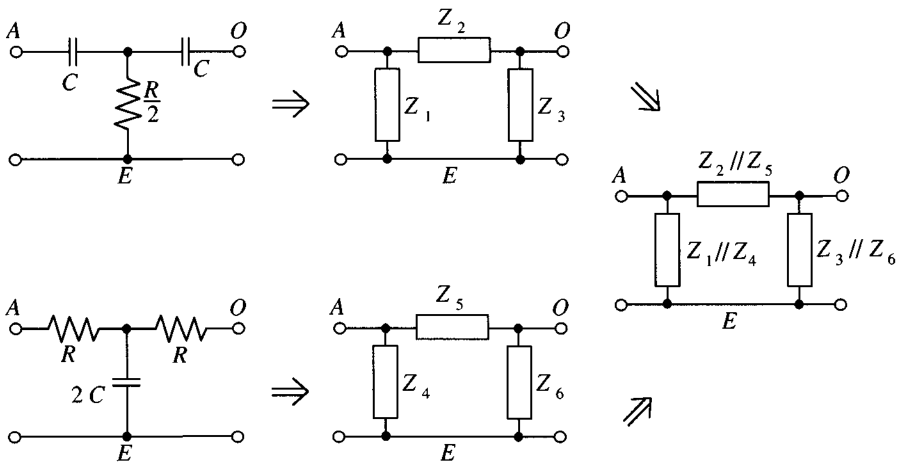
$$\frac{2}{j\omega_0 C} - \frac{2}{\omega_0{}^2 C^2 R} + 2R + 2j\omega_0 CR^2 = 0$$

which gives, on equating real or imaginary parts to zero,

$$\omega_0 = 1/CR$$

(b) At the frequency of the mains power lines, $\omega_0 = 2\pi \times 50 = 314$, therefore, the product $CR = 1/314$. The actual values of C and R are chosen on practical grounds: capacitors with large stable values of capacitance (with

Fig. 3.27. The T-π transformation applied to the twin-T filter of fig. 3.26; by symmetry, $Z_1 = Z_3$ and $Z_4 = Z_6$.



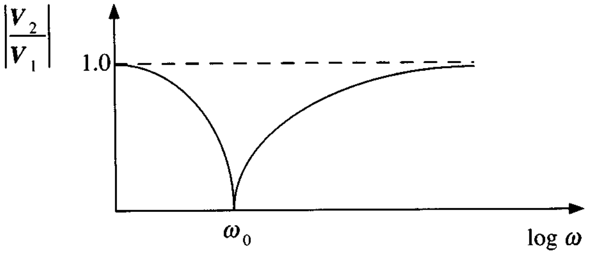
https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

the requisite low leakage resistance necessary for this application) are bulky and expensive; very small values of C would make the circuit too sensitive to stray capacitances between components. With these considerations in mind a suitable value for C would be 0.1 μF, giving $R = 33$ kΩ. In practice R would be adjusted to produce the precise CR product required by using shunt or series trimming resistors.

(c) The general form of the modulus of the transfer function $|V_2/V_1|$, may be found without calculation by considering the extremes of the frequency range; from $\omega = 0$ (d.c.) to $\omega \to \infty$. At $\omega = 0$, all the capacitances are effectively open circuit and the output node O is connected to the input node A through series resistance $2R$. On the assumption that no current is drawn from node O, there can be no voltage drop across this resistance, therefore, the input and output voltages are the same and $|V_2/V_1| = 1$. Similar considerations apply for $\omega \to \infty$: in this case the capacitances act as short circuits thus connecting input and output terminals directly together. We conclude that the plot of $|V_2/V_1|$ versus ω must take the general form shown in fig. 3.28; the curve being asymptotic to the value unity for high frequencies. Because of the characteristic shape of the transfer function, the term *notch-filter* is often used in connection with this circuit.

It will be obvious that this circuit can be effective only if the wanted bridge frequency is well separated from the notch frequency. Typically bridge circuits operate at 1 kHz or so and the notch filter frequency is arranged to be at 50 Hz. The reactance of the capacitive elements in the circuit are therefore some twenty times smaller at the wanted frequency than at the notch frequency, consequently, the attenuation produced by the filter at the bridge frequency is small. The reader may care to confirm that for the filter in this example. $|V_2/V_1| = 0.98$ at 1 kHz.

Fig. 3.28. Modulus of the transfer function, as a function of frequency, for the circuit of Fig. 3.26.



3.12 Inductively coupled circuits

The a.c. voltage–current relationships for a mutual inductance may be readily established using the methods of section 3.3. We take as our starting point the general relationship (1.49) between the instantaneous voltage v_1 and currents i_1 and i_2 indicated in fig. 3.29(a), namely,

$$v_1 = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{1.49}$$

It will be recalled (section 1.10) that the ($\pm$) signs in the second term of this equation arise because of the two alternative ways of connecting the coils: either with fluxes aiding ((+) sign) or fluxes opposing (($-$) sign).

Since all currents and voltages are sinusoidal they may be represented in complex exponential form by:

$$v_1 = \boldsymbol{V}_{1\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t}; \qquad i_1 = \boldsymbol{I}_{1\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t}; \qquad i_2 = \boldsymbol{I}_{2\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t}$$

Substituting in (1.49) and taking the real part as understood, we obtain

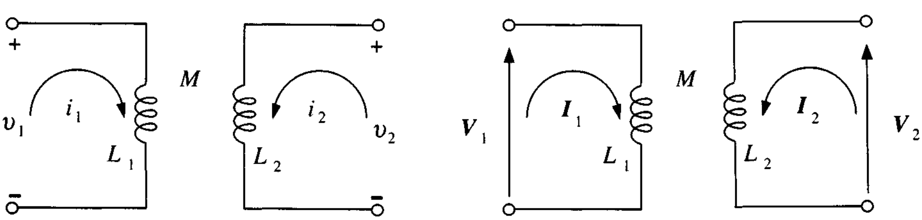
$$\boldsymbol{V}_{1\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t} = L_1\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{I}_{1\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t}) \pm M\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{I}_{2\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t})$$

or

$$\boldsymbol{V}_{1\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t} = \mathrm{j}\omega L_1\boldsymbol{I}_{1\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t} \pm \mathrm{j}\omega M\boldsymbol{I}_{2\mathrm{m}}\mathrm{e}^{\mathrm{j}\omega t}$$

By cancelling out $\mathrm{e}^{\mathrm{j}\omega t}$ from each term in this expression, and dividing throughout by $\sqrt{2}$, we obtain the required relationship in terms of stationary phasors and r.m.s. magnitudes:

Fig. 3.29. Voltage–current relationships for mutual inductance.



$$v_1 = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M\frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$v_2 = L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \pm M\frac{\mathrm{d}i_1}{\mathrm{d}t}$$

(a) Instantaneous values

$$\boldsymbol{V}_1 = \mathrm{j}\,\omega\, L_1\boldsymbol{I}_1 \pm \mathrm{j}\,\omega M\boldsymbol{I}_2$$

$$\boldsymbol{V}_2 = \mathrm{j}\,\omega\, L_2\boldsymbol{I}_2 \pm \mathrm{j}\,\omega M\boldsymbol{I}_1$$

(b) Phasors

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$V_1 = j\omega L_1 I_1 \pm j\omega M I_2 \tag{3.64}$$

A similar expression for the voltage V_2 is obtained:

$$V_2 = j\omega L_2 I_2 \pm j\omega M I_1 \tag{3.65}$$

These results are illustrated in fig. 3.29(*b*).

The analysis of circuits containing mutual inductance is almost invariably carried out using the method of mesh analysis rather than nodal analysis because of the difficulty of defining reference nodes and node potentials in circuits containing separate and distinct parts coupled only by mutual inductance. For a similar reason the relationship (2.31), relating numbers of nodes and elements in a circuit, does not apply directly, although it may be extended to include circuits containing separate parts (see reference 4).

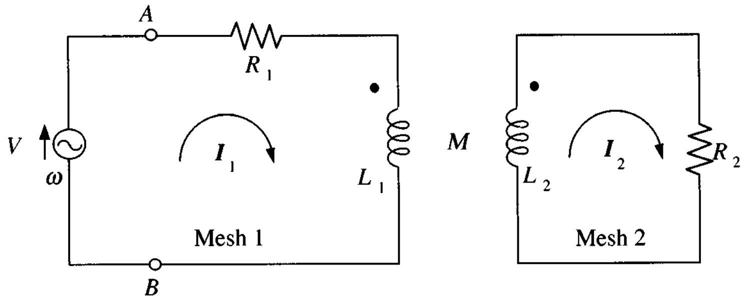
The application of (3.64) and (3.65), and the general method of analysis of a.c. circuits containing mutual inductance, may be illustrated with reference to the circuit shown in fig. 3.30. This consists of two separate meshes linked only by mutual inductance; corresponding ends of the coils are indicated by the dots (see section 1.10). Currents I_1 and I_2 are assigned to meshes (1) and (2) as shown, and these have been chosen in a clockwise direction.

For mesh (1) the self-impedance is $(R_1 + j\omega L_1)$, and the voltage drop is $(R_1 + j\omega L_1)I_1$; to this we must add, according to (3.64), the mutual inductance term $j\omega M I_2$. Hence the mesh equation is

$$(R_1 + j\omega L_1)I_1 - j\omega M I_2 = V \tag{3.66}$$

The sign of the mutual inductance term follows from the dot convention discussed in section 1.10: if one assigned current *enters* a dotted end and the other *leaves* a dotted end, fluxes oppose and the mutual inductance term is negative. The voltage on the right-hand side of (3.66) takes a (+) sign

Fig. 3.30. Mesh analysis of mutually-coupled circuits.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

because the assigned current I_1 leaves the voltage source in the direction of the reference polarity arrow (working rule 2, section 2.3).

For mesh (2) the voltage drop across the self-impedance is $(R_2+j\omega L_2)I_2$, and the mutual inductance term is $-j\omega MI_1$. There is no voltage source in this mesh so that the equation is

$$-j\omega MI_1+(R_2+j\omega L_2)I_2=0 \tag{3.67}$$

Notice that the mutual inductance term in this equation is treated as a voltage drop (in conformity with (3.65)) and not as a source voltage, even though it is the e.m.f. due to the mutual inductance effect that causes current to circulate in mesh (2). It is clear from the circuit model discussed in section 1.10.1, and illustrated in fig. 1.26, that it would be equally valid to write (3.67), *ab initio*, as

$$(R_2+j\omega L_2)I_2=j\omega MI_1$$

where the term on the right is regarded as a source of voltage. However, the formulation (3.67) is more consistent with the equations of mesh analysis developed in chapter 2.

We shall now use (3.66) and (3.67) to find the impedance of the complete circuit at the terminals AB, as seen from the voltage source. This impedance, called the *driving point impedance*, is discussed in more general terms in chapter 8.

From (3.67)

$$I_2=\frac{j\omega MI_1}{R_2+j\omega L_2}$$

and substituting in (3.66) we obtain

$$(R_1+j\omega L_1)I_1-\frac{j^2\omega^2M^2I_1}{R_2+j\omega L_2}=V$$

hence

$$I_1\left[R_1+j\omega L_1+\frac{\omega^2M^2}{R_2+j\omega L_2}\right]=V$$

The impedance at AB is then given by

$$Z_{AB}=\frac{V}{I_1}=R_1+j\omega L_1+\frac{\omega^2M^2}{R_2+j\omega L_2}$$

Rationalizing the last term in this expression we obtain

$$Z_{AB}=R_1+j\omega L_1+\frac{\omega^2M^2R_2-j\omega(\omega^2M^2L_2)}{R_2{}^2+(\omega L_2)^2}$$

or

$$Z_{AB}=\left[R_1+\frac{\omega^2 M^2 R_2}{R_2{}^2+(\omega L_2)^2}\right]+\mathrm{j}\omega\left[L_1-\frac{\omega^2 M^2 L_2}{\mathrm{R}_2+(\omega L_2)^2}\right] \tag{3.68}$$

Examination of (3.68) shows that if the mutual inductance between the two meshes is negligibly small the driving point impedance becomes identical to that of mesh (1) alone, that is, $(R_1 + \mathrm{j}\omega L_1)$. With increasing M, the effective resistance, as seen at the terminals AB (first term in brackets), is increased while the effective inductance (second term in brackets) is decreased. This result is unaffected by the relative winding directions of the two coils since the final expression contains M only as a squared term; the sign associated with M in the original mesh equations is therefore immaterial.

Another frequently encountered network configuration involving mutual inductance is shown in fig. 3.31. In this the mutually coupled coils are connected in series. Taking voltage drops in sequence round the circuit, and with due regard for the dot convention, we obtain

$$\mathrm{j}\omega L_1\boldsymbol{I}-\mathrm{j}\omega M\boldsymbol{I}+\mathrm{j}\omega L_2\boldsymbol{I}-\mathrm{j}\omega M\boldsymbol{I}=\boldsymbol{V} \tag{3.69}$$

or

$$\mathrm{j}\omega(L_1+L_2-2M)\boldsymbol{I}=\boldsymbol{V}$$

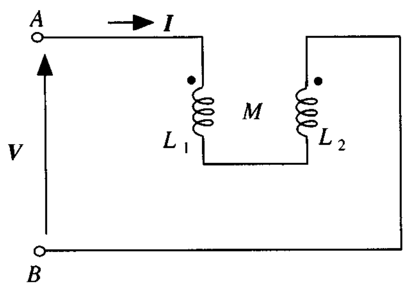
This shows that the effective driving point impedance at AB is

$$\boldsymbol{Z}_{AB}=\mathrm{j}\omega(L_1+L_2-2M) \tag{3.70}$$

That is, the effective inductance is $(L_1 + L_2 - 2M)$. If the connections to one of the coils are reversed, the signs of the mutual inductance terms in (3.69) become positive and the effective inductance is then $(L_1 + L_2 + 2M)$.

These relationships provide the basis of a method for determining the mutual inductance between two coupled coils. Measurements are made, usually by means of an a.c. bridge method, of the effective inductance of

Fig. 3.31. Series-connected coupled coils.



coils connected in series in the two alternative circuit configurations; first, with fluxes aiding and then with fluxes opposing. If these measurements yield effective values of, say, L_3 and L_4 ($L_3 > L_4$), then we have, according to the above theory,

$$L_3 = L_1 + L_2 + 2M$$

and

$$L_4 = L_1 + L_2 - 2M$$

Combining these two equations gives

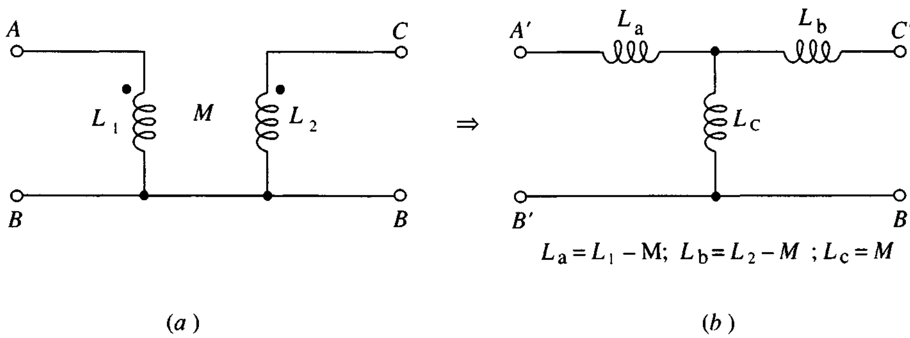
$$4M = L_3 - L_4$$

hence, M is determined in terms of the measured values.

We complete this discussion of inductively coupled circuits by considering a useful circuit transformation that resembles in some ways the star–delta (T-π) transformation discussed previously. This is the so-called T-equivalent of two coupled coils and is illustrated in fig. 3.32. Fig. 3.32(a) shows two coupled coils, similar to those shown in fig. 3.29, but with one corresponding end of each coil joined to form a common connection (terminal B). The circuit configuration shown in fig. 3.32(a) can be arranged to be electrically equivalent to fig. 3.32(b), so far as any external connections are concerned, by choosing suitable values for the inductances L_a, L_b and L_c. The utility of this transformation lies in the fact that the inductances in fig. 3.32(b) are separate and distinct having no mutual coupling between them, and this can lead to analytical simplification in some circuits of practical interest.

To obtain the connecting relationships for the two circuits we consider the inductance that appears at corresponding terminal pairs (the remaining

Fig. 3.32. The T-equivalent of two coupled coils.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

terminal being open circuit). For equivalence we therefore require the same inductances at AB and $A'B'$, hence,

$$L_1 = L_a + L_c$$

Similarly at CB and $C'B'$

$$L_2 = L_b + L_c$$

Also, using similar arguments to those leading to (3.70), we have at AC and $A'C'$

$$L_1 + L_2 - 2M = L_a + L_b$$

Combining and rearranging these equations we obtain:

$$L_a = L_1 - M; \qquad L_b = L_2 - M; \qquad L_c = M \tag{3.71}$$

If non-corresponding ends are joined at B, the connecting equations become:

$$L_a = L_1 + M; \qquad L_b = L_2 + M; \qquad L_c = -M \tag{3.72}$$

A final point to note is that although the circuit transformation illustrated in fig. 3.32 has here arisen in connection with a.c. circuits, it is valid for source excitations of any form.

3.13 Resonant circuits

3.13.1 Losses in inductors and capacitors

(a) *Inductors*

A practical inductor consists typically of a length of wire wound into the form of a coil to enhance the self flux linking effect and therefore the inductance. The wire will possess some resistance so that it is natural to think of an inductor as having a lumped linear circuit model of the form shown in fig. 3.33(a). It might be supposed that the resistance r could be regarded as a fixed characteristic parameter, but it is found that to describe satisfactorily the electrical behaviour of an inductor, it is necessary to adjust the value of r according to the frequency at which the inductor is to operate. The reason for this is that the power loss in an inductor arises with increasing frequency because of eddy currents induced in the wire itself. The net effect of this is to cause the current in the wire to concentrate near the surface, producing the so-called 'skin effect' and thereby increasing the conduction loss (see reference 6).

Eddy currents induced in any conducting material in proximity to the

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

inductor, for example a screening can, will also contribute to the losses, as will hysteresis losses in any magnetic materials that may be present. All these sources of loss affect the value of r to be ascribed to the circuit model.

The effective inductance will also vary with the frequency because of stray capacitances between turns of the coil. We must conclude therefore that the constants L_s and r in the circuit model of fig. 3.33(a) are true only for one particular frequency, or at best for a narrow band of frequencies.

We shall see later in our study of resonant circuits that there are good reasons for making inductors as loss free as possible; in terms of the circuit model, this implies that for a given value of L_s, r should be as small as possible. We find it convenient therefore to take as a measure of the excellence of an inductor the dimensionless ratio $\omega L_s/r$, where L_s and r are values appropriate to the particular angular frequency ω at which the inductor is to be operated. This ratio is termed the quality factor or Q-factor of the inductor. Thus, by definition,

$$Q=\frac{\text{Reactance}}{\text{Series resistance}}\equiv\frac{\omega L_s}{r} \tag{3.73}$$

At radio frequencies the Q-factor of an inductor may typically be of the order of 100, and when Q is this large, certain approximations may be made in the theory of resonant circuits which greatly simplifies their analysis. Furthermore, we are generally concerned with only narrow bands of frequency which means that Q can be regarded as being substantially constant.

Although the series circuit model shown in fig. 3.33(a) is the most natural representation for an inductor, it is often analytically convenient to use the parallel representation shown in fig. 3.33(b). The series and parallel representations will both uniquely describe the electrical behaviour of a linear inductor at one particular frequency if they both present the same complex impedance at their terminals at that frequency. The transformation relations may therefore be derived by equating complex impedances.

Fig. 3.33. Lumped circuit models for an inductor.

(a) Series model (b) Parallel model

For fig. 3.33(*a*) the impedance is $r+\mathrm{j}\omega L_s$; for fig. 3.33(*b*) the impedance is $\mathrm{j}\omega L_p//R$, thus,

$$r+\mathrm{j}\omega L_s=\frac{\mathrm{j}\omega L_p R}{R+\mathrm{j}\omega L_p}$$

Rationalizing the right-hand side of this expression gives

$$\begin{aligned}r+\mathrm{j}\omega L_s&=\frac{\mathrm{j}\omega L_p R(R-\mathrm{j}\omega L_p)}{R^2+(\omega L_p)^2}\\&=\frac{\omega^2 L_p{}^2 R}{R^2+(\omega L_p)^2}+\mathrm{j}\omega\frac{L_p R^2}{R^2+(\omega L_p)^2}\end{aligned}$$

and by equating real and imaginary parts we obtain

$$r=\frac{\omega^2 L_p{}^2 R}{R^2+(\omega L_p)^2} \tag{3.74}$$

and

$$\omega L_s=\frac{\omega L_p R^2}{R^2+(\omega L_p)^2} \tag{3.75}$$

Dividing (3.75) by (3.74) gives

$$\frac{\omega L_s}{r}=\frac{R}{\omega L_p}=Q \tag{3.76}$$

Equation (3.76) provides an alternative definition of the Q-factor: the ratio of the equivalent parallel resistance of the inductor to its equivalent parallel reactance.

Now (3.74) and (3.75) may be re-written in terms of Q, thus,

$$r=\frac{R}{\dfrac{R^2}{\omega^2 L_p{}^2}+1}=\frac{R}{Q^2+1}\simeq\frac{R}{Q^2}\qquad(Q^2\gg 1) \tag{3.77}$$

and

$$\omega L_s=\frac{\omega L_p}{1+\dfrac{\omega^2 L_p{}^2}{R^2}}=\frac{\omega L_p}{1+\dfrac{1}{Q^2}}\simeq\omega L_p\qquad(Q^2\gg 1) \tag{3.78}$$

We see from these relations that, in the transformation from series to parallel equivalent form, the inductance remains substantially unchanged but that the resistance is altered by a factor equal to Q^2. Hence we may write, to a very good approximation,

$$L_s = L_p = L \tag{3.79}$$

and

$$R = Q^2 r \tag{3.80}$$

(b) *Capacitors*

Power loss in a capacitor arises because of leakage and other effects in the dielectric between the plates and because of resistance in the plates themselves. To account for leakage, it is natural to model the capacitor by the parallel combination of fig. 3.34(*a*). However, as for the inductor, losses from all sources can be accounted for, at one frequency, by either a series or a parallel model. A similar procedure to that carried out above for the inductor leads to the following relations:

$$Q = \frac{\text{Reactance}}{\text{Series resistance}} = \frac{1/\omega C_s}{r} \tag{3.81}$$

$$= \frac{R}{1/\omega C_p} \left(\frac{\text{Parallel resistance}}{\text{Reactance}} \right)$$

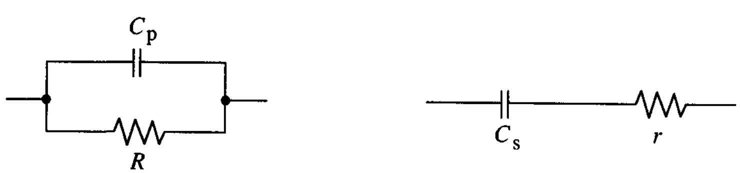
$$C_s = C_p = C \tag{3.82}$$

and

$$R = Q^2 r \tag{3.83}$$

The quality of a capacitor can also be expressed in terms of the phase angle ϕ between current and applied voltage (fig. 3.35). For a pure capacitance the phase angle is 90° but any losses cause a reduction in the phase angle by a small amount (angle δ_d in the diagram). It is usual to take the cosine of the phase angle as a suitable measure of the losses since this is

Fig. 3.34. Lumped circuit models for a capacitor.



(*a*) Parallel model (*b*) Series model

the factor by which the product VI must be multiplied in order to calculate the power dissipated in the capacitor. We call this the *power factor* (see section 4.4).

Figs. 3.35(b) and (c) show the phasor relationships appropriate to the parallel circuit model of fig. 3.35(a). The two components of the total current I are $I_R(=V/R)$ flowing through R, and $I_C(=V/(1/\omega C))$ flowing through C. Hence,

$$\text{Power factor} = \cos\phi = \frac{V/R}{\sqrt{[(V/R)^2 + (\omega CV)^2]}} = \frac{1}{\sqrt{[1 + (\omega CR)^2]}} \tag{3.84}$$

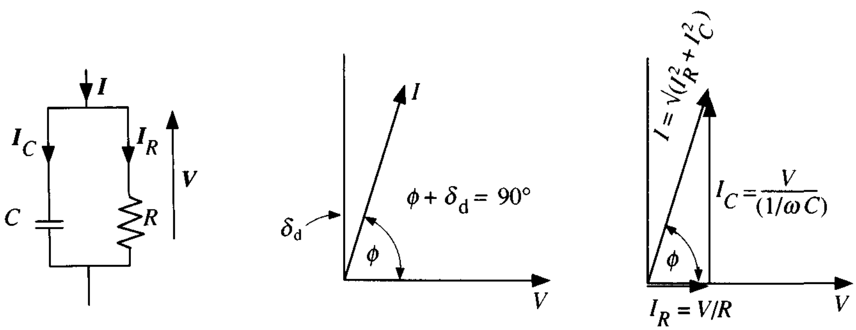
But, for a good quality capacitor, $\omega CR \gg 1$, therefore,

$$\text{Power factor} = \cos\phi = \frac{1}{\omega CR} = \frac{1}{Q} \tag{3.85}$$

We see that there is a simple reciprocal relationship between power factor and Q-factor. Precisely the same relationship may be derived on the basis of the series circuit model of fig. 3.34(b).

In the type of capacitor commonly used in resonant circuits at radio frequencies, the losses occur mainly in the dielectric material of the capacitor; the greater these dielectric losses, the greater the angle δ_d in fig. 3.35(b). Consequently, the angle δ_d is sometimes referred to as the *loss-angle* of the dielectric. This loss-angle can be determined for a dielectric material by making measurements on a capacitor containing the dielectric using an a.c. bridge technique. By this means the parameters R and C are found at a particular operating frequency ω and the loss-angle may then be calculated from the expression

Fig. 3.35. Phasor relationships for the parallel circuit model of a capacitor. The power factor of the capacitor is $\cos\phi$, and the loss angle is δ_d.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$\tan\delta_d = \frac{1}{\omega CR} \tag{3.86}$$

This expression is readily derived by considering the geometry of the phasor diagrams in fig. 3.35.

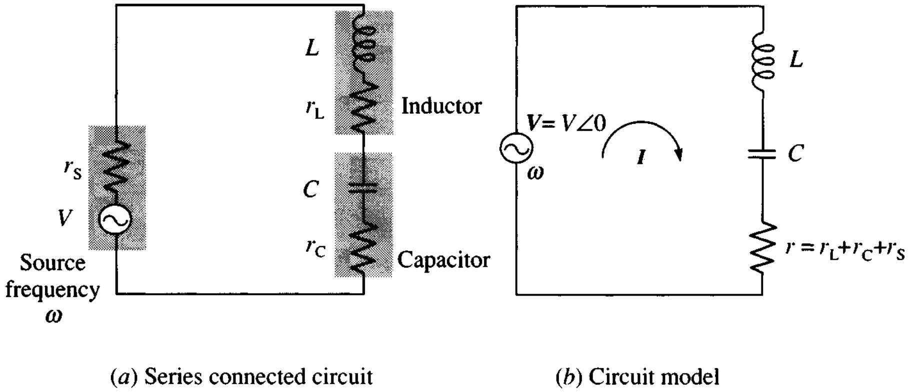
3.13.2 The series resonant circuit

When the frequency at which a circuit is excited is such that the reactances of individual inductive and capacitive elements are comparable, the branch impedances of the circuit change rapidly with change of frequency and so do the currents and voltages. The phenomena that are exhibited when this takes place are classed under the general heading of resonance, and the frequencies at which these phenomena occur are called the resonant frequencies. Of the many forms of resonant circuit the simplest is the series resonant circuit shown in fig. 3.36(*a*). This comprises an inductor, a capacitor and a practical voltage source, all connected in series.

The losses of the inductor and capacitor are represented by r_L and r_C respectively, while r_S includes the resistance of the source together with any other resistances that may have been included in the circuit for any purpose. (The current in a series resonant circuit may be monitored, for example, by using an oscilloscope to detect the voltage developed across a small-value resistor connected into the circuit.) For the purposes of analysis, the separate resistances may be represented by a single equivalent resistance r as shown in fig. 3.36(*b*).

The impedance of the circuit is $r + j(\omega L - 1/\omega C)$ hence the current is given by

Fig. 3.36. The series resonant circuit.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$I = \frac{V}{r + j\left(\omega L - \frac{1}{\omega C}\right)} \tag{3.87}$$

The magnitude of the current is therefore

$$I = \frac{V}{\sqrt{\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}} \tag{3.88}$$

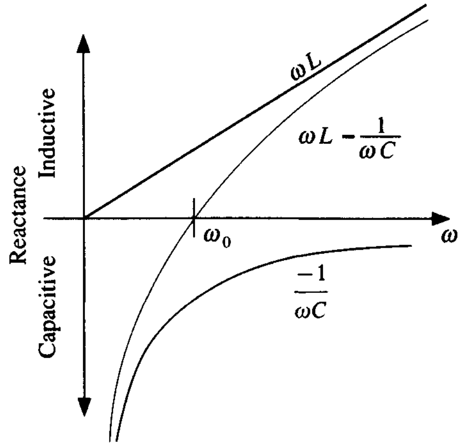
and the phase angle ϕ of the current with respect to the voltage is given by

$$\phi = -\tan^{-1}\frac{(\omega L - 1/\omega C)}{r} \tag{3.89}$$

Let us now consider the situation where the voltage of the source is maintained at a constant value whilst the angular frequency is varied. Fig. 3.37 illustrates how the inductive and capacitive reactances, and their difference, vary with frequency. Clearly there is an angular frequency $\omega = \omega_0$ at which the total reactance of the circuit is zero. This occurs when

$$\omega_0 L = \frac{1}{\omega_0 C}$$

Fig. 3.37. The reactance of a series resonant circuit as a function of angular frequency ω; ω_0 is the frequency at which *current resonance* occurs.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

or

$$\omega_0 = \frac{1}{\sqrt{(LC)}} \tag{3.90}$$

At this frequency the circuit is purely resistive and the current has a maximum value $I_0 = V/r$. The circuit is said to exhibit *current resonance* and $\omega_0/2\pi$ is called the *resonant frequency*. (We shall see later that other definitions for the resonant frequency of a series circuit are possible.)

The magnitude and phase of the current, given respectively by (3.88) and (3.89), are shown plotted as functions of angular frequency (in the vicinity of ω_0) in fig. 3.38.

At frequencies below resonance the capacitive reactance is greater than the inductive reactance and the circuit is therefore predominantly capacitive. In consequence the current leads the source voltage, the angle of lead increasing asymptotically to the value $\pi/2$ with decreasing frequency. Above resonance the circuit is inductive and the current lags the source voltage.

The shape of the resonance curves shown in fig. 3.38 are governed by the relative values of the reactance and the resistance of the circuit and we now define a quality factor Q_0, applicable to the complete resonant circuit, which enables us to describe in a succinct fashion the shapes of the resonance curves and other properties characteristic of the resonant circuit. The quality factor Q_0 of the *complete* circuit is defined by:

$$Q_0 = \frac{\omega_0 L}{r} \equiv \frac{1}{\omega_0 C r} \tag{3.91}$$

Now $r = r_S + r_L + r_C$, therefore,

$$Q_0 = \frac{\omega_0 L}{r_S + r_L + r_C} = \frac{1/\omega_0 C}{r_S + r_L + r_C} \tag{3.92}$$

and

$$\frac{1}{Q_0} = \frac{r_S}{\omega_0 L} + \frac{r_L}{\omega_0 L} + \frac{r_C}{1/\omega_0 C}$$

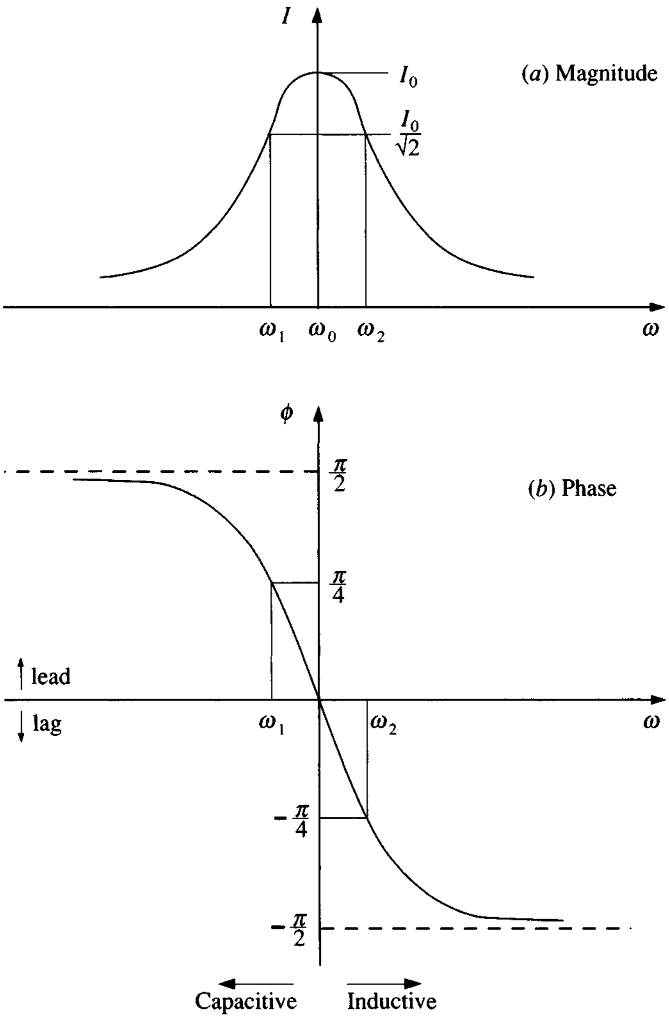
that is

$$\frac{1}{Q_0} = \frac{1}{Q_S} + \frac{1}{Q_L} + \frac{1}{Q_C} \tag{3.93}$$

where $Q_S = \omega_0 L/r_S$, and Q_L and Q_C are the quality factors of the inductor and capacitor respectively at the resonant frequency. Thus, given the quality factors of each component separately, and a knowledge of the source resistance, the quality factor of the complete circuit may be found using (3.93). In most practical resonant circuits the quality factor of the capacitor is much greater than that of the inductor and Q_0 depends mainly on Q_S and Q_L. It must be remembered that although the quality factor for the complete circuit is defined strictly at the resonant frequency, it is substantially constant over the narrow band of frequencies with which we are concerned in the theory of resonant circuits.

We now take as a measure of the sharpness of the resonance peak (that is,

Fig. 3.38. Variation of magnitude and phase of the current in a series resonant circuit in the vicinity of resonance.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

the sensitivity of the current amplitude to changes in frequency above and below ω_0), the increment of frequency between the two points at which the amplitude of the resonance curve (fig. 3.38(*a*)) falls to $1/\sqrt{2}$ of its peak value. The frequencies at these two points are designated ω_1 and ω_2.

At these frequencies the power dissipated in r is just half the power dissipated when $\omega=\omega_0$, and they are referred to as the *half-power frequencies.* The frequency increment $\omega_2-\omega_1$ is called the *half-power* bandwidth or, simply, the *bandwidth.*

Rewriting (3.88) we obtain

$$I=\frac{V/r}{\left[1+\left(\dfrac{\omega L-\dfrac{1}{\omega C}}{r}\right)^2\right]^{\frac{1}{2}}}=\frac{I_0}{\left[1+\left(\dfrac{\omega L-\dfrac{1}{\omega C}}{r}\right)^2\right]^{\frac{1}{2}}} \tag{3.94}$$

But at the half-power frequencies $I/I_0=1/\sqrt{2}$ thus, the denominator in (3.94) is equal to $\sqrt{2}$ and it follows that

$$\omega_2 L-\frac{1}{\omega_2 C}=r$$

or

$$\omega_2{}^2-\frac{r}{L}\omega_2-\frac{1}{LC}=0$$

Similarly

$$\frac{1}{\omega_1 C}-\omega_1 L=r$$

or

$$\omega_1{}^2+\frac{r}{L}\omega_1-\frac{1}{LC}=0$$

Hence

$$\omega_1=-\frac{r}{2L}+\sqrt{\left[\left(\frac{r}{2L}\right)^2+\frac{1}{LC}\right]} \tag{3.95}$$

$$\omega_2=\frac{r}{2L}+\sqrt{\left[\left(-\frac{r}{2L}\right)^2+\frac{1}{LC}\right]} \tag{3.96}$$

Therefore, in terms of the angular frequency, the bandwidth is given by

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$\omega_2 - \omega_1 = \frac{r}{L} = \omega_0 \frac{r}{\omega_0 L} = \frac{\omega_0}{Q_0} \tag{3.97}$$

In terms of the resonant frequency $f_0 = \omega_0/2\pi$:

$$\text{Bandwidth} = \frac{f_0}{Q_0}\,\text{Hz} \tag{3.98}$$

A circuit having a large Q-factor (described as a high-Q circuit) has a small bandwidth; for a low-Q circuit the bandwidth is large. In fig. 3.38(*a*) the resonance curve is shown as being substantially symmetrical about ω_0, but detailed examination of (3.88) shows that the curve starts at the origin with zero amplitude and falls to zero again only at a theoretically infinite frequency. Considered over all frequencies the resonance curve is not therefore symmetrical, although near resonance the slight departure from symmetry is not normally noticeable.

Multiplying (3.95) by (3.96) we obtain, after some manipulation and reduction,

$$\omega_1 \omega_2 = \omega_0{}^2 \tag{3.99}$$

Thus, we see that the resonant frequency is the geometric mean of the half-power frequencies, and it follows that $(\omega_0 - \omega_1) \neq (\omega_2 - \omega_0)$. However, as the bandwidth decreases with increasing Q, the geometric mean and the arithmetic mean approach the same value; for most practical purposes the half-power frequencies can be considered as being symmetrically disposed about ω_0.

As indicated in fig. 3.38(*b*), at the half-power frequencies the phase of the current with respect to voltage is just $\pm\pi/4$. This follows from (3.89) and the fact that at these frequencies the net reactance is equal to the resistance.

An important property of the resonant circuit arises from the fact that the voltages developed individually across the inductor and the capacitor, at or near resonance, can be many times larger than the voltage of the source itself. (The *sum* of the voltages across the capacitance and inductance in the circuit of fig. 3.36(*b*) is, of course, zero at resonance since these voltages are of equal magnitude and opposite in phase.) An expression for the voltage V_C across the capacitor at resonance may be obtained by taking the product of the magnitude of the current at resonance and the impedance of the capacitor, that is,

$$V_C = \frac{V}{r}\sqrt{\left[r_C{}^2 + \left(\frac{1}{\omega_0 C}\right)^2\right]}$$

$$\approx \frac{V}{r\omega_0 C} \text{ for } \frac{1}{\omega_0 C} \gg r_C$$

or

$$V_C = VQ_0 \tag{3.100}$$

We see that the magnitude of the capacitor voltage is approximately Q_0 times the source voltage. For this reason Q_0 is also known as the *circuit magnification factor.*

When energy is taken from a mains power line, which represents a source of practically zero resistance, a resonant circuit can produce dangerously high voltages. Suppose, for example, $V=240$ volts, $r=40$ ohms, and $f=50$ hertz ($\omega = 2\pi \times 50 = 314$ radians/s), $L=1$ henry: a capacitor of about 10 microfarads will cause resonance. The current at resonance is $240/40 = 6$ amps, and the reactance of the capacitor is $1/(314 \times 10^{-5} = 318$ ohms. The voltage across the capacitor is then approximately $6 \times 318 = 1908$ volts. The same voltage appears across the inductor.

The situation is different when the source of energy has an appreciable resistance. A common type of low-frequency laboratory oscillator has an internal resistance of 600 ohms and a maximum output of about 30 volts. Even with a short circuit across its terminals this device can supply only about 0.05 amps. So for the same circuit and frequency considered in the preceding paragraph, the capacitor and inductor voltages could not exceed about $318 \times 0.05 = 16$ volts.

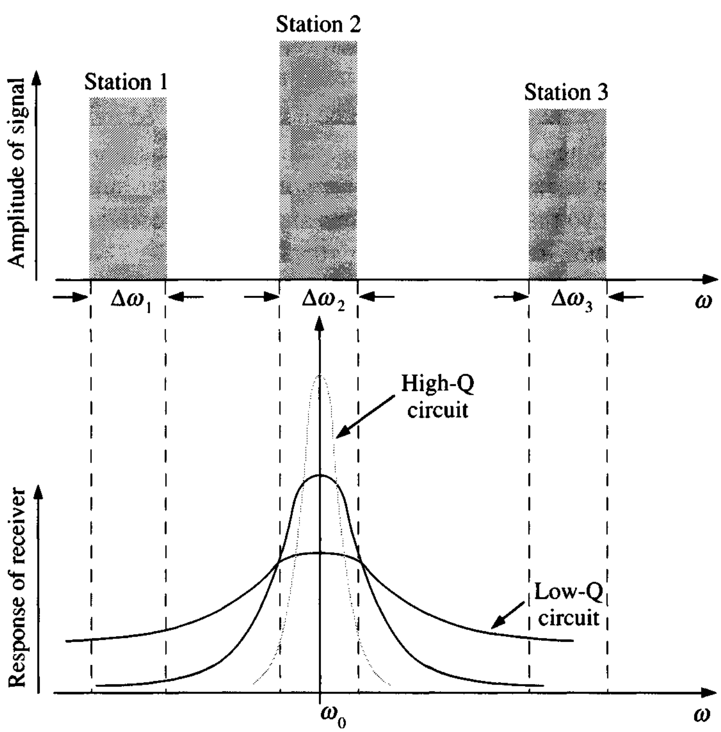
The frequency selective and magnification properties of resonant circuits play an important part in all forms of telecommunications equipment, and we have seen that these properties can be neatly described by means of the quality factor. When, for instance, a resonant circuit is used in a broadcast receiver to select one station from among many others, it is important to use a high-Q circuit so that adjacent stations are not also received at the same time. This situation is illustrated schematically in fig. 3.39. Each broadcasting station transmits information over a narrow band of frequencies $\Delta\omega$ (typically 6 kHz on the medium wave a.m. broadcasting band of 0.5–1.5 MHz).

Let us suppose that the resonant circuit of a receiver is 'tuned' so that its resonant frequency ω_0 coincides with the centre frequency of the $\Delta\omega_2$ band transmitted by station 2. Using a high-Q circuit means that the response to stations 1 and 3 will be negligible because these frequency bands coincide with parts of the resonance curve where the response has fallen practically to zero. A low-Q circuit, on the other hand, will be unselective and allow stations 1 and 3 to be received also. A high quality factor is, therefore, a desirable property for this type of application, but it must also be remembered that if the resonant circuit is given too high a quality factor (dotted line in fig. 3.39), wanted frequencies in the selected signal will be attenuated and information will be lost. It should be stressed that the situation shown in fig. 3.39 is highly simplified; in practice not one but several resonant circuits would be used in a broadcast receiver, each tuned to a slightly different frequency so as to achieve an overall response tailored to meet the requirements of a particular broadcasting band.

So far we have considered the criterion for resonance in a series circuit as being the frequency for which the current rises to its maximum value. However, in many cases it is the voltage developed across the capacitor that is of interest. At first sight it might be expected that since the capacitor voltage is the product of current and impedance, this too would rise to a maximum at the same frequency ω_0 given by (3.90). We now show that this is very nearly but not quite the case.

To find the critical frequency at which the voltage V_C across the capacitor reaches its maximum value we express V_C as a function of ω, making the simplifying assumption that the losses in C are small and that r_C can therefore be neglected. We then have

Fig. 3.39. Use of a resonant circuit in broadcast reception.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$V_C = I\frac{1}{\omega C} = \frac{V}{\sqrt{\left[r^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2\right]}} \cdot \frac{1}{\omega C}$$

This is a maximum when

$$\omega^2 C^2 \left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right] \tag{3.101}$$

is a minimum. Expanding (3.101), differentiating with respect to ω, and equating to zero; we find that V_C is a maximum when $\omega = \omega_0{}'$ such that

$$\omega_0{}' = \frac{1}{\sqrt{LC}}\sqrt{\left(1 - \frac{r^2 C}{2L}\right)}$$

or

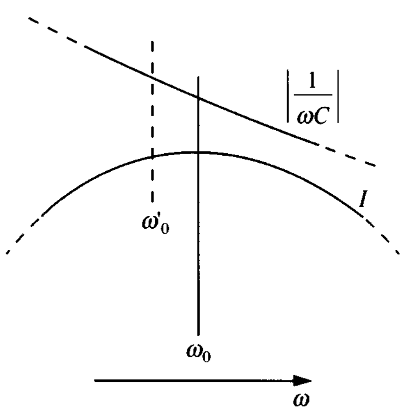
$$\omega_0{}' = \omega_0 \sqrt{\left(1 - \frac{1}{2Q_0{}^2}\right)} \tag{3.102}$$

It is clear from this expression that, if Q is large ($Q \geqslant 10$), ω_0 and ω_0 are very nearly the same.

The reason for this slight difference between the values of ω corresponding to maxima of I and V_C will be apparent if we consider the shape of the resonance curve of fig. 3.38(a) in the vicinity of its maximum.

Fig. 3.40 shows the top of the resonance curve, on an enlarged scale, together with a curve depicting the variation of the reactance of the capacitance. From (3.94) we see that, over the range of frequencies for which

Fig. 3.40. Variation of current and capacitive reactance near resonance in a series resonant circuit.



$\left(\omega L-\dfrac{1}{\omega C}\right)\ll r$, the current I is virtually equal to V/r and is independent of frequency; in other words, the curve is rather flat. Thus, although I is a maximum at ω_0 we shall get a higher value of V_C by going to a slightly lower value of ω, where the current has not decreased appreciably and the reactance of the capacitor is slightly larger.

In the whole of the above theory it has been assumed that the circuit parameters L and C are fixed and that the circuit is being brought to resonance by varying the angular frequency ω of the source. However, in broadcast reception, it is nearly always the capacitor that is varied to bring the circuit to resonance at a particular incoming signal frequency. By differentiating (3.101) with respect to C as the variable, we find that in this case the capacitor voltage is maximum at a frequency given by:

$$\omega_0\sqrt{\left(1-\frac{1}{Q_0{}^2}\right)}$$

Other criteria for resonance may be found, giving slightly different resonant frequencies, depending on which quantity is being measured or detected and which parameter is being varied. The term resonance must therefore be taken to refer to a class of phenomena and not simply to a unique condition. When, however, the Q-factor of the circuit is large ($Q\geqslant 10$), the different conditions for resonance are so nearly the same that for most purposes we need not bother to distinguish between them.

It may have become apparent from the discussion so far that fig. 3.38 represents in a general way the behaviour of any series RLC circuit. Rather than having curves that apply to circuits with specific component values, it is useful to construct universal curves in which the quantities plotted are dimensionless ratios. The first step toward obtaining such curves is to define a new dimensionless quantity δ, the *fractional mistuning*:

$$\delta=\frac{\omega-\omega_0}{\omega_0}\quad(\delta\ll 1) \tag{3.103}$$

This quantity allows us to effect a considerable simplification in the equations describing the behaviour of a circuit near resonance. Re-writing (3.87) we obtain an expression for the current in the series resonant circuit expressed as a dimensionless ratio:

$$\frac{\boldsymbol{I}}{I_0}=\frac{1}{1+\dfrac{\mathrm{j}}{r}\left(\omega L-\dfrac{1}{\omega C}\right)} \tag{3.104}$$

Now remembering that $\omega_0{}^2 = 1/LC$ and that $Q_0 = \omega_0 L/r$, the term $(\omega L - 1/\omega C)$ in the denominator of (3.104) may be written

$$\omega L - \frac{1}{\omega C} = L\left(\omega - \frac{1}{\omega LC}\right)$$

$$= \omega_0 L\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \qquad (3.105)$$

But from (3.103) $\omega/\omega_0 = 1 + \delta$ hence

$$\omega L - \frac{1}{\omega C} = \omega_0 L[1 + \delta - (1 + \delta)^{-1}]$$

$$\approx \omega_0 L[1 + \delta - (1 - \delta)]$$

or to a very good approximation,

$$\omega L - \frac{1}{\omega C} = 2\omega_0 L\delta \qquad (3.106)$$

Using this result in (3.104) we have, near resonance,

$$\frac{\boldsymbol{I}}{I_0} = \frac{1}{1 + \mathrm{j}2\left(\dfrac{\omega_0 L}{r}\right)\delta}$$

or in terms of the Q-factor

$$\frac{\boldsymbol{I}}{I_0} = \frac{1}{1 + \mathrm{j}2Q_0\delta} \qquad (3.107)$$

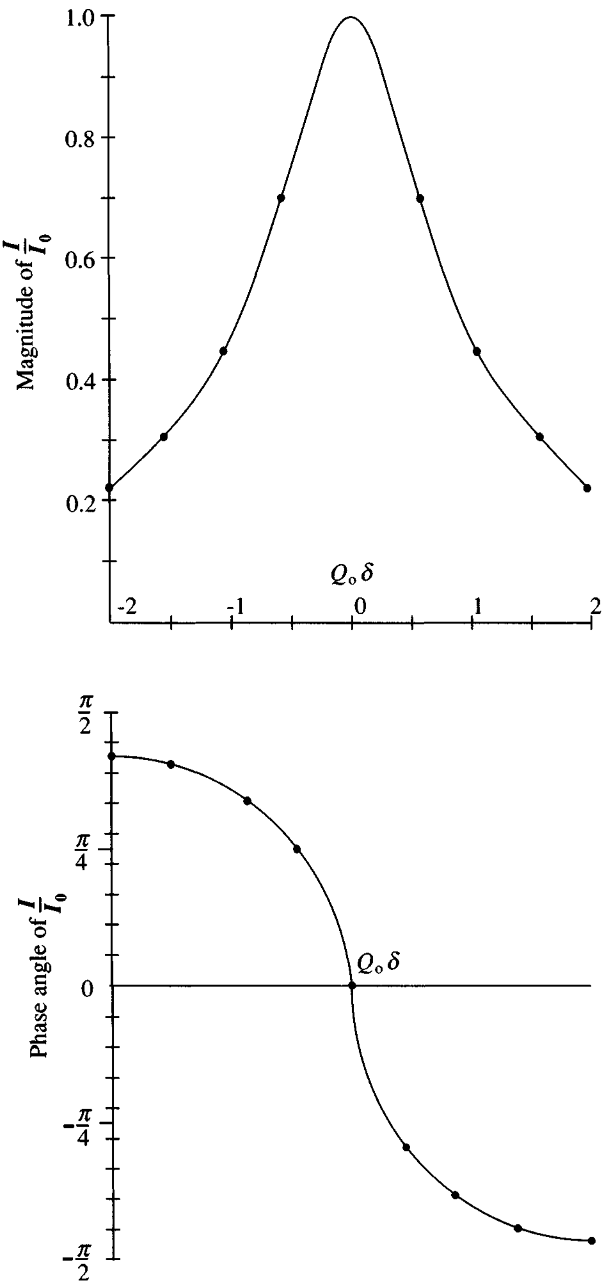
The magnitude of $\boldsymbol{I}/I_0$ is

$$\frac{I}{I_0} = \frac{1}{\sqrt{[1 + (2Q_0\delta)^2]}} \qquad (3.108)$$

Then, if we plot I/I_0 versus $Q_0\delta$ we obtain the universal resonance curve. Similarly, a graph of $\tan^{-1}(-2Q_0\delta)$ yields a curve showing the phase of the current as a function of $Q_0\delta$. The half-power frequencies are those for which $Q_0\delta = 1/2$. Fig. 3.41 shows such universal curves derived from (3.107). They should be compared with figs. 3.38(*a*) and (*b*).

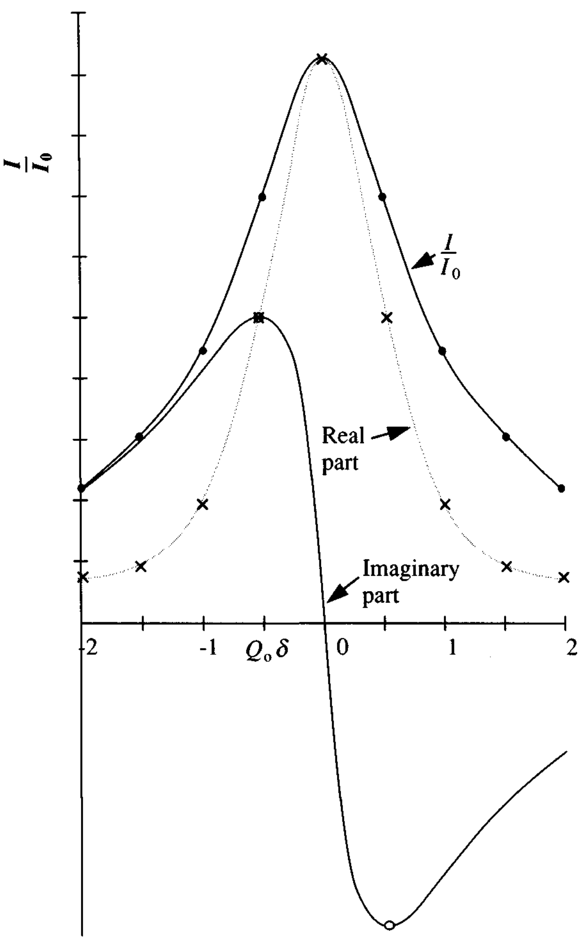
Alternative dimensionless representations of the series resonant circuit characteristics may be obtained by using the rationalized form of (3.104). Thus, the real and imaginary components of $\boldsymbol{I}/I_0$ may be plotted against $Q_0\delta$ as shown in fig. 3.42.

Fig. 3.41. Universal resonance curves: magnitude and phase.



It should be noted that the simplification achieved by the use of the fractional mistuning parameter δ in the equations describing resonance with frequency as the variable, can also be achieved when some other parameter in the circuit is varied. For example, if the circuit is being brought to resonance by varying C, at a fixed ω, we may write $\delta=(C-C_0)/C_0$, where C_0 is the capacitance required for resonance. In this case, instead of the approximation (3.106), we obtain $\left(\omega L-\dfrac{1}{\omega C}\right)=\delta/\omega C_0$.

Fig. 3.42. Universal resonance curves: real and imaginary components.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

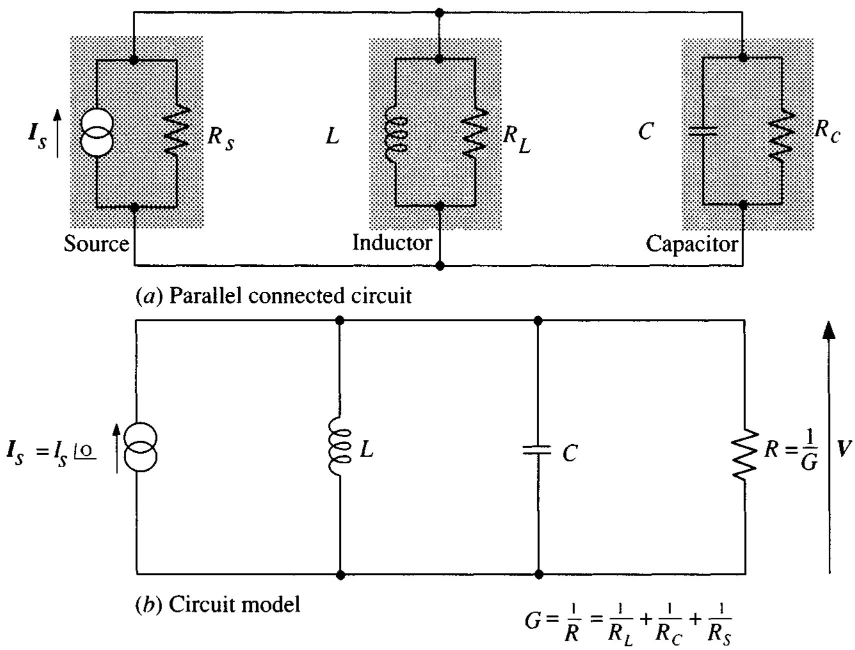
3.13.3 The parallel resonant circuit

The basic form of the parallel resonant circuit is shown in fig. 3.43. The inductor and capacitor are in this case represented by their parallel equivalents and the circuit is driven by a practical current source. As shown in fig. 3.43(*b*), the three parallel resistances may be combined to form a single resistance R (or conductance $G=1/R$). The voltage V across the circuit is given by

$$V=\frac{I_s}{G+\mathrm{j}\left(\omega C-\dfrac{1}{\omega L}\right)} \tag{3.109}$$

Comparing this equation with the corresponding equation for the series resonant circuit (3.87), we recognize that the two circuits are duals; with the appropriate change of symbolism, therefore, the theory developed for the series resonant circuit applies in its entirety to the parallel resonant circuit. The frequency at which voltage resonance occurs in a parallel circuit is the same as that for current resonance in a series circuit (both containing the same L and C), namely, $\omega_0=1/\sqrt{(LC)}$, the curves of V and phase angle versus ω are similar to those in fig. 3.38, and the universal resonance curves

Fig. 3.43. The parallel resonant circuit.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

of fig. 3.41 likewise apply if the dimensionless ratio V/V_0 is used instead of I/I_0. The parallel resonant circuit exhibits *current magnification*; the current in either L or C of fig. 3.43 being Q_0 times the current delivered by the source. This phenomenon is of importance in the type of resonant circuit used in the output stage of a broadcast transmitter. The very heavy currents circulating in such a circuit (the so-called 'tank' circuit) necessitate the use of massive water-cooled conductors.

The principle of duality renders it unnecessary to discuss the parallel circuit of fig. 3.43 in any greater detail, however, we conclude by considering a special case of practical interest for a parallel circuit in which the only significant losses arise in the inductor. In this case the inductor may be represented conveniently by means of its series model, as shown in fig. 3.44. The admittance across the source is

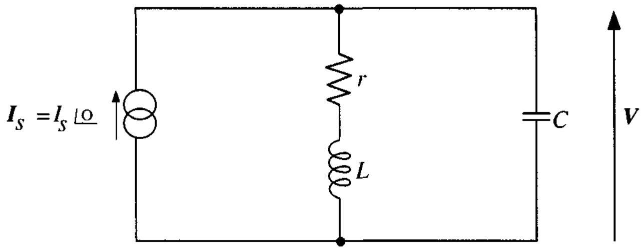
$$
\begin{aligned}
Y &= \mathrm{j}\omega C + \frac{1}{r+\mathrm{j}\omega L} \\
&= \mathrm{j}\omega C + \frac{r-\mathrm{j}\omega L}{r^2+\omega^2L^2} \\
&= \frac{r-\mathrm{j}(\omega L-\omega C(r^2+\omega^2L^2))}{r^2+\omega^2L^2}
\end{aligned}
\tag{3.110}
$$

Then

$$V=\frac{I_s}{Y}=I_s\left[\frac{r^2+\omega^2L^2}{r-\mathrm{j}(\omega L-\omega C(r^2+\omega^2L^2))}\right] \tag{3.111}$$

For this circuit, zero phase angle (indicating that the circuit is a pure conductance) and the maximum value of V do not occur at exactly the same frequency. If we define resonance as the conditions for which V and I_s are in phase, then the imaginary part of the denominator in (3.111) must be zero.

Fig. 3.44. Parallel resonant circuit with lossless capacitor.



The resonant frequency is then found from

$$\omega_0{}'L = \omega_0{}'C(r^2 + \omega_0{}'^2L^2)$$

which gives

$$\omega_0{}' = \frac{1}{\sqrt{LC}}\sqrt{\left(1 - \frac{r^2C}{L}\right)} = \omega_0\sqrt{\left(1 - \frac{1}{Q_0{}^2}\right)} \tag{3.112}$$

So the resonant frequency defined in this way differs from the previous definition by the factor $\sqrt{(1 - 1/Q_0{}^2)}$, but for $Q_0 \geqslant 10$, ω_0 and ω_0 differ by less than 1%.

Equation (3.112) shows that if $r^2C > L$, $\omega_0{}'$ becomes imaginary. For this condition, corresponding to $Q < 1$, there will be no frequency for which V and I_s are in phase. Of course, as Q decreases below the numerical value of 1 the assumptions made in the preceding derivation become less valid. For example, if $Q_0 = 2$, then from (3.112), $\omega_0{}' = \omega_0\sqrt{(1 - \frac{1}{4})} = 0.87\omega_0$. It is not usually the practice to use coils at frequencies so high that the Q-factor of the coil has a value much below 1.

The impedance Z_0 of the circuit at resonance may be found from (3.110) by putting the imaginary term equal to zero with $\omega = \omega_0{}'$ thus

$$Z_0 = \frac{r^2 + (\omega_0{}'L)^2}{r} = r + \frac{(\omega_0{}'L)^2}{r}$$

But from (3.112),

$$(\omega_0'L)^2 = \frac{L}{C} - r^2$$

hence

$$Z_0 = r + \frac{1}{r}\left(\frac{L}{C} - r^2\right) = \frac{L}{Cr}$$

The quantity L/Cr, which is called the *dynamic resistance*, may be also expressed in terms of the Q-factor:

$$\text{Dynamic resistance} = \frac{L}{Cr} = Q_0\omega_0 L = \frac{Q_0}{\omega_0 C} \tag{3.113}$$

3.13.4 Worked example

A coil and a variable capacitor are connected to a voltage generator to form a series resonant circuit. The coil has an inductance of 0.2 mH and a Q-factor of 150; the power factor of the capacitor is 4×10^{-4} The frequency of the voltage generator is 1 MHz, its internal resistance is

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$2\,\Omega$, and its unloaded output voltage is 2 V. Find (a) the value of the capacitor required to tune the circuit to resonance, (b) the effective resistance and Q-factor of the complete circuit at resonance, (c) the complex voltage across the inductor both at the resonant frequency and at a frequency 10 kHz above resonance.

Solution

(a) At the resonant frequency of 1 MHz we have from (3.90)

$$\omega_0 = \frac{1}{\sqrt{(LC)}} = 2\pi \times 10^6$$

therefore,

$$C = \frac{1}{(2\pi \times 10^6)^2 \times 0.2 \times 10^{-3}} = 126.5\,\mathrm{pF}$$

(b) At 1 MHz the reactance of the inductor is $\omega_0 L = 2\pi \times 10^6 \times 0.2 \times 10^{-3} = 1256\,\Omega$. From the definition of Q-factor (equation (3.73)) we have

$$Q_L = \frac{\omega_0 L}{r_L} = \frac{1256}{r_L} = 150$$

hence

$$r_L = \frac{1256}{150} = 8.373\,\Omega$$

For the capacitor, using (3.85),

$$Q_C = \frac{1}{\text{power factor}} = \frac{1}{4 \times 10^{-4}} = 2500$$

but

$$Q_C = \frac{1/\omega_0 C}{r_C} = \frac{1256}{r_C} = 2500$$

hence

$$r_C = \frac{1256}{2500} = 0.502\,\Omega$$

The total resistance of the circuit is therefore

$$r = r_L + r_C + r_S = 8.373 + 0.502 + 2.0 = 10.875\,\Omega$$

From the definition (3.91), the Q-factor for the complete circuit is

https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$Q_0 = \frac{\omega_0 L}{r} = \frac{1256}{10.875} = 115.5$$

Alternatively the Q-factor of the complete circuit may be found using (3.93):

$$\frac{1}{Q_0} = \frac{1}{Q_S} + \frac{1}{Q_L} + \frac{1}{Q_C}$$

In this expression

$$Q_S = \frac{\omega_0 L}{r_S} = \frac{1256}{2} = 628,$$

hence

$$\frac{1}{Q_0} = \frac{1}{628} + \frac{1}{150} + \frac{1}{2500}$$

giving

$$Q_0 = 115.5 \text{ as before.}$$

(c) The complex voltage across the inductor is given by

$$\boldsymbol{V}_L = \boldsymbol{I}(r_L + \mathrm{j}\omega L)$$

where

$$\boldsymbol{I} = \frac{V}{r + \mathrm{j}\left(\omega L - \dfrac{1}{\omega C}\right)} = \frac{V}{r + \mathrm{j}2\omega_0 L\delta}$$

using the approximation (3.106) involving the fractional mistuning δ. We note that the Q-factor of the inductor is high hence $r_L \ll \omega L$. For frequencies near resonance the above expressions may therefore be written more simply as

$$\boldsymbol{V}_L \simeq \frac{V}{r\left(1 + \mathrm{j}2\dfrac{\omega_0 L}{r}\delta\right)} \mathrm{j}\omega L \simeq \frac{V}{(1 + \mathrm{j}2Q_0\delta)} \frac{\mathrm{j}\omega_0 L}{r}$$

or

$$\boldsymbol{V}_L \simeq \frac{\mathrm{j}VQ_0}{1 + \mathrm{j}2Q_0\delta}$$

At resonance the fractional mistuning $\delta = 0$, hence,

$$\boldsymbol{V}_L = \mathrm{j}2 \times 115.5 = \mathrm{j}231 = 231\underline{/90}$$

That is, the magnitude of the voltage across the inductor is 231 V leading the generator e.m.f. by 90°.

At 10 kHz above resonance, $\delta = 10 \times 10^3/10^6 = 10^{-2}$, hence

$$V_L = \frac{\mathrm{j}2 \times 115.5}{1 + \mathrm{j}(2 \times 115.5 \times 10^{-2})} = \frac{\mathrm{j}231}{1 + \mathrm{j}2.31}$$

$$V_L = \frac{231\underline{/90}}{2.52\underline{/66.6}} = 91.66\underline{/90 - 66.6} = 91.77\underline{/23.4}$$

†3.13.5 Definition of Q-factor in terms of stored energy

The phenomenon of resonance in an electrical circuit involves the continuous interchange of energy between the inductances and capacitances in the circuit, and during this interchange some energy is lost in the resistances of the circuit. Taking as a specific example the series resonant circuit of fig. 3.36 the current at resonance is $I_0 = V/r$, therefore, the power in r is $(V/r)^2 r$. The energy dissipated per cycle is then given by

$$\left(\frac{V}{r}\right)^2 r \cdot \frac{2\pi}{\omega_0} = \frac{2\pi V^2}{r\omega_0}$$

Now the maximum energy stored in the inductor is, from (1.42),

$$\tfrac{1}{2}L\left(\surd 2\frac{V}{r}\right)^2 = L\frac{V^2}{r^2}$$

therefore,

$$\frac{2\pi\ (\text{Maximum energy stored})}{\text{Energy dissipated per cycle}} = \frac{\omega_0 L}{r} = Q_0 \tag{3.114}$$

This definition of the quality factor, although derived for a special case, is of general application. It can be applied to mechanical and acoustical vibrating systems as well as to more complicated electrical ones.

†3.13.6 Multiple resonance

Networks containing combinations of more than two independent reactive elements can resonate at more than a single frequency; a phenomenon known as *multiple resonance.* Three examples of such circuits are shown in fig. 3.45. For the purpose of this discussion it is easier to consider the components in these circuits as being lossless so that the resonant frequencies may be deduced from consideration of the reactances only. The results so obtained will be substantially correct provided the Q-factors of the components exceed 10 or so.

Considering first the circuit of fig. 3.45(a), this consists of two sections: first the inductance L_1 with reactance X_1, second the parallel resonant circuit, formed by L_2 and C_1, with reactance X_p. The variation of reactances X_1 and X_p is shown in fig. 3.46. The curve of reactance X_1 with frequency is, of course, a straight line through the origin (fig. 3.46(a)). The reactance X_p (fig. 3.46(b)) is zero at $\omega = 0$, because the inductance L_2 forms a short circuit, and is asymptotic to zero at $\omega \to \infty$ because the capacitor forms a short circuit. Resonance occurs at some intermediate frequency $\omega = \omega_p$ where the reactance rises to a theoretically infinite value. The form of the variation of the reactance $X = (X_1 + X_p)$ for the complete circuit is obtained by combining the individual curves, as shown in fig. 3.46(c). It is seen that a second resonant frequency ω_2 occurs where the reactance of X_1 (inductive)

Fig. 3.45. Circuits exhibiting multiple resonance.

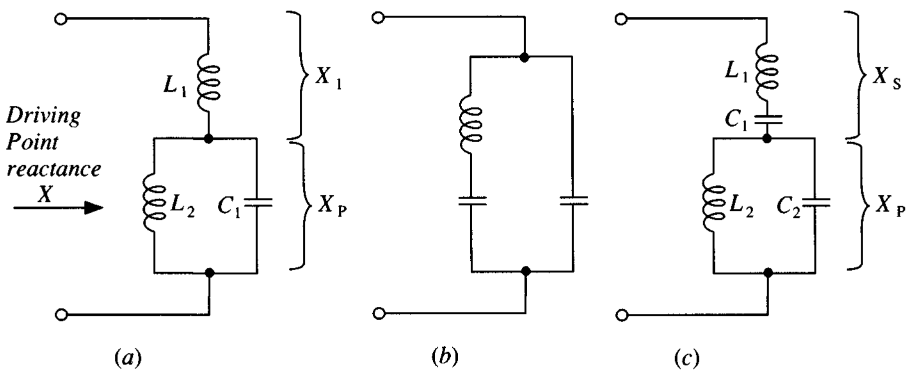
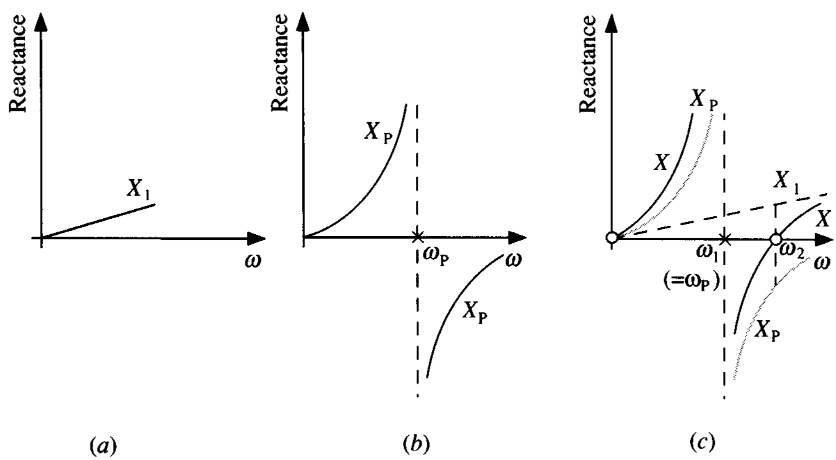


Fig. 3.46. Variation of reactance in the circuit of fig. 3.45(a).



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

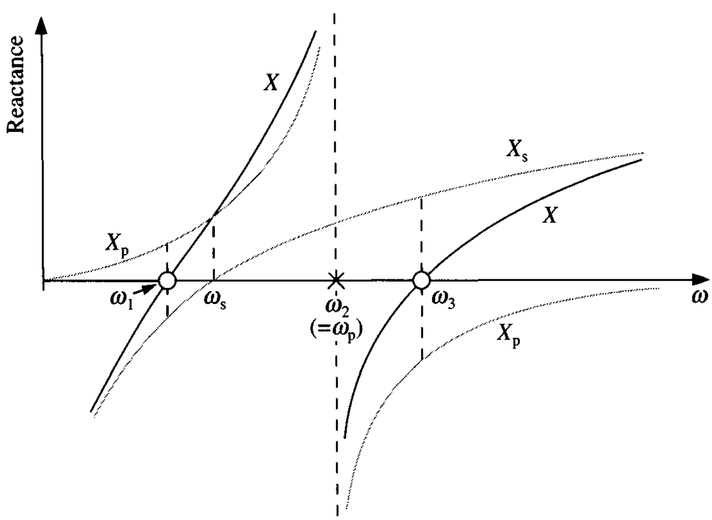
and the reactance X_p (capacitive) cancel. The resonant frequency $\omega_1 = \omega_p$ is unchanged. Similar behaviour is exhibited by the circuit of fig. 3.45(*b*) since this is the dual of the circuit in fig. 3.45(*a*); for the dual circuit, however, all the curves shown in fig. 3.46 must be interpreted in terms of susceptance rather than reactance.

We now consider the circuit shown in fig. 3.45(*c*). This circuit again has two sections: L_2 and C_2 forming a parallel resonant circuit with resonant frequency ω_p, and L_1 and C_1 forming a series resonant circuit with resonant frequency ω_s. The variation of the total reactance $X = (X_s + X_p)$ of this circuit is indicated by the solid line in fig. 3.47, which is obtained by combining the curve for the series circuit (fig. 3.37) with that for the parallel circuit (fig. 3.46(*b*)). We see that the complete circuit resonates at three different frequencies: at ω_1 and ω_3 the reactance is zero, and at $\omega_2 (= \omega_p)$ the reactance is infinite. The curves have been drawn on the assumption that $\omega_s < \omega_p$ but it is easy to see that the same conclusions will apply for the conditions $\omega_s \geqslant \omega_p$. Three resonant frequencies will always occur, the two frequencies for which the reactance is zero lying on either side of the frequency for which the reactance is infinite.

Expressions for the resonant frequencies $\omega_1, \omega_2, \omega_3$, are now derived in terms of the circuit parameters. The driving-point reactance function of the circuit in fig. 3.45(*c*) is

$$X = X_s + X_p$$

Fig. 3.47. Variation of reactance in the circuit of fig. 3.45(*c*).



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

$$X = \left(\omega L_1 - \frac{1}{\omega C_1}\right) + \frac{\omega L_2\left(-\dfrac{1}{\omega C_2}\right)}{\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}$$

$$= \frac{L_1}{\omega}\left(\omega^2 - \frac{1}{L_1 C_1}\right) - \frac{\dfrac{L_2}{C_2}}{\dfrac{L_2}{\omega}\left(\omega^2 - \dfrac{1}{L_2 C_2}\right)}$$

$$= \frac{\left(\omega^2 - \dfrac{1}{L_1 C_1}\right)\left(\omega^2 - \dfrac{1}{L_2 C_2}\right) - \dfrac{\omega^2}{L_1 C_2}}{\dfrac{\omega}{L_1}\left(\omega^2 - \dfrac{1}{L_2 C_2}\right)}$$

$$= \frac{\omega^4 - \omega^2\left(\dfrac{1}{L_1 C_1} + \dfrac{1}{L_2 C_2} + \dfrac{1}{L_1 C_2}\right) + \dfrac{1}{L_1 C_1 L_2 C_2}}{\dfrac{\omega}{L_1}\left(\omega^2 - \dfrac{1}{L_2 C_2}\right)}$$

In this equation the numerator is a quadratic in ω^2. If $\omega_1{}^2$ and $\omega_3{}^2$ are solutions of this, we may write:

$$X = \frac{(\omega^2 - \omega_1{}^2)(\omega^2 - \omega_3{}^2)}{\dfrac{\omega}{L_1}(\omega^2 - \omega_2{}^2)} \tag{3.115}$$

where $\omega_2{}^2 = 1/L_2 C_2$.

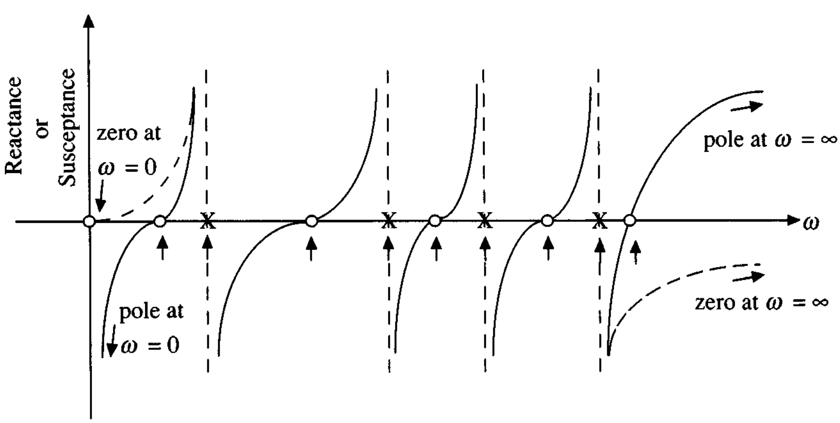
The expression (3.115) shows that the reactance function X is zero at $\omega = \omega_1$ and at $\omega = \omega_3$, and that it is infinite at $\omega = \omega_2$. We describe this by saying that *zeros* of X occur at ω_1 and ω_3, and that a *pole* occurs at ω_2. It will be observed that a pole also occurs at the origin ($\omega = 0$) for this circuit. Poles and zeros are indicated by the crosses and circles on the ω-axis in figs. 3.46(*c*) and 3.47.

Circuits of greater complexity give rise to a greater number of poles and zeros but the driving-point reactance (or susceptance) function of a circuit always conforms to the general pattern of behaviour described above and which is illustrated in fig. 3.48. The slopes of the curves of reactance or susceptance are everywhere positive, and poles and zeros occur alternately along the frequency axis. If there is a continuous path through the circuit from terminal to terminal formed by one or more inductive elements (as in fig. 3.45(*a*)), then a zero occurs at the origin, otherwise there is a pole. Likewise, if there is a continuous path formed by capacitive elements, there is a zero at $\omega = \infty$, otherwise there is a pole. The alternative possibilities of either a pole or a zero occurring at the extremes of frequency is indicated in fig. 3.48 by the dotted lines. Note, however, that whatever happens at these extremes does not change the alternating pattern of poles and zeros. The poles and zeros other than those at the two extremes are called the *internal* poles and zeros, and it may be shown rigorously that the driving-point reactance function is uniquely specified, apart from a vertical scaling factor, by the location of its internal poles and zeros. If the value of the function at a single frequency is also specified thus determining the scaling factor, then the function is completely specified. This is known as *Foster's reactance theorem* (see reference 2). It is useful for the purposes of synthesizing networks having specified characteristics.

†3.13.7 Inductively coupled resonant circuits

In telecommunications equipment, radio receivers for example, high-frequency signals are amplified by means of active devices such as transistors. The signals usually occupy a relatively narrow frequency band disposed about some large central frequency, and to amplify this narrow band it is usual to employ tuned coupling networks between successive stages of amplification. Such networks often take the form of two inductively coupled coils each coil forming part of a resonant circuit. A typical arrangement is shown in fig. 3.49. In this circuit the output of the transistor is represented by a practical current source. Coupling to the next stage is achieved by means of L_1, L_2 and M, which together constitute a *radio-frequency* or *high-frequency transformer*.

Fig. 3.48. Poles and zeros of a driving-point reactance (or susceptance) function. (Internal poles and zeros are indicated by arrows.)



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

The frequency response characteristic of the complete circuit is governed by the frequencies to which the primary and secondary circuits are tuned, their Q-factors, and the degree of coupling between them. Tuning is achieved by fine adjustment of the values of the capacitors or inductors; the mutual inductance is usually fixed once the designer has decided upon a suitable value.

If we apply a variable frequency drive voltage of constant amplitude to the input of the circuit of fig. 3.49, with the primary and secondary circuits tuned to the *same* resonant frequency, we observe an output voltage of the form shown in fig. 3.50. Here we see curves for three different values of the mutual inductance. When the mutual inductance is very small, the normal type of resonance curve with a single peak at the resonant frequency is

Fig. 3.49. Basic circuit for one stage of a tuned radio-frequency amplifier.

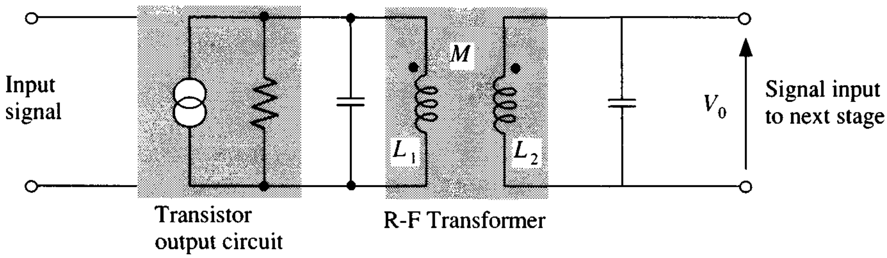
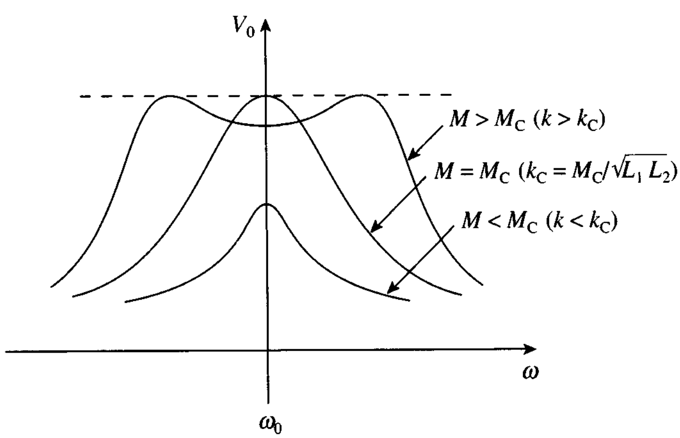


Fig. 3.50. Curves of output voltage as a function of frequency for the circuit of fig. 3.49 with mutual inductance (or coupling coefficient) as a parameter. M_c is the critical value of mutual inductance.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

observed. As the mutual inductance is increased the peak of the curve rises until a critical value is reached when no further increase is obtained, instead the peak splits into two separate peaks which move further apart as the mutual inductance is increased beyond the critical value. This phenomenon is known as *double-humping.*

For reasons that will become apparent it is convenient to express the degree of coupling in terms of the coupling coefficient $k = M/\sqrt{(L_1 L_2)}$. The critical value for the coupling coefficient is approximately 0.01, when the Q-factors of the primary and secondary circuits are of the order 100.

We shall now proceed to investigate the behaviour of this circuit analytically but in order to do so we first simplify the circuit by applying the Thévenin–Norton transformation to the left-hand portion of the circuit consisting of the current generator and the capacitor. When this is done we obtain the circuit shown in fig. 3.51 in which R_1 accounts for both the primary coil loss-resistance and the output resistance of the transistor. R_2 accounts for all loss resistance in the secondary circuit. The circuit now consists of two separate series resonant circuits coupled by the mutual inductance. It will be sufficient for our purpose to solve for the current $\boldsymbol{I}_2$ since for small frequency changes about resonance, the output voltage will be very nearly proportional to the modulus of this current. Using a procedure similar to that established in section 3.12 we obtain by mesh analysis:

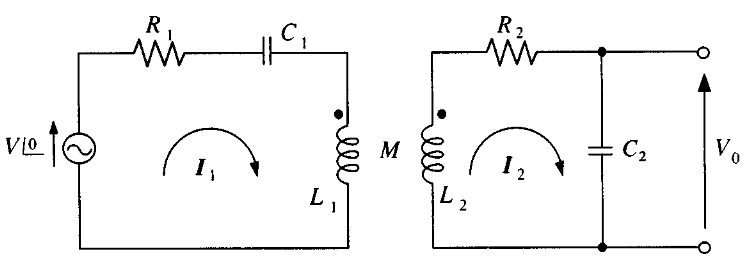
for the primary circuit

$$\left(R_1 + \mathrm{j}\omega L_1 + \frac{1}{\mathrm{j}\omega C_1}\right)\boldsymbol{I}_1 - \mathrm{j}\omega M\boldsymbol{I}_2 = V$$

and for the secondary circuit

$$-\mathrm{j}\omega M\boldsymbol{I}_1 + \left(R_2 + \mathrm{j}\omega L_2 + \frac{1}{\mathrm{j}\omega C_2}\right)\boldsymbol{I}_2 = 0$$

Fig. 3.51. Circuit model for fig. 3.49.



Putting $X_1=\left(\omega L_1-\dfrac{1}{\omega C_1}\right)$, $X_2=\left(\omega L_2-\dfrac{1}{\omega C_2}\right)$ and solving for the modulus of I_2 we obtain

$$I_2=\frac{V}{\left\{\left(\dfrac{R_1X_2+R_2X_1}{\omega M}\right)^2+\left(\omega M+\dfrac{R_1R_2}{\omega M}-\dfrac{X_1X_2}{\omega M}\right)^2\right\}^{\frac{1}{2}}} \tag{3.116}$$

Now consider the denominator of this expression at the resonant frequency; both X_1 and X_2 will be zero since primary and secondary circuits are tuned alike to the same frequency. In this case (3.116) reduces to

$$I_2=\frac{V}{\omega_0 M+\dfrac{R_1R_2}{\omega_0 M}}$$

Regarding M as the variable parameter we see that there will be a minimum value of the denominator, and therefore a maximum in I_2, when M is equal to a critical value $M=M_c$ such that

$$\frac{\mathrm{d}}{\mathrm{d}M}\left(\omega_0 M+\frac{R_1R_2}{\omega_0 M}\right)=0$$

that is when $\omega_0{}^2M_c{}^2=R_1R_2$

The above relationship may be expressed alternatively in terms of the coupling coefficient $k_c=M_c/\surd(L_1L_2)$. We have in this case

$$\omega_0{}^2M_c{}^2=\omega_0{}^2k_c{}^2L_1L_2=R_1R_2$$

or

$$k_c{}^2=\frac{R_1R_2}{\omega_0{}^2L_1L_2}$$

Recalling our definition of Q-factor (3.91) this may be expressed as

$$k_c=\frac{1}{\surd(Q_1Q_2)} \tag{3.117}$$

where Q_1 and Q_2 are the quality factors of the primary and secondary circuits. We see that the use of the coupling coefficient leads to a particularly simple expression.

It is possible to show that if the coupling is less than critical, then I_2 falls monotically either side of the resonance frequency. If the coupling is greater than critical ($k\geqslant k_c$), then the current rises to a maximum on either side of the resonant frequency; the location of these maxima may be found by differentiation of the complete denominator of (3.116). The procedure is simplified if the assumption is made that ωM remains substantially constant, with value $\omega_o M$, over the narrow band of frequencies of interest around ω_0. It is convenient also to introduce the fractional mistuning δ, (3.103); then, by (3.106), $X_1 = 2\omega_0 L_1 \delta$ and $X_2 = 2\omega_0 L_2 \delta$. The denominator of (3.116) may then be written:

$$\frac{1}{\omega_0 M}\Big\{[R_1(2\omega_0 L_2\delta) + R_2(2\omega_0 L_1\delta)]^2 + [\omega_0{}^2 M^2 + R_1 R_2 - (2\omega_0 L_1\delta)(2\omega_0 L_2\delta)]^2\Big\}^{1/2}$$

Differentiating the square of this expression and equating the result to zero gives

$$[2R_1\omega_0 L_2 + 2R_2\omega_0 L_1]^2 - 8\omega_0{}^2 L_1 L_2[\omega_0{}^2 M^2 + R_1 R_2 - 4\omega_0{}^2 L_1 L_2 \delta^2] = 0$$

We may express this equation in terms of the quality factors for the primary and secondary circuits: multiplying through by the factor $1/(\omega_0{}^2 L_1 L_2)^2$ and putting $M^2 + k^2 L_1 L_2$ gives

$$\left[\frac{2}{Q_1} + \frac{2}{Q_2}\right]^2 - 8\left[k^2 + \frac{1}{Q_1 Q_2} - 4\delta^2\right] = 0$$

Further simplification is obtained if it is assumed that the quality factors for the primary and secondary circuits are equal. Then $1/Q_1 = 1/Q_2 = k_c$, and the above expression becomes

$$16k_c{}^2 - 8(k^2 + k_c{}^2 - 4\delta^2) = 0$$

which gives, finally,

$$4\delta^2 = k^2 - k_c{}^2$$

or

$$\delta = \pm\tfrac{1}{2}\sqrt{(k^2 - k_c{}^2)} \qquad (k \geqslant k_c) \tag{3.118}$$

The over-coupled condition with primary and secondary circuits tuned to the same frequency produces a desirable band-pass characteristic but it is found in practice to be difficult to set up the circuit correctly because the primary and secondary circuits interact strongly in the over-coupled condition. It is more usual therefore to design cascaded radio-frequency amplifiers using under-coupled transformers with primary and secondary circuits each tuned to slightly different frequencies to produce the desired overall band-pass characteristic. This is known as *stagger tuning.*

The phenomenon of double-humping described in this section is not confined to tuned coupled circuits; the same phenomenon arises in many other physical systems which exhibit oscillatory characteristics. Such systems fall under the general heading of *coupled oscillators.*

3.14 Summary

A.C. networks consist of interconnected elements of resistance, inductance, mutual inductance and capacitance excited by ideal sources producing sinusoidal voltage and current waveforms of the same frequency. The theory developed for such networks refers to steady-state conditions in which the amplitudes and phases of all waveforms are time invariant.

Representation of sinusoidal voltages and currents by means of the complex exponential leads to a succinct notation ($\boldsymbol{V} = V\underline{/\theta}$ or $\boldsymbol{I} = I\underline{/\beta}$) for describing the amplitudes and phases of currents and voltages in every branch of a network. Each branch in a network is then characterized by a complex impedance ($\boldsymbol{Z} = Z\underline{/\phi}$), being the ratio of complex voltage to complex current at that branch. Complex voltages, currents and impedances may be combined and manipulated according to the normal rules of linear network analysis to provide a complete description of network behaviour at any terminal pair or port. The use of the complex exponential (or phasor) notation also provides a convenient means of illustrating graphically (using the Argand diagram as a basis) the relationships between voltages and currents in parts of a network.

For many important types of circuit, including filters and bridge circuits, it is of primary interest to determine the behaviour of the circuit as a function of frequency. Such circuits often have two ports in which case the relationship between input and output port parameters can be described by a transfer function $H(\mathrm{j}\omega) = |H(\mathrm{j}\omega)|\underline{/\phi(\omega)}$. The transfer function is conveniently illustrated by means of the polar or locus diagram (based on the Argand diagram), or by the Bode diagram in which amplitude and phases are plotted separately on graphs having linear–log scales.

Some types of circuit exhibit marked changes in their impedance properties at certain critical frequencies known as the resonant frequencies. In a circuit containing a single inductance L and a single capacitance C, there is one resonant frequency only given by $\omega_0 = 1/\sqrt{(LC)}$.

The properties of a resonant circuit are determined by the inductance capacitance and resistance associated with the circuit elements of which it is composed. The Q-factor is a mathematically convenient parameter for describing these properties. If, in a series connected circuit the effective resistance is R, the Q-factor is given by $Q_0 = \omega_0 L/R$ where ω_0 is the resonant frequency. Multiple resonances can occur in circuits containing several inductive and capacitive elements, or in circuits containing mutual inductance.

3.15 Problems

1. A d.c. generator of e.m.f. E and an a.c. generator of e.m.f. $E_m \sin\omega t$, each having negligible internal impedance, are connected in series to a load resistance R. Obtain an expression for the power dissipated in R, and hence an expression for the equivalent r.m.s. e.m.f. in the circuit.

2. Calculate the r.m.s. value and form factor for:
(a) a triangular wave of unit amplitude;
(b) a sine wave of unit amplitude

3. Find the equivalent complex impedance and admittance of the circuits shown in fig. 3.52, in both Cartesian and polar forms.

4. Represent the following voltages on a phasor diagram, hence find their sum, both graphically and by calculation.

(i) $200\sin(\omega t+\pi/6)$ V, (ii) $150\cos(\omega t+\pi/6)$ V,
(iii) $200\sin(\omega t+5\pi/6)$ V, (iv) $-150\cos(\omega t+5\pi/6)$ V.

5. The three elements shown in fig. 3.53 are connected in series across a 50 Hz supply, and the current drawn from the supply is found to be 7 A. The voltages are $V_R=120$, $V_L=240$ and $V_C=160$. Find the supply voltage and

Fig. 3.52. Circuit for problem 3.

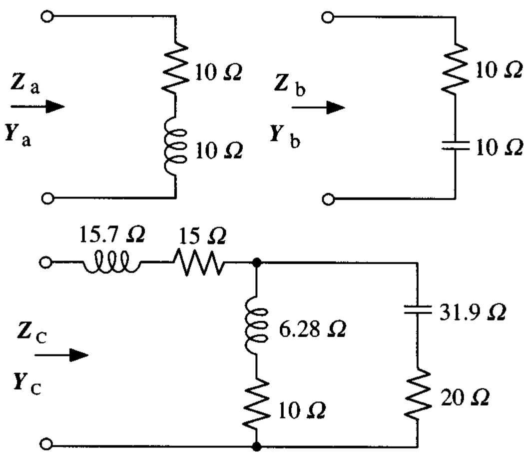
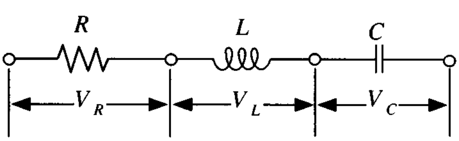


Fig. 3.53. Circuit for problem 5.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

its phase relative to the current. Find also the values of the three elements.
6. What 50 Hz voltage V must be applied to the circuit of fig. 3.54 to produce a steady-state current of 5 A in the capacitor?
(Manchester University)
7. Figure 3.55 shows a circuit used in TV and radar amplifiers. Show that for $L=0$ the ratio V_o/V_i falls from 0.5 at very low frequencies to 0.35 at a frequency of 1.59 MHz, but that if $L=400\,\mu$H this drop is reduced, the ratio then being 0.47 at 1.59 MHz. What are the phase angles between V_o and V_i at this frequency when $L=0$ and when $L=400\,\mu$H?
(Hint: apply voltage divider principle (admittance formulation, table 3.1).)
8. In fig. 3.56 V is a variable frequency, constant-voltage source. Find an expression for the frequency at which the voltage across AB is in phase with V.
(Liverpool University)
9. Show for the circuit of fig. 3.57 that

$$\text{as } \omega\to 0,\ \frac{V_o}{V_i}\to\frac{R_2}{R_1+R_2}=m_0$$

$$\text{as } \omega\to\infty,\ \frac{V_o}{V_i}\to\frac{C_1}{C_1+C_2}=m_\infty$$

Fig. 3.54. Circuit for problem 6.

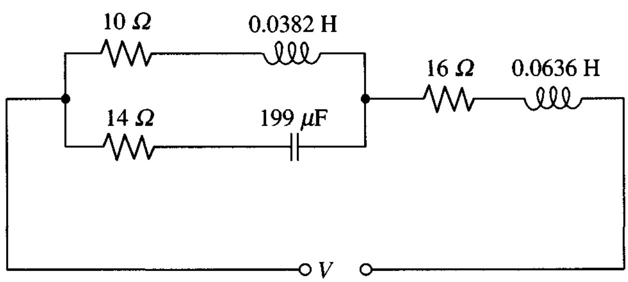
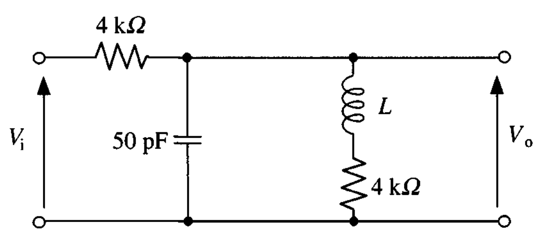


Fig. 3.55. Circuit for problem 7.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

that $\left|\frac{V_o}{V_i}\right| = \sqrt{m_o m_\infty}$ when $\omega^2 = \frac{R_1 + R_2}{C_1(C_1 + C_2)R_1{}^2 R_2}$

and that the phase lead of V_o on V_i is

$$\theta = \tan^{-1}(\omega C_1 R_1) - \tan^{-1}\left(\frac{m_0}{m_\infty}\omega C_1 R_1\right)$$

Discuss the case of $R_1 C_1 = R_2 C_2$
(Manchester University)

10. Show for the circuit of fig. 3.58 that as the frequency is raised from zero, $\left|\frac{V_o}{V_i}\right|$ increases from $\frac{r}{2R+r}$ at $\omega = 0$ to 1 at $\omega \to \infty$, without a maximum or minimum value between these extremes.

Show that the output lags the input by $\theta = \tan^{-1}\omega CR + \tan^{-1}\frac{\omega CRr}{2R+r}$, or leads by $(\pi - \theta)$.
(Manchester University)

11. Sketch the straight line approximation for the frequency response of i_2/i_1 where $i_1 = I\sin\omega t$ in the circuit of fig. 3.59. You may assume that C is of

Fig. 3.56. Circuit for problem 8.

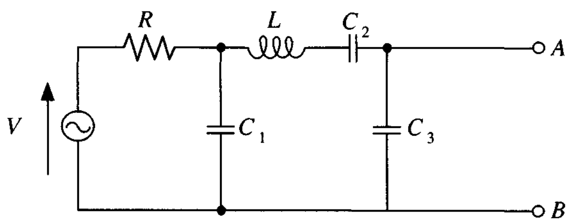
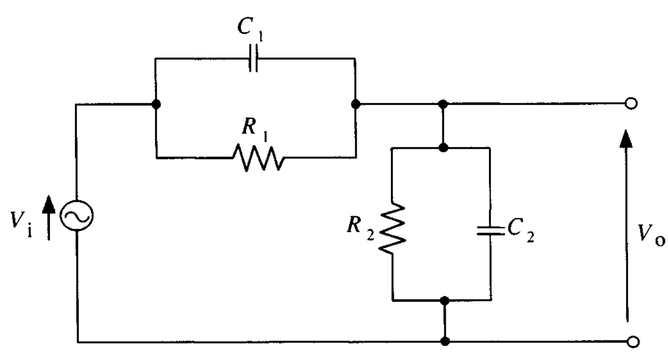


Fig. 3.57. Circuit for problem 9.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

such a value that any frequency-dependent effects it introduces occur at frequencies which are very much higher than those at which frequency-dependent effects due to L are evident. The effects of C and L on the frequency response may thus be considered separately. Give expressions for any 'corner frequencies' shown in the response and values of the ratio $|i_2/i_1|$ on any horizontal parts of the response.
(Lancaster University)

12. The Schering bridge shown in fig. 3.60 is used for measuring the power loss in dielectrics. The specimens are in the form of discs 0.3 cm thick and have a dielectric constant of 2.3. The area of each electrode is 314 cm^2 and the bridge frequency is 50 Hz. The bridge balances when $R_3 = 1000\,\Omega$, $C_1 = 50$ pF, and $C_4 = 1960$ pF. Find the value of R_4, C_2, R_2 and the power factor for the dielectric.
(Liverpool University)

13. Show that there is zero output from the bridged-T section of fig. 3.61(*a*) if

$$Z = -(Z_1 + Z_2) - \frac{Z_1 Z_2}{Z_3}$$

Fig. 3.58. Circuit for problem 10.

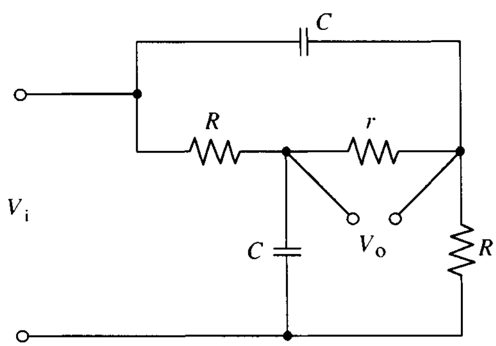
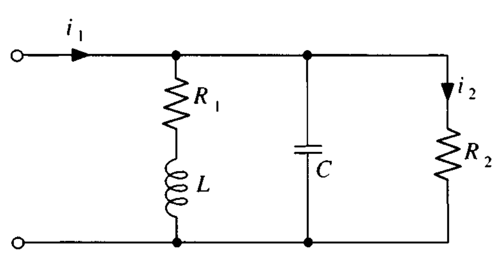


Fig. 3.59. Circuit for problem 11.



Determine the form of Z in the section of fig. 3.61(b) if there is to be zero output at angular frequency ω_0.
(Manchester University)

14. Show for the circuit of fig. 3.62 that

$$I_1=\left[\frac{R_2+j\omega(L_2\pm M)}{R_1R_2+j\omega(L_1R_2+L_2R_1)-\omega^2(L_1L_2-M^2)}\right]V$$

$$I_2=\left[\frac{R_1+j\omega(L_1\pm M)}{R_1R_2+j\omega(L_1R_2+L_2R_1)-\omega^2(L_1L_2-M^2)}\right]V$$

and that the p.d. between A and B is zero when

$$\frac{R_1}{R_2}=\frac{L_1\pm M}{L_2\pm M}$$

(Manchester University)

Fig. 3.60. Circuit for problem 12.

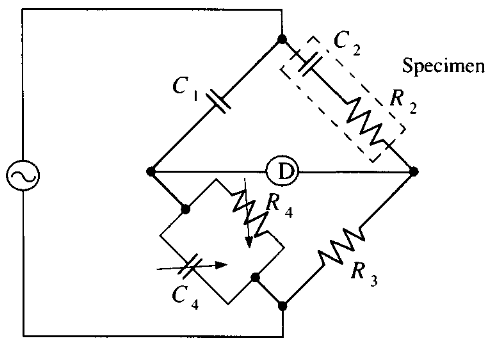
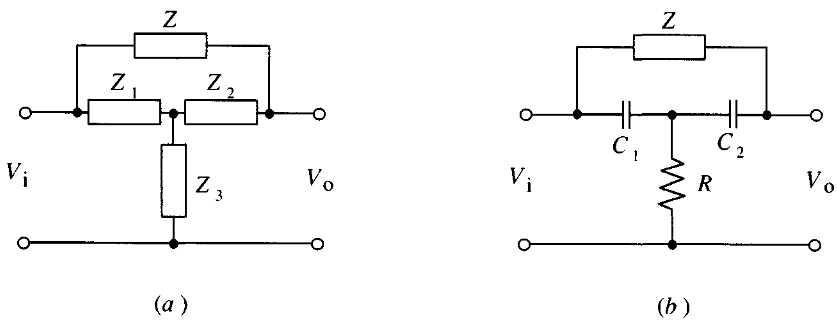


Fig. 3.61. Circuit for problem 13.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

15. Give expressions for the self and mutual impedances of the two meshes in the circuit of fig. 3.63.

16. A parallel circuit consists of a coil of resistance R and inductance L and a variable capacitance C. A fixed-frequency voltage source, of angular frequency ω, is connected to this circuit. If $Q=\omega L/R$ and $C_0=1/(\omega^2 L)$ calculate the value of C, in terms of Q and C_0, such that the current drawn from the supply is a minimum.
(Newcastle University)

17. Why is a parallel resonant circuit sometimes known as a rejector circuit?

Show that the resonant frequency f_0 and dynamic resistance R_d of a parallel resonant circuit consisting of a capacitor C in parallel with a coil of inductance L and resistance R are given by

$$f_0=\frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)}; \qquad R_d=\frac{L}{CR}$$

Fig. 3.62. Circuit for problem 14.

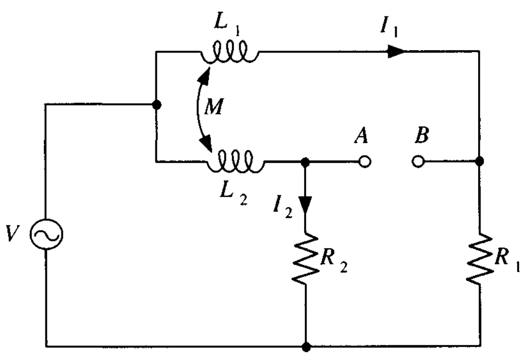
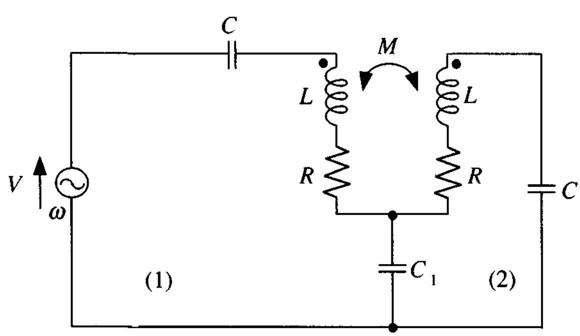


Fig. 3.63. Circuit for problem 15.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

In the circuit of fig. 3.64, the capacitor C has been adjusted until the voltage E_L is at its minimum value. Calculate the source frequency and E_L under these conditions.
(Newcastle University)

18. In the circuit of fig. 3.65, $R=20\,\Omega$, $L_1=2\,\text{H}$, $C_1=10^{-2}\,\text{F}$, $C_2=5\times10^{-3}\,\text{F}$.
(a) Calculate the complex impedance $Z(\mathrm{j}\omega)$ seen by a voltage source $v(t)=80\cos 10t$. Find $i(t)$.
(b) Construct the phasor diagram for $\boldsymbol{I}$ and $\boldsymbol{V}$ and the impedance diagram for $Z(\mathrm{j}\omega)$.
(c) What are the resonant frequencies of this circuit, that is $X(\mathrm{j}\omega_1)=0$ and $X(\mathrm{j}\omega_2)=\infty$? [$X(\mathrm{j}\omega)$ is the reactance.]
(Sheffield University)

19. The response of two non-identical tuned circuits coupled together by

Fig. 3.64. Circuit for problem 17.

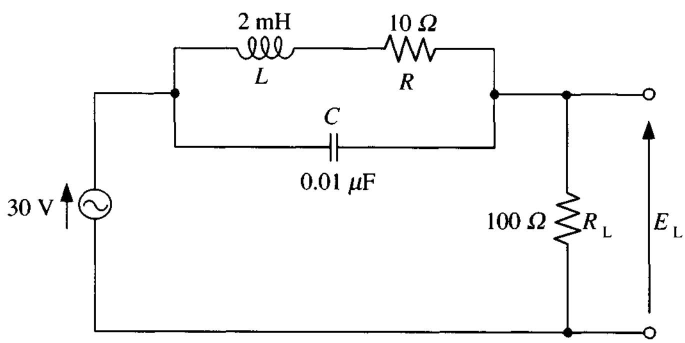
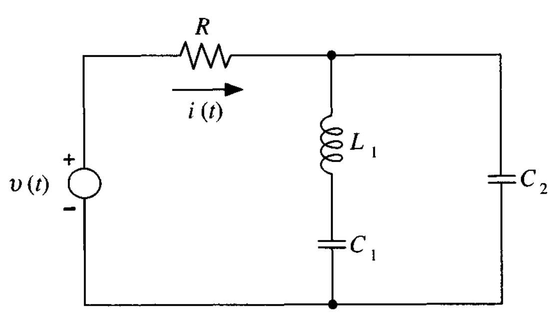
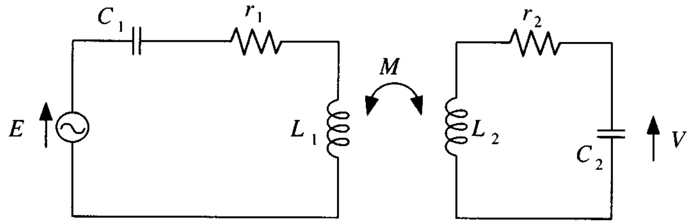


Fig. 3.65. Circuit for problem 18.



https://doi.org/10.1017/CBO9781139170093.005 Published online by Cambridge University Press

Fig. 3.66. Circuit for problem 19.



the mutual inductance $M=k\sqrt{(L_1L_2)}$, as shown in fig. 3.66, is given by

$$\left|\frac{V}{E}\right|=\frac{k\sqrt{(C_1/C_2)}}{\sqrt{\left[\left(\frac{1}{Q_1Q_2}+k^2-x^2\right)^2+x^2\left(\frac{1}{Q_1}+\frac{1}{Q_2}\right)^2\right]}}$$

where

$$x=\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}; \qquad Q_1=\frac{\omega_0 L_1}{r_1} \text{ and } Q_2=\frac{\omega_0 L_2}{r_2}$$

Show that for critical coupling

$$k^2=\tfrac{1}{2}\left(\frac{1}{Q_1}+\frac{1}{Q_2}\right)^2-\frac{1}{Q_1Q_2}$$

and the -3 dB points on the critically coupled response curve are given by

$$x=\pm\frac{1}{\sqrt{2}}\left(\frac{1}{Q_1}+\frac{1}{Q_2}\right)$$

Find the 3 dB bandwidth if $f_0=1$ MHz, $Q_1=100$ and $Q_2=80$.
(Glasgow University: Third year)