4

Power and transformers in single-phase circuits

4.1 Introduction

For the sinusoidal steady state, one can calculate the total power supplied to a circuit consisting of linear elements by adding directly the power absorbed by each individual resistive element in the circuit. However, it is often more convenient to express power in terms of the voltage across and the current supplied to the input terminals of a circuit whose detailed configuration is unknown or is of no interest.

In all electric power distribution networks voltage and frequency are maintained substantially constant. A given load will draw a current whose amplitude and phase (relative to the power line voltage) depend upon the load impedance. On the other hand in electronic and telecommunication networks, signal power rather than voltage is fixed and we are concerned more with arranging source and load conditions to achieve maximum power transfer from one part of a circuit to another.

For the above reasons the treatment of power in electrical circuits depends to a marked extent on the type of circuit under consideration. In this chapter we develop general methods for determining the power and total energy supplied to, or dissipated within, a circuit. We also consider one of the most important components involved in the utilization and transmission of power; namely, the transformer.

4.2 Average power

Consider a network or load as shown in fig. 4.1, supplied at voltage $\boldsymbol{V}$ (r.m.s. magnitude V) and drawing current $\boldsymbol{I}$ (r.m.s. magnitude I). If the network contains reactive elements, voltage and current will differ in phase by some angle ϕ. It is convenient, particularly in power distribution calculations, to take the voltage as phase reference. The instantaneous

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current and voltage are then described by $v = V_m \sin\omega t$ and $i = I_m \sin(\omega t + \phi)$, and the instantaneous power to the load is

$$p = vi = V_m I_m \sin\omega t \sin(\omega t + \phi) \tag{4.1}$$

In fig. 4.2, p, v, and i are shown as functions of time. Instantaneous power is sinusoidal with a frequency double that of the supply voltage. By our previously established sign convention (section 1.4) p is positive when energy flows into the load.

The total energy W supplied during the interval t_0 to t is (see section 1.4)

$$W = \int_{t_0}^{t} p\,\mathrm{d}t$$

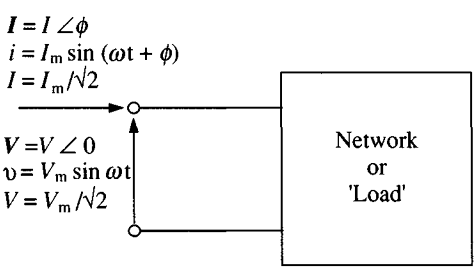
therefore the net area under one complete cycle of the instantaneous power curve represents the energy supplied to the load during a time equal to this period (equal to half the period of the supply frequency). When $\phi = 0$ (corresponding to a pure resistive load), this area is everywhere positive. As ϕ increases in magnitude, the negative area increases until for $|\phi| = \pi/2$ (corresponding to a pure reactive load), positive and negative areas are equal and there is no net transfer of energy to the load. For passive linear elements ϕ is always within the range $-\pi/2 \leqslant \phi \leqslant \pi/2$, so that the net area will never become negative.

The *average power* P supplied to the load is equal to the net area under one cycle of p in fig. 4.2 divided by the period of p. Thus the average power is maximum when $\phi = 0$ and zero when $|\phi| = \pi/2$.

An expression for P in terms of conveniently measurable quantities may be derived by expanding (4.1) using the appropriate trigonometrical identities, viz.,

$$p = \tfrac{1}{2} V_m I_m (\cos\phi - \cos 2\omega t \cos\phi + \sin 2\omega t \sin\phi) \tag{4.2}$$

Fig. 4.1. Instantaneous and phasor voltages and currents at the terminals of a reactive network or 'load'. Current leads voltage by phase angle ϕ.



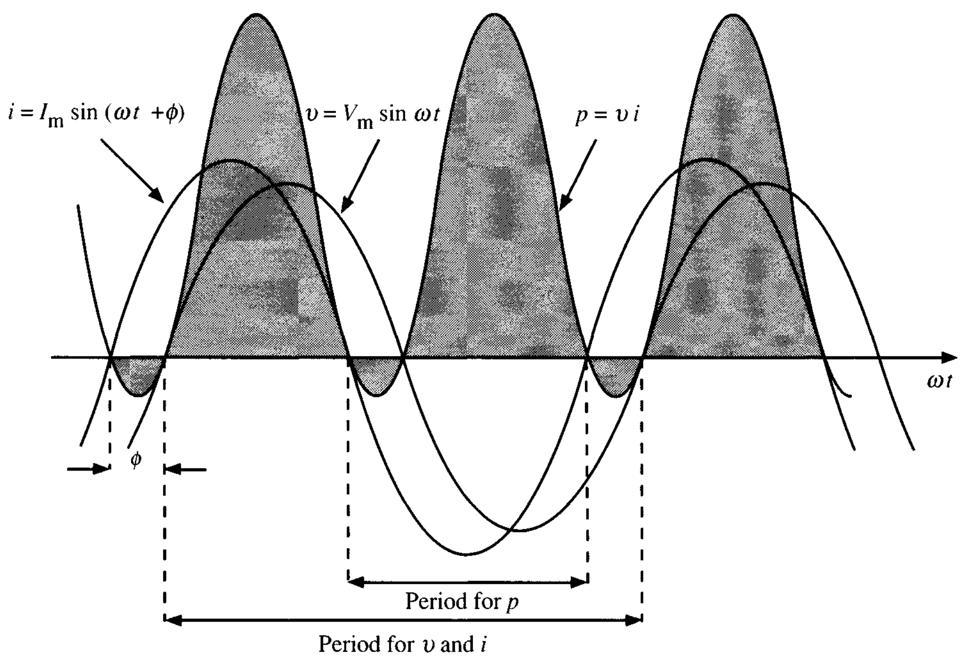
If T is the period of p, the average power is

$$P=\frac{1}{T}\int_0^T p\,\mathrm{d}t=\tfrac{1}{2}V_\mathrm{m}I_\mathrm{m}\cos\phi=VI\cos\phi \qquad (4.3)$$

In this expression V and I are the r.m.s. values. It is an important convention that all power line voltages and all voltages and currents marked on a.c. appliances such as a motor, for example, are r.m.s. values. Thus the European standard voltage of 240 V had a maximum value of $240\sqrt{2}=364$ V, similarly the standard in many parts of the U.S.A. is 120 V, which rises to $120\sqrt{2}=169$ V. The same applies to the ratings of fuses and circuit breakers; a fuse marked 5 A must carry a peak current of just over 7 A. In all a.c. problems the data are assumed to refer to r.m.s. quantities unless specifically stated otherwise.

Because, as noted above, $-\pi/2\leqslant\phi\leqslant\pi/2$, $\cos\phi$ in (4.3) is always positive or zero, so P is never negative. Note also that the expression (4.3) is independent of the choice of the quantity taken as phase reference.

Fig. 4.2. Instantaneous values of voltage, current and power for the circuit of fig. 4.1.



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4.3 Reactive power and apparent power

Further insight into the details of power flow comes from examination of (4.2). The three terms on the right and their sum are shown in fig. 4.3. The sum corresponds, of course, to the instantaneous power curve in fig. 4.2.

The first two terms on the right can be written

$$\tfrac{1}{2}V_m I_m(\cos\phi - \cos 2\omega t\cos\phi) = VI\cos\phi(1-\cos 2\omega t)$$

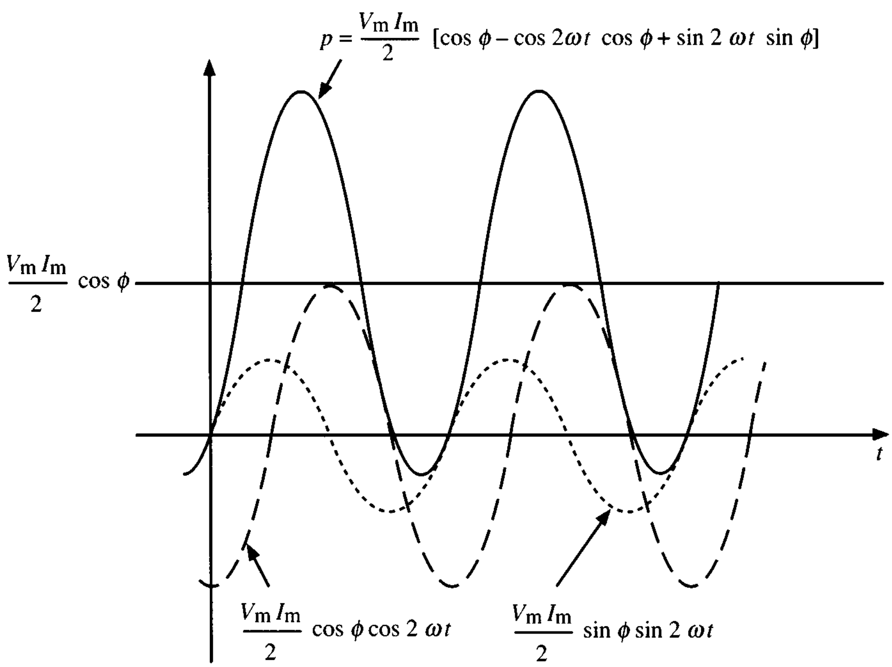
The sum of just these two terms represents the *instantaneous real power*; it never becomes negative, oscillating in amplitude between $2VI\cos\phi$ and zero. The average value is $VI\cos\phi$, which we have identified as the average power P. This quantity is often referred to as the *real power* or *active power*.

The third term on the right, namely,

$$\tfrac{1}{2}V_m I_m \sin 2\omega t\sin\phi = VI\sin\phi\sin 2\omega t,$$

represents energy that oscillates between the power source and the purely reactive elements in the load. Its average value is zero and its magnitude is $VI\sin\phi$. This quantity is referred to as *reactive power*; however, since it does

Fig. 4.3. The three components of instantaneous power (equation 4.2).



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not represent energy actually absorbed by the load, the designation *volt–amperes reactive* (abbreviated vars) is also commonly employed.

The sign of this term will depend on the sign of the phase angle ϕ, which in turn will depend on whether voltage or current is chosen as phase reference. In this instance, with voltage as phase reference, a capacitive load will draw a leading current, that is, ϕ will be positive and such a load is said to draw positive vars. An inductive load will draw negative vars. This convention has been adopted by the International Electrotechnical Convention, but the reader should be aware that the alternative convention (capacitive vars: negative; inductive vars: positive) is often used.

The symbol for reactive power or volt–amperes reactive is Q, thus

$$\text{Reactive power } Q = VI\sin\phi \tag{4.4}$$

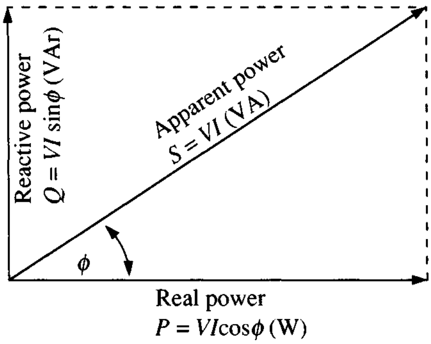
When a load on a power system has a phase angle other than zero, the resulting reactive power represents a requirement in system current capacity in addition to that necessary to supply the average power, that is, the actual power used to produce work or heat. It is customary, therefore, to specify the *apparent power* required by a load. This is simply the product of the effective (r.m.s.) voltage and effective current at the terminals of the load; it is usually designated by the symbol S. Thus,

$$\text{Apparent power } S = VI \tag{4.5}$$

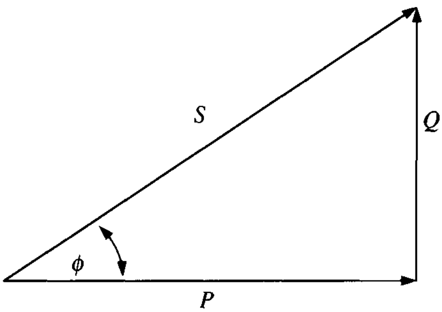
From the definitions of P, Q and S we see that the apparent power is given by quadrature addition of real and reactive power, that is,

$$P^2 + Q^2 = (VI\cos\phi)^2 + (VI\sin\phi)^2 = (VI)^2 = S^2$$
$$S = \sqrt{(P^2 + Q^2)} \tag{4.6}$$

Fig. 4.4. Diagrams illustrating the relationships between real, apparent and reactive powers.



(*a*) Power diagram



(*b*) Power triangle

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The relationship of these quantities may be represented diagrammatically by means of either the power diagram or power triangle as shown in fig. 4.4. Observe that only the power P is properly expressed in watts. Reactive power is expressed in vars (unit symbol VAr) and apparent power is expressed in volt–amperes (unit symbol VA). In the following sections and throughout the remainder of this book, when reference is made simply to *power* (without qualification) we shall mean the average, real, or active power.

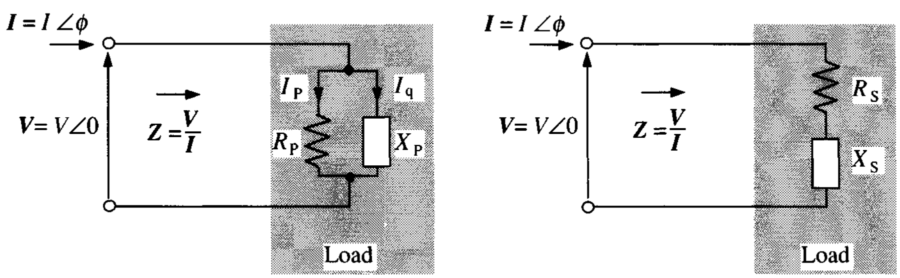
Any two-terminal circuit containing resistive and reactive elements may, according to Thévenin's theorem, be reduced to a single equivalent complex impedance or admittance. This implies that a load consuming energy may be characterized by either of the elementary series or parallel circuits shown in fig. 4.5. in which the R are pure resistances and the X are pure reactances. For either circuit, values for the components can be found such that any given impedance $\boldsymbol{Z} = \boldsymbol{V}/\boldsymbol{I}$ is produced at the terminals.

For power distribution calculations the parallel equivalent circuit of fig. 4.5(*a*) is often convenient. The phasor diagram for this circuit is shown in fig. 4.6(*a*). As before, the terminal voltage is chosen as reference, and ϕ is the angle by which the current leads the voltage. We see that the current I may be resolved into two components: (a) a current $I_p = I\cos\phi$, called the *in-phase* component, which flows through R_p; (b) a current $I_q = I\sin\phi$, called the *quadrature* component, which flows through X_p. The power diagram fig. 4.6(*c*) may be derived directly from this phasor diagram simply by multiplying the total current I and each of its components by the magnitude of the terminal voltage V.

Since $I_p = V/R_p$, $I_q = V/X_p$, and $I = V/Z$, the real, reactive and apparent powers may be expressed as:

$$\text{Power } P = VI\cos\phi = V\left(\frac{V}{R_p}\right) = \frac{V^2}{R_p}$$

Fig. 4.5. Equivalent circuits representing a complex load $\boldsymbol{Z}$.



(*a*) Parallel (*b*) Series

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$$\text{Reactive power } Q = VI\sin\phi = V\left(\frac{V}{X_\text{p}}\right) = \frac{V^2}{X_\text{p}} \tag{4.7}$$

$$\text{Apparent power } S = VI = V\left(\frac{V}{Z}\right) = \frac{V^2}{Z}$$

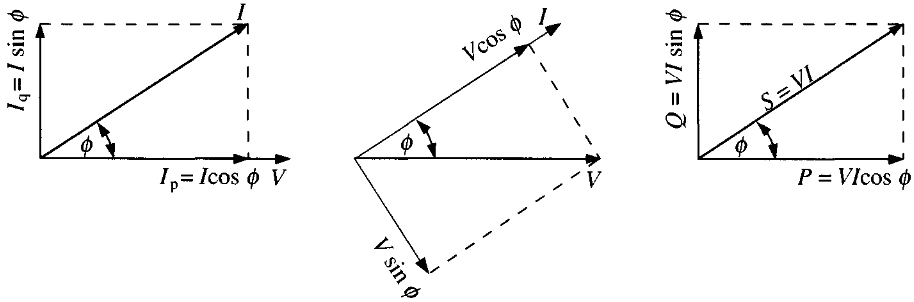
We now turn to the series equivalent circuit of fig. 4.5(*b*). Since this circuit presents the same complex impedance $\boldsymbol{Z}$ at its terminals as that of the parallel circuit of fig. 4.5(*a*), the same terminal voltage will cause the same current to flow; the power diagram of fig. 4.6(*c*) must therefore apply also to this circuit.

Using a similar procedure to that adopted in relation to the current we may resolve the voltage into two components (fig. 4.6(*b*)): that across R_s (in phase with I) is $IR_\text{s} = V\cos\phi$; and that across X_s (in quadrature with I) is $IX_\text{s} = V\sin\phi$. Expressions analogous to (4.7) are then

$$\begin{aligned} \textit{Power } P &= VI\cos\phi = I(IR_\text{s}) = I^2R_\text{s} \\ \text{Reactive power } Q &= VI\sin\phi = I(IX_\text{s}) = I^2X_\text{s} \\ \text{Apparent power } S &= VI = I(IZ) = I^2Z \end{aligned} \tag{4.8}$$

Although the expressions (4.7) and (4.8) have been derived above in relation to the parallel and series equivalent circuits of a general complex two-terminal network, it will be readily apparent that they apply also to the individual elements of which the network is composed if voltages and currents are interpreted appropriately. For example, if a current I_b flows through a series branch within the network containing a resistance R and a capacitance C, with reactance $1/\omega C$, then the power in the resistance is $I_\text{b}{}^2R$ and the reactive power in the capacitance is $I_\text{b}{}^2(1/\omega C)$. If the currents in

Fig. 4.6. Phasor and power diagrams for the circuits of fig. 4.5. (*a*) and (*b*). Resolution of current and voltage into in-phase and quadrature components. (*c*) Power diagram.



(*a*) (*b*) (*c*)

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every branch of a network are known, then the real and reactive powers associated with every element of that network can be simply calculated using the appropriate expressions.

Now we have seen that the real power at the terminals of a network is a measure of the average rate of energy flow associated with the resistive elements of the network, this energy being dissipated in the form of heat. The reactive power is a measure of the average rate of energy flow associated with the reactive elements, this energy oscillating between source and network. It follows from the principle of conservation of energy that the power, in watts, absorbed by all the resistive elements in a network must equal the total power supplied at its terminals. Likewise, the reactive power, in vars, associated with all of the reactive elements must equal the reactive power supplied at the terminals. Symbolically we have for any network:

$$\begin{aligned} \text{Total power (watts)} &= \Sigma(I^2R)_{\text{branch}} \\ \text{Total reactive power (vars)} &= \Sigma(I^2X)_{\text{branch}} \end{aligned} \tag{4.9}$$

The expressions (4.9), are sometimes loosely referred to as the *principle* of *conservation of watts* and *vars*.* (The use of this principle is illustrated, in relation to three-phase power systems, in the worked example of section 5.5.)

4.4 Power factor

The quantity $\cos\phi$ in (4.3) is the *power factor*. When energy is being drawn from a distribution system, the most desirable condition is $\cos\phi = 1$, because then the apparent power and the real power are identical and the current requirement for a given amount of delivered power is a minimum. The excess current requirement represented by $\cos\phi \neq 1$ means that the power distribution lines must be capable of supplying the additional current. It follows that there will be increased heating in the conductors and a corresponding decrease in the efficiency of the overall system. It follows further that an installation having a low power factor may reasonably be required to pay more for each unit of energy delivered than would an installation having a power factor close to unity.

Operators of establishments that use substantial amounts of power find it worthwhile to 'improve' the power factor, that is, to install equipment that will bring the power factor closer to unity. This improvement may be accomplished by installing in parallel with the load a device that draws a

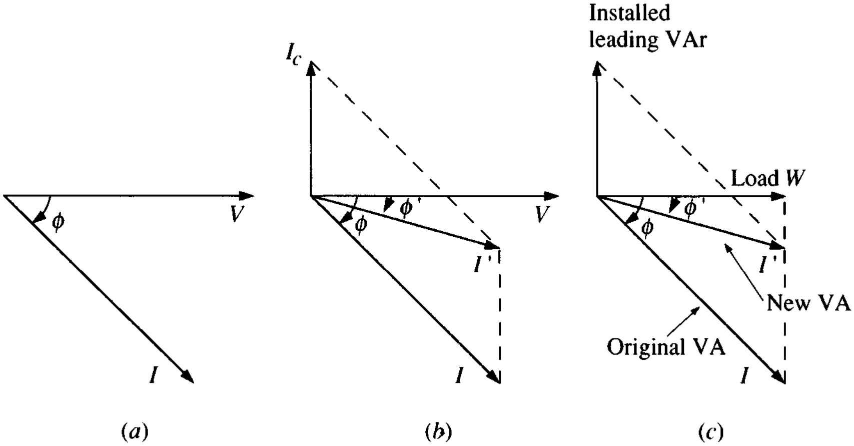
* The expressions (4.9) may be derived also from a more general theorem known as Tellegen's Theorem. This states that if v_k and i_k are the voltage across and the current through the kth element of a network, then $\Sigma v_k i_k = 0$ where the sum is taken over all elements of the network including sources. (For further details see reference 4.)

quadrature current of a sign opposite to that drawn by the load. Most industrial loads are predominantly inductive and draw a lagging current as shown in the phasor diagram of fig. 4.7(*a*). Such a load is said to be a *lagging load* and to possess a *lagging power factor*. The corrective measure usually consists in adding capacitance at the terminals of the load; the resulting arrangement being electrically similar to the parallel tuned circuit discussed in section 3.13.3. In fig. 4.7(*b*), I_c is the current drawn by the capacitance, the new (reduced) line current is I' and the new power factor is $\cos\phi'$. Looked at in another way, the usual load draws negative vars; therefore, one places in parallel a circuit element that draws positive vars. The power diagram shown in fig. 4.7(*c*), corresponding to the phasor diagram of fig. 4.7(*b*), illustrates power factor improvement from this viewpoint. We see that the total apparent power is reduced; the actual load power is, of course, unchanged. Usually one does not attempt to make the overall power factor unity (the circuit is not quite tuned to resonance) because, (1) the power factor of the load may vary as load conditions in a large installation change, so such exact correction would require continual adjustment, and (2) the cost of the added capacitance must be weighed against the saving that may be expected as a result of its installation.

Where large values of capacitance are required the capacitors may be rotating machines. An over-excited synchronous motor draws a leading current and so has the electrical characteristics of a capacitor.

The following worked example is intended to illustrate the principles

Fig. 4.7. Power factor correction. (*a*) Phasor diagram for inductive (lagging) load. (*b*) Modified phasor diagram showing the result of power factor correction. (*c*) Power diagram corresponding to (*b*).



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discussed above, but it should be remembered that power factor improvement is usually applied to three-phase rather than single-phase systems. Power factor and its improvement in relation to three-phase system is discussed in sections 5.4 and 5.5.

4.5 Worked example

A factory draws 18 kW of power from a 240 V, 50 Hz distribution system. The power factor is 0.75 lagging. The feeder wires to the factory have a total resistance of 0.35 Ω. The factory operates 24 hours, every day.
(a) Calculate the current supplied to the factory. If energy costs 1p/kW hr, determine the annual cost of the energy lost in heating the feeder wires. (Note: It is still common practice in power systems analysis to measure energy in units of kilowatt hours rather than joules.)
(b) If the power factor is improved to 0.95, how much money will be saved annually by the reduction of feeder wire loss?
(c) Determine the size and nature of the unit required to correct the power factor to 0.95.

Solution
(a)

$$\text{Power } P = VI\cos\phi$$

therefore

$$\text{current } I = \frac{P}{V\cos\phi} = \frac{18\times10^3}{240\times0.75} = 100\,\text{A}$$

$$\text{Power loss in feeder} = I^2R_f = 100^2\times0.35 = 3.5\,\text{kW}$$

$$\begin{aligned}\text{Annual cost} &= 3.5\,\text{kW}\times(24\,\text{hr/day})\times(365\,\text{day/yr})\times(1\text{p/kW hr})\\ &= £306\end{aligned}$$

(b) With a new power factor of 0.95 the current becomes

$$I = \frac{18\times10^3}{240\times0.95} = 78.9\,\text{A}$$

Power loss in feeder is then $78.9^2\times0.35 = 2.18$ kW.
New annual cost:

$$306\times\frac{2.18}{3.5} = £190.7$$

Saving:

$$£(306-191) = £115 \text{ per annum}$$

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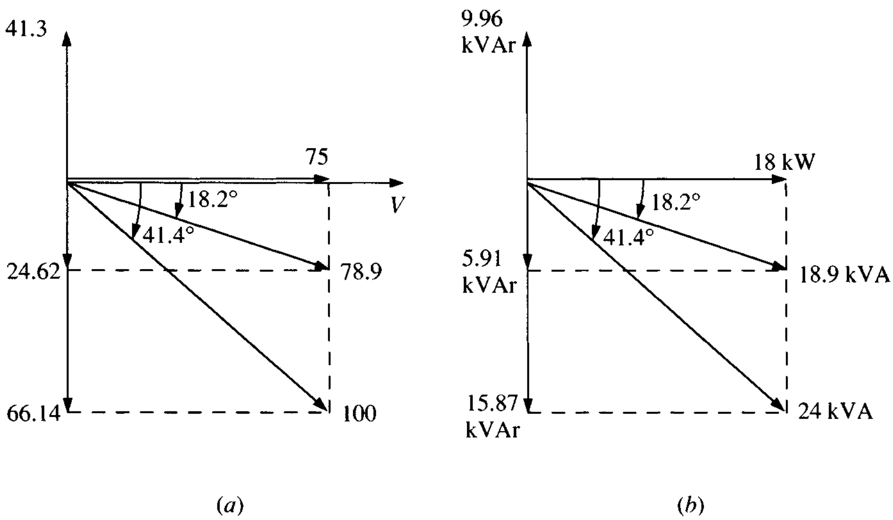
(c) The line voltage and initial current are shown in the phasor diagram of fig. 4.8(*a*). The in-phase current is 75 A and the quadrature current is 66.14 lagging. The initial phase angle is 41.4°. With the power factor improved to 0.95 the new phase angle is $\cos^{-1} 0.95 = 18.2°$. The new total current is 78.9 A and the new quadrature current is $78.9 \sin 18.2 = 24.62$ A. To achieve this improvement a capacitor must be connected across the load terminals which will draw a current of $66.14 - 24.62 = 41.3$ A. The reactance of the capacitor will be $X_c = 240/41.3 = 5.8\,\Omega$. But $X_c = 1/(2\pi fC)$ hence

$$C = 1/(2\pi \times 50 \times 5.8) = 550\,\mu\text{F}$$

An alternative approach to this problem is shown in fig. 4.8(*b*). Here the calculation is carried out in terms of real, apparent and reactive powers. We see that the capacitor is required to draw 9.96 kVAr, hence, from (4.7), $X_c = 240^2/9960 = 5.8\,\Omega$ as before.

Observe that addition of the capacitor does not reduce the reactive current *in the load*. The inductive load still draws vars equal to 15.87 kVAr. However, with the capacitor in place most of the quadrature current flows between the capacitor and the load. Power loss in the feeder wire conductors is thus greatly reduced.

Fig. 4.8. Diagrams for worked example (section 4.5).



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4.6 Complex power

For the purposes of calculation it is often convenient to express the relationship between power, reactive power and apparent power in the form of a complex number, as follows:

$$S = P + \mathrm{j}Q = VI\cos\phi + \mathrm{j}VI\sin\phi = VI\mathrm{e}^{\mathrm{j}\phi} \tag{4.10}$$

The power diagram of fig. 4.4(*a*) may then be interpreted in terms of the Argand diagram as shown in fig. 4.9. Now suppose that the complex voltage and current are known at the terminals of a load; these quantities having been found by previous calculation. Let

$$\boldsymbol{V} = V\underline{/\alpha} = V\mathrm{e}^{\mathrm{j}\alpha} \qquad \text{and} \qquad \boldsymbol{I} = I\underline{/\alpha + \phi} = I\mathrm{e}^{\mathrm{j}(\alpha+\phi)}$$

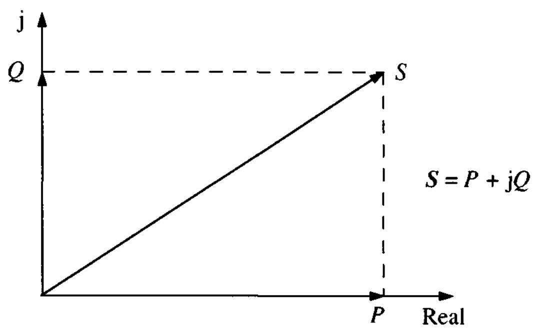
If we form the product $\boldsymbol{VI}(=VI\mathrm{e}^{\mathrm{j}(2\alpha+\phi)})$, we fail to obtain the complex power as defined in (4.10). However, if we form the product $\boldsymbol{V}^*\boldsymbol{I}$, where $\boldsymbol{V}^*$ indicates the complex conjugate of $\boldsymbol{V}$, namely, $V\mathrm{e}^{-\mathrm{j}\alpha}$, we obtain

$$\boldsymbol{V}^*\boldsymbol{I} = V\mathrm{e}^{-\mathrm{j}\alpha}I\mathrm{e}^{\mathrm{j}(\alpha+\phi)} = VI\mathrm{e}^{\mathrm{j}\phi} = \boldsymbol{S} \tag{4.11}$$

Therefore, if we have the complex expressions for the voltage $\boldsymbol{V}$ across a load and the current $\boldsymbol{I}$ drawn by the load and if we form the product $\boldsymbol{V}^*\boldsymbol{I}$, the real part of this product is the power P and the imaginary part is the reactive power Q.

Note that (4.11) gives the correct sign for reactive power using the convention adopted here, namely, that capacitive vars are positive, inductive vars negative. If the alternative convention is used, (capacitive vars negative, inductive vars positive), then the product $\boldsymbol{VI}^*$ must be formed to obtain the correct sign for the reactive power. The real part of either $\boldsymbol{V}^*\boldsymbol{I}$ or $\boldsymbol{VI}^*$ will give the power P.

Fig. 4.9. Apparent power expressed as a complex quantity.



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An alternative expression for P, which is sometimes useful, is obtained as follows:

$$V^*I = VIe^{j\phi}$$

and

$$VI^* = Ve^{j\phi}Ie^{-j(\alpha+\phi)} = VIe^{-j\phi}$$

Adding

$$V^*I + VI^* = VI(e^{j\phi} + e^{-j\phi}) = 2VI\cos\phi = 2P$$

Hence

$$P = \tfrac{1}{2}(V^*I + VI^*) \tag{4.12}$$

4.7 The ideal transformer

In section 1.10 we described inductively coupled circuits. A useful application of inductive coupling is the transformer, a device that transfers energy from one circuit to another without direct connection between the two. A *real* transformer consists of two or more coils of wire wound on a common core. The core usually is of ferromagnetic material in order to achieve as nearly as possible a coefficient of coupling of unity. Such a device has losses in the resistance of the windings and in the magnetic core material. Real transformers will be considered in detail in a later section.

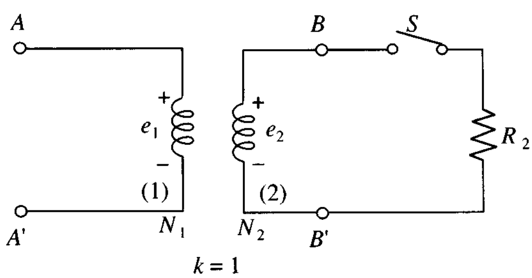
Useful information about transformer characteristics results from a study of the *ideal* transformer which is assumed to have

(a) negligible energy loss in the windings and in the core material
(b) perfect coupling so that the coupling coefficient $k = 1$
(c) very large self-inductance in each coil.

A transformer may have any number of independent windings. We shall consider the two-winding device shown schematically in fig. 4.10.

Let the supply voltage be connected at terminals AA' of winding number

Fig. 4.10. Schematic diagram for an ideal two-winding transformer.



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(1). It is customary to refer to the input windings as the *primary* winding (or simply the primary) and to the other windings as *secondary* windings (or secondaries). In general, energy may be supplied to any winding, so 'primary' may not always refer to the same winding of a particular transformer.

In fig. 4.10 with switch S *open*, an applied voltage e_1 requires, according to Faraday's Law, a flux ϕ_1 such that $e_1 = N_1 \mathrm{d}\phi_1/\mathrm{d}t$, $N_1\phi_1$ being the flux linkage in winding (1). But if the coupling coefficient is unity, the flux in winding (2) will also be ϕ_1, and the induced voltage $e_2 = \mathrm{N}_2\mathrm{d}\phi_1/\mathrm{d}t$. It follows then that:

$$\frac{e_1}{e_2} = \frac{N_1}{N_2} \tag{4.13}$$

and the voltage ratio is just equal to the ratio of numbers of turns. Because we have specified very large coil inductance, we may assume that with S open a negligibly small amount of current is required to establish the flux ϕ_1.

Now let the switch be *closed*. The secondary voltage e_2 will then give rise to a current $i_2 = e_2/R_2$ in winding (2), which will establish a flux ϕ_2. We recall, from Amperes circuital law of electromagnetic theory, that this flux ϕ_2 is proportional to the product $N_2 i_2$ so that we may write $\phi_2 = cN_2 i_2$ where c is a constant of proportionality. Further, from Lenz's law, the current i_2 will be in such a direction as to oppose the change of magnetic flux producing it, in other words, the direction of the flux ϕ_2 will be opposite to that of ϕ_1. Because e_1 is still applied to winding (1), the net flux in the core must still be ϕ_1, consequently there must now be a current i_1 in winding (1) of sufficient magnitude to produce a forward flux (that is, in the direction of the initial flux ϕ_1) just equal in magnitude to ϕ_2. Therefore, $\phi_2 = cN_1 i_1$ and we have the relation:

$$\phi_2 = cN_2 i_2 = cN_1 i_1$$

or

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{4.14}$$

With the switch closed, the secondary current and voltage are related by $e_2/i_2 = R_2$, hence, from (4.13) and (4.14) we obtain for the primary:

$$\frac{e_1}{i_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{e_2}{i_2} = \left(\frac{N_1}{N_2}\right)^2 R_2 \tag{4.15}$$

The ratio e_1/i_1 is the effective resistance at the primary, thus a resistance R_2

in the secondary is reflected into the primary as a resistance whose value depends upon R_2 and upon the square of the ratio of primary turns to secondary turns.

Equation (4.15) describes the resistance-transforming property of the transformer. Because the relation holds equally for reactive circuit elements, we may conclude that the transformer has impedance transforming properties. This property is useful in applications such as the following.

4.8 Worked example

Loudspeakers have resistance of the order of 10 ohms. Neither vacuum tube nor many simple transistor amplifier circuits operate satisfactorily with such small values of load resistance. Therefore a transformer is employed to raise the impedance level of the speaker to a value compatible with the requirements of the amplifier circuit. What turns ratio n is required to match a $16\,\Omega$ speaker to a transistor circuit that is designed to have a load resistance of $400\,\Omega$?

Solution: When the $16\,\Omega$ resistance is connected to the secondary, it should be reflected into the primary as $400\,\Omega$, therefore,

$$(N_1/N_2)^2 = R_1/R_2 \text{ or } (N_1/N_2)^2 = 400/16 = 25$$

and

$$n = (N_1/N_2)^{\frac{1}{2}} = 25 = 5$$

4.9 Single-phase power transformers

Although some transmission lines operate at high (hundreds of kilovolts) direct voltage, all power distribution systems use alternating voltage. Power transformers are essential parts of these systems. Such devices, which may operate at high voltage and carry large currents, cannot be represented by the simple model of fig. 4.10. In this section we develop a model for a power transformer. In succeeding sections we see how the transformer is used in a real circuit and describe methods of determining experimentally the characteristics of a power transformer.

In a power transformer, a core of high permeability ferromagnetic material provides a path for magnetic flux. The separate circuits are wound in such a fashion that practically all the flux produced by one winding links with all the other windings. Sufficient ferromagnetic material is provided so that it does not become saturated under normal operating conditions. Fig. 4.11 shows three frequently used transformer constructions. We shall confine our discussion to transformers having two windings, although often several independent secondary circuits may be supplied from a single primary winding.

The simple ideal transformer model of fig. 4.10 is inappropriate for the power transformer. We require a model that takes into account: (1) the *exciting current* which is defined as the primary current when the secondary current is zero; (2) the energy loss in the conductors that comprise the windings; and (3) the presence of *leakage* flux (the magnetic flux from one winding that does not link the other winding).

Let the transformer consist of a primary and secondary wound upon a high permeability core as shown schematically in fig. 4.12. Let the secondary be open circuit so that the secondary current is zero. If the flux in the primary is sinusoidal, of the form $\phi = \Phi_m \sin\omega t$, then the primary induced voltage is

$$e_1 = N_1 \frac{d\phi}{dt} = N_1 \frac{d}{dt}(\Phi_m \sin\omega t)$$

$$= \omega N_1 \Phi_m \cos\omega t \tag{4.16}$$

Fig. 4.11. Transformer construction.

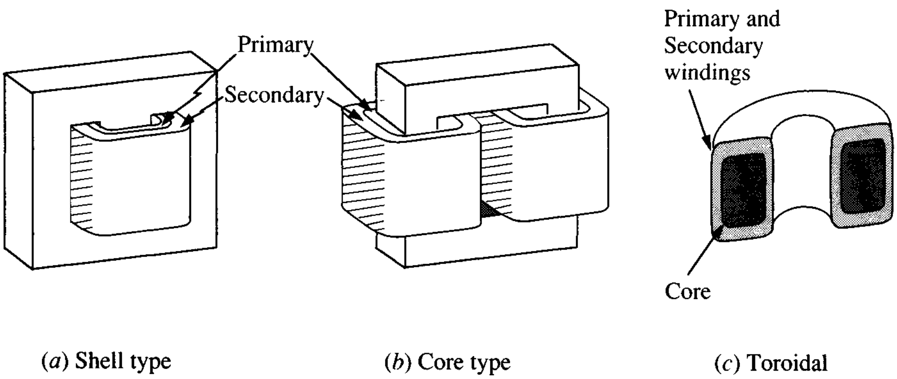
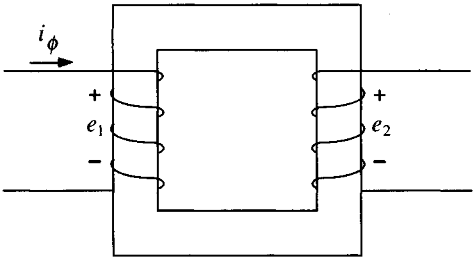


Fig. 4.12. Two-winding transformer with ferromagnetic core. With the secondary open circuit a small exciting current i_ϕ flows in the primary.



and e_1 is, therefore, also of sinusoidal form leading ϕ by 90° (fig. 4.13(*a*)). The r.m.s. magnitude of the induced voltage is

$$E_1 = \frac{2\pi f}{\sqrt{2}} N_1 \Phi_m = 4.44 f N_1 \Phi_m \tag{4.17}$$

Thus, an applied sinusoidal voltage of this magnitude will cause a sinusoidal flux to be established whose maximum value (if the effects of conductor resistance and leakage flux are negligible) will be given by

$$\Phi_m = \frac{E_1}{4.44 f N_1}$$

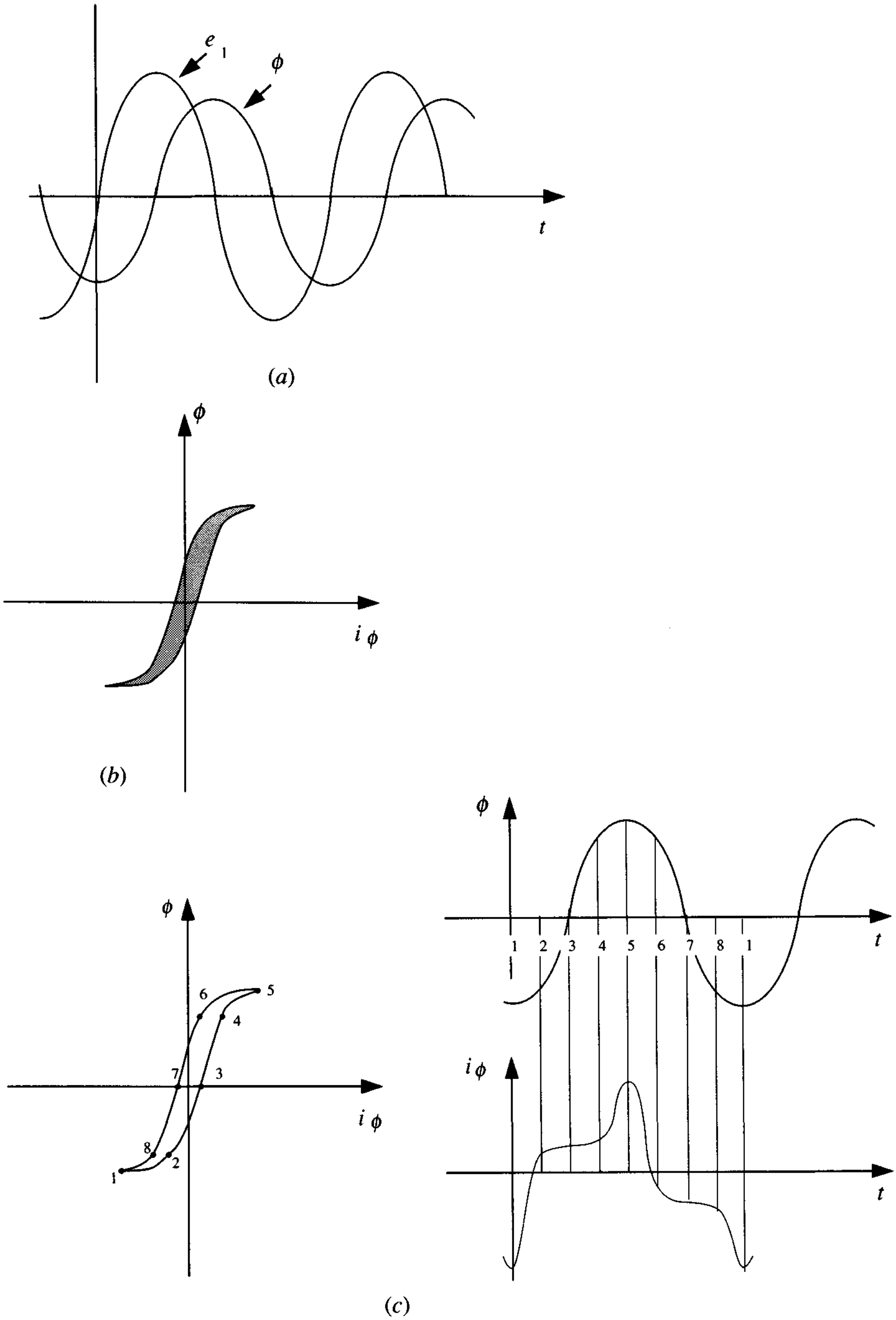
In order for the flux to exist, there must be an exciting current i_ϕ. For an air-cored coil, flux is directly proportional to current, so with a sinusoidal applied voltage, i_ϕ will be sinusoidal. In a closed ferromagnetic core, *flux density B* and *magnetizing force H* are not linearly related and one must describe the core's magnetic properties by the familiar B–H curve. Now

$$B = \frac{\phi}{A} \text{ and } H = \frac{Ni}{\ell} \tag{4.18}$$

where N is the number of turns in the winding, and A and ℓ are, respectively, the cross-sectional area and the length of the flux path in the magnetic material. For a specific device, then, one may plot ϕ v. i_ϕ as shown in fig. 4.13(*b*). Then we may use a graphical method as shown in fig. 4.13(*c*) to derive the graph of i_ϕ v. time. This current is the exciting current of the transformer. Because of the non-linear magnetic characteristic of the transformer core, i_ϕ is non-sinusoidal, consisting of a fundamental and a set of odd harmonics (see section 7.8 for a discussion of harmonic (Fourier) analysis). Thus a sinusoidal applied voltage results in a non-sinusoidal exciting current.

In comparison with the rated load current of the transformer the exciting current is small. Unless we are interested specifically in its harmonic content, we may confine our attention to the fundamental, sinusoidal frequency component of i_ϕ. This fundamental may be resolved into two components. One component, called the *magnetizing current*, is in phase with ϕ, and so in quadrature with e_1. The other component, called the *core loss current*, leads ϕ by 90° and so is in phase with e_1. The core loss current thus represents power supplied to the transformer. It accounts for the work required periodically to magnetize the core first in one direction and then in the other. On the B–H plane, the area enclosed by the hysteresis loop is a measure of the work required to carry the magnetic material through one

Fig. 4.13. Primary voltage e_1, core flux ϕ, and primary exciting current i_ϕ for the transformer of fig. 4.12.



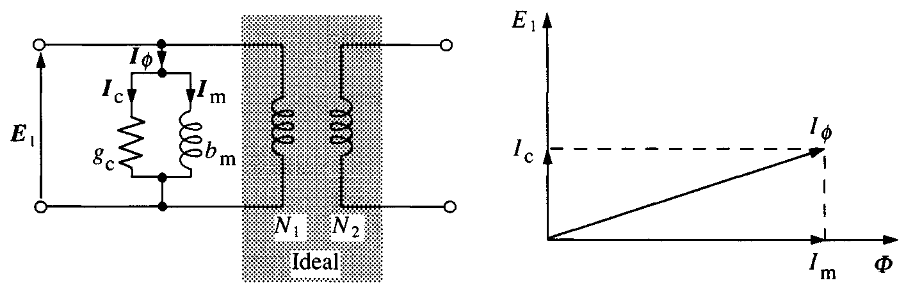
complete cycle of magnetization. The core loss current also accounts for the power loss due to eddy currents induced in the transformer core.

The phasor diagram of fig. 4.14(b) shows the applied voltage E_1 and the exciting current I_ϕ along with its two components, I_m and I_c. The exciting current may be accounted for in the transformer model by adding the conductance g_c and the susceptance b_m to the ideal transformer model as shown in fig. 4.14(a), where

$$g_c = I_c/E_1 \qquad \text{and} \qquad b_m = I_m/E_1 \tag{4.19}$$

When an impedance is connected to the secondary, currents will flow in secondary and primary circuits 4.15(a). The total current $\boldsymbol{I}_1$ flowing in the primary circuit will be the sum of the load current $\boldsymbol{I}_1'$ and the exciting current $\boldsymbol{I}_\phi$ as shown in the phasor diagram of fig. 4.15(b). In this diagram the length of the $\boldsymbol{I}_\phi$ phasor has been exaggerated in relation to that of the $\boldsymbol{I}_1'$

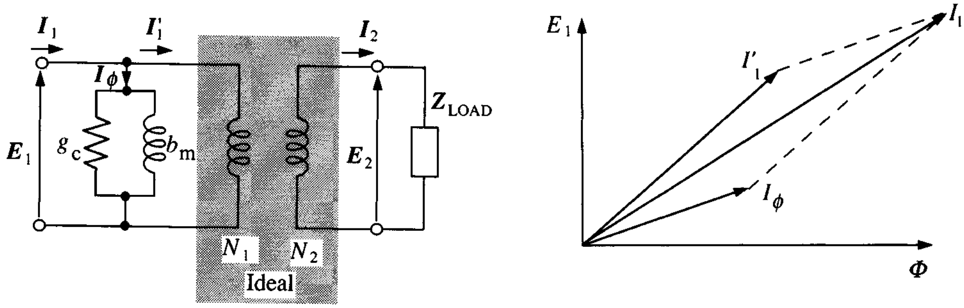
Fig. 4.14. Circuit model of the transformer including elements g_c and b_m to account for the exciting current I_ϕ.



(a) Circuit model

(b) Phasor diagram

Fig. 4.15. Illustrating the effect of adding a load to the secondary winding in the circuit model of fig. 4.14(a).



(a) Circuit model

(b) Phasor diagram

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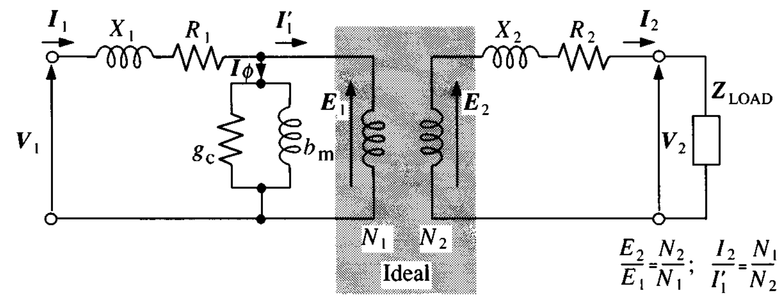
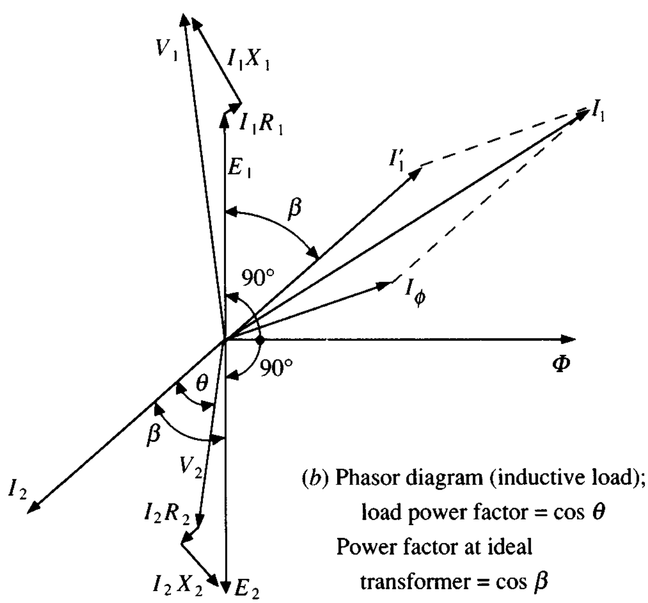
phasor; the full load current may be twenty times the exciting current in a power transformer.

To complete the circuit model of the transformer we must introduce elements which account for the resistance of the windings and the leakage flux. Consider first the primary leakage flux, that is, the flux which links with the primary winding but not the secondary winding; let this flux be ϕ_{l1}. The path which this flux takes lies partly in the core and partly in the air space surrounding the core; the primary winding will, therefore, as far as this flux is concerned, behave very nearly like an air-cored inductor, and the

Fig. 4.16. Transformer circuit model including elements X_1, X_2, R_1, R_2 to account for leakage reactances and winding resistances.



(*a*) Circuit model



(*b*) Phasor diagram (inductive load); load power factor = cos θ
Power factor at ideal transformer = cos β

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flux linkage $N_1\phi_{\ell 1}$ will be proportional to the primary current. (This is in contradistinction to the mutual flux which, as we have seen, is independent of the load currents flowing in the windings.) Hence, from the definition of inductance given in section 1.9, we may write

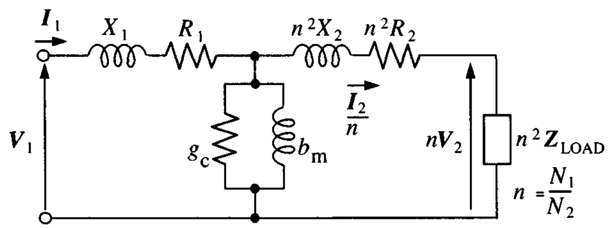
$$N_1\phi_{\ell 1} = L_1 i_1 \tag{4.20}$$

where the constant of proportionality L_1 is called the *primary leakage inductance.*

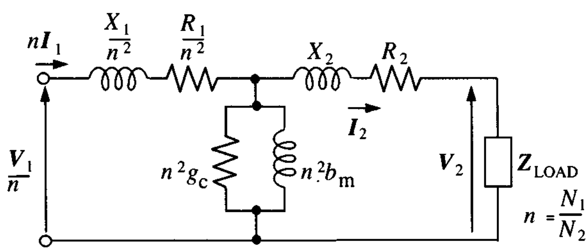
This inductance is represented in the circuit model of fig. 4.16(*a*) by the series reactance $X_1 = \omega L_1$. Similarly a reactance X_2 in the secondary circuit represents the secondary leakage inductance. Resistances R_1 and R_2 account for the resistances of the windings. The complete phasor diagram, drawn for an inductive load (lagging current) corresponding to this circuit model is shown in fig. 4.16(*b*). Note that the lengths of the phasors representing reactive and resistive voltage drops have been exaggerated, in relation to the main voltage phasors, for the sake of clarity. The relative phase of E_1 and E_2 can be either zero or 180°, depending on the relative winding directions of the transformer; the latter has been chosen – again for clarity in the diagram.

We may employ the impedance transforming property of the transformer to simplify the circuit of fig. 4.16(*a*). Resistance and reactance parameters are transferred across the ideal transformer by multiplying them by the square of the turns ratio. Fig. 4.17(*a*) shows the simplified circuit model

Fig. 4.17. Alternative transformer circuit models.



(*a*) Quantities referred to the primary



(*b*) Quantities referred to the secondary resulting when secondary quantities are ‘referred to the primary’. Fig. 4.17(*b*) is the circuit model with quantities ‘referred to the secondary’. Of course, in any calculation, both models give the same final result; the choice of model will depend on the particular problem in hand.

Calculations using the circuit models are simplified if the shunt elements representing the exciting current are shifted to the left in fig. 4.17 so that they are directly across the input voltage as shown in fig. 4.18. A small but usually insignificant error is introduced by neglect of the voltage drops in R_1 and X_1 caused by the exciting current.

4.10 Worked example

A single-phase transformer with turns ratio $n = N_1/N_2 = 10$ has an output of 200 volts when supplying a load of 10 kVA at 0.8 power factor lagging. Resistance and leakage reactance are 4 Ω and 7 Ω respectively for the primary and 0.04 Ω and 0.08 Ω for the secondary. The exciting current is 0.5 A at 0.2 power factor lagging. Calculate the input voltage and the efficiency of the transformer with this load.

Solution: The appropriate circuit is found by calculating numerical values for the elements shown in fig. 4.18, and combining resistances and reactances in series.

$$R = R_1 + n^2 R_2 = 4 + (10)^2(0.04) = 8\,\Omega$$
$$X = X_1 + n^2 X_2 = 7 + (10)^2(0.05) = 15\,\Omega$$

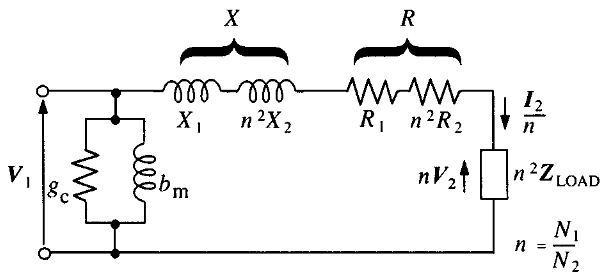
The load current is

$$I_2 = 10 \times 10^3/200 = 50\,\text{A}$$

Then,

$$nV_2 = 2000\,\text{V} \qquad \text{and} \qquad I_2/n = 5\,\text{A}$$

Fig. 4.18. Simplified transformer circuit model with quantities referred to primary.



Assume $I_2/n = 5 + j0$ (phase reference). Then, since the power factor of the load is 0.8,

$$nV_2 = 2000(0.8) + j2000(0.6) = 1600 + j1200\ \text{V}$$

Hence,

$$V_1 = (I_2/n)(R + jX) + nV_2 = 5(8 + j5) + (1600 + j1200) = 1640 + j1275$$

and

$$V_1 = (1640^2 + 1275^2)^{\frac{1}{2}} = 2080\ \text{V}$$

The efficiency of the transformer is found by dividing the output power by the input power. Furthermore, the input power may be written as (output power + losses). Now,

$$\text{Output power} = V_2 I_2 \cos\theta = 10^4(0.8) = 8 \times 10^3 \text{ watts}$$
$$\text{Copper losses} = (I_2/n)^2 R = 5^2 \times 8 = 200\ \text{W}$$
$$\text{Core loss} = V_1 I_e \cos\theta' = (2080)(0.5)(0.2) = 208\ \text{W}$$

Therefore,

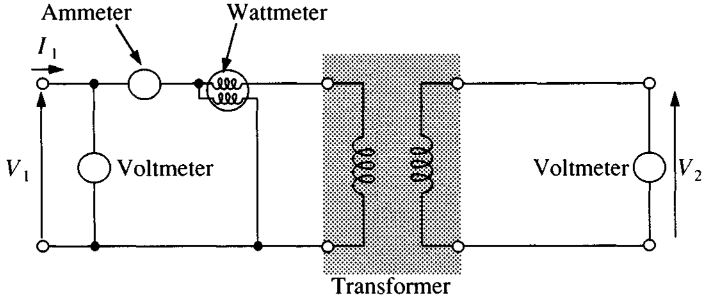
$$\text{Efficiency} = (8000)/(8000 + 408) = 0.951 \text{ or } 95.1\%$$

4.11 Transformer tests

All the elements in the circuit model of fig. 4.18 may be determined experimentally by performing two simple tests using voltmeters, an ammeter and a wattmeter. Connections are shown in fig. 4.19.

Open circuit test: The appropriate circuit model for this test is shown in fig. 4.20(a). Rated voltage V_{1o} is applied at the primary terminals, and the output voltage V_{2o}, the input current I_{1o}, and the input power P_o are measured.

Fig. 4.19. Connections for transformer tests.



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Since the secondary is open circuit, $I_2 = 0$ and so $I_1' = 0$. Therefore $E_1 = V_{1o}$ and $E_2 = V_{2o}$ and the turns ratio is, by (4.13),

$$\frac{N_1}{N_2} = \frac{V_{1o}}{V_{2o}} \tag{4.21}$$

Since $I_1 = 0$, the current I_{1o} is the exciting current. Therefore

$$g_c = \frac{P_o}{V_{1o}^2} \quad \text{and} \quad \frac{I_{1o}}{V_{1o}} = \sqrt{(g_c^2 + b_m^2)} \tag{4.22}$$

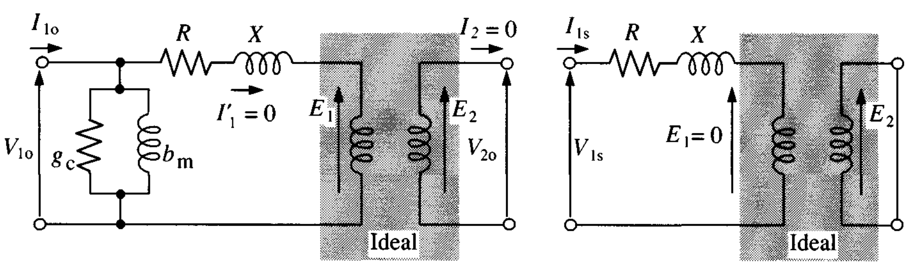
from which b_m may be calculated.

Short circuit test: The secondary winding is shorted (fig. 4.20(*b*)) and the primary voltage raised carefully from a low value to give rated current. The input voltage V_{1s}, the input current I_{1s} and the power P_s are measured. Because all impedances are referred to the primary in fig. 4.20, when $V_2 = 0$ then $E_2 = 0$ and $E_1 = 0$. So the voltage V_{1s} is across the impedance $R + jX$. For a typical short circuit test the input voltage may be only one-tenth the rated value. Then the loss in the magnetizing circuit will be only $(0.1)^2 = 0.01$ of the loss at rated voltage and may be neglected in these measurements. R and X may be determined from:

$$R = \frac{P_s}{I_{1s}^2} \quad \text{and} \quad \frac{V_{1s}}{I_{1s}} = \sqrt{(R^2 + X^2)} \tag{4.23}$$

For most transformer calculations it is not necessary to subdivide R and X to represent the separate contributions of the primary and the secondary windings.

Fig. 4.20. Circuit models for open and short circuit tests on the transformer. R and X are total resistance and reactance referred to the primary, as in fig. 4.18.



(a) Open circuit test

(b) Short circuit test

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4.12 Voltage regulation

An important characteristic of a transformer is the *voltage regulation*, defined as

$$\text{Regulation} = \frac{V_{2o} - V_2}{V_{2o}} \tag{4.24}$$

where V_{2o} is the no-load output voltage and V_2 is the output voltage under a specified load. We shall derive an expression for regulation in terms of the transformer properties and the phase angle of the load. We use the circuit model of fig. 4.21 where resistance and reactance are referred to the secondary. In fig. 4.21

$$R' = R_2 + (N_2/N_1)^2 R_1 \qquad X' = X_2 + (N_2/N_1)^2 X_1$$

If

$$I_2 = 0,$$

then

$$V_{2o} = (N_2/N_1)V_1$$

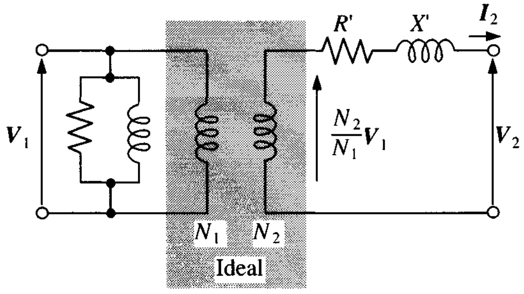
If there is a load current I_2 such that the phase angle for the secondary is ϕ, the appropriate phasor diagram is as shown in fig. 4.22. In fig. 4.22

$$a = I_2 R' \cos\phi \quad ; \quad b = I_2 R' \sin\phi$$
$$c = I_2 X' \cos\phi \quad ; \quad d = I_2 X' \sin\phi$$

Then

$$\left[\frac{N_2}{N_1} V_1\right]^2 = (V_2 + a + d)^2 + (c - b)^2$$

Fig. 4.21. Circuit model for calculation of regulation.



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In a well-designed transformer the losses are small (the voltage drops I_2R' and I_2X' have been greatly exaggerated in relation to V_2 in fig. 4.22) and so we neglect the difference term $(c-b)^2$. Then

$$\frac{N_2}{N_1}V_1 = V_2 + I_2(R'\cos\phi + X'\sin\phi)$$

therefore

$$\frac{N_2}{N_1}V_1 - V_2 = I_2(R'\cos\phi + X'\sin\phi)$$

But

$(N_2/N_1)V_1 = V_{2o}$, hence from (4.24)

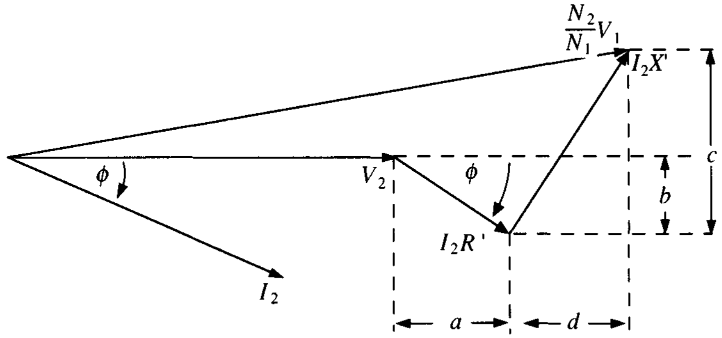
$$\text{Regulation} = \frac{I_2(R'\cos\phi + X'\sin\phi)}{V_{2o}} \tag{4.25}$$

It is often convenient to express regulation in terms of quantities referred to the primary. Multiplying numerator and denominator of (4.25) by $(N_1/N_2)^2$ gives

$$\text{Regulation} = \frac{I_2\left[R'\left(\frac{N_1}{N_2}\right)^2\cos\phi + X'\left(\frac{N_1}{N_2}\right)^2\sin\phi\right]\frac{N_2}{N_1}}{\frac{N_1}{N_2}V_{2o}}$$

Now from (4.21), $V_1 = (N_1/N_2)V_{2o}$ and, if the exciting current is small, we have $I_1 = (N_2/N_1)I_2$. Furthermore, $R_1'(N_1/N_2)^2 = R$ and $X'(N_1/N_2)^2 = X$. Hence,

Fig. 4.22. Phasor diagram, corresponding to fig. 4.21, for determining regulation.



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$$\text{Regulation} = \frac{I_1(R\cos\phi + X\sin\phi)}{V_1} \tag{4.26}$$

For a lagging power factor, ϕ (as defined in fig. 4.22) is positive and so the regulation is positive meaning that the voltage falls with increasing load current. However, for a leading power factor ϕ is negative in (4.26). There is then the possibility that the regulation may be negative, corresponding to an increased output voltage under load.

4.13 Conditions for maximum efficiency

The efficiency of a transformer can be expressed as

$$\eta = \frac{\text{output power}}{\text{input power}} = \frac{\text{input power} - \text{losses}}{\text{input power}}$$

or

$$\eta = 1 - \frac{\text{losses}}{\text{input power}} \tag{4.27}$$

Losses are of two kinds: core losses (hysteresis and eddy current) and copper losses. The core losses depend on the magnitude of the flux in the core, which in turn depends on the applied voltage, and on the frequency. For fixed voltage and frequency the core losses in a transformer will be substantially constant irrespective of load conditions. The copper losses depend on the current carried by the windings of the transformer and will vary with the load.

Let the fixed core losses be P_0, and the variable copper losses be $I_1{}^2R$, where R is the total equivalent resistance referred to the primary and I_1 is the primary current.

Then if the primary voltage is V_1 and the power factor is $\cos\phi$, (4.27) becomes

$$\eta = 1 - \frac{P_0 + I_1{}^2R}{V_1 I_1 \cos\phi} = 1 - \frac{P_0}{V_1 I_1 \cos\phi} - \frac{I_1 R}{V_1 \cos\phi} \tag{4.28}$$

Clearly, when the transformer is unloaded, the input power will be P_0 (neglecting the small copper loss due to the exciting current) and the efficiency will be zero. As the load is increased from zero the efficiency improves but copper losses become an increasingly large proportion of the total losses. Eventually, the third term in (4.28) predominates and the efficiency falls. The load current at which maximum efficiency occurs is obtained by differentiating (4.28) and equating to zero:

$$\frac{d\eta}{dI_1} = \frac{P_0}{V_1 I_1{}^2 \cos\phi} - \frac{R}{V_1 \cos\phi} = 0$$

whence

$$P_0 = I_1^2 R \tag{4.29}$$

Therefore, at a given power factor, maximum efficiency is obtained when the fixed core losses equal the variable copper losses. Under these conditions the primary load current is, from (4.29), $I_1 = \sqrt{(P_0/R)}$ and the efficiency is then

$$\eta_{max} = 1 - \frac{2\sqrt{(P_0 R)}}{V_1 \cos\phi} \tag{4.30}$$

The maximum possible efficiency is attained when the power factor, $\cos\phi$, becomes unity in the above expression.

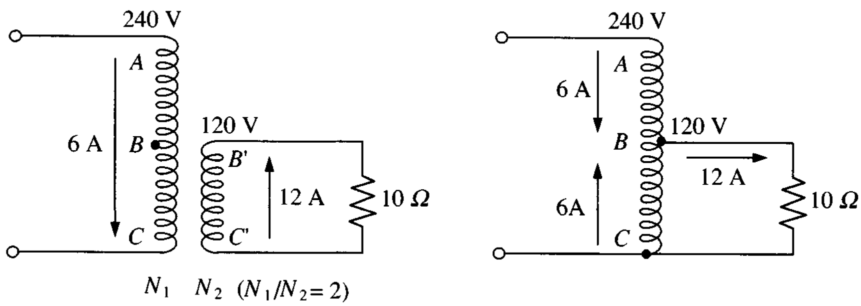
In practice, transformer efficiencies range between 95% for small, single-phase units to better than 98% for large, three-phase units of the type employed in power distribution systems.

4.14 The autotransformer

Figure 4.23(*a*) shows a conventional two-winding transformer, with a primary/secondary turns ration of 2. We assume that the transformer is ideal. The primary is supplied from the 240 V a.c. line. The voltage across the secondary winding $B'C'$ is then 120 V and a 10 ohm load resistor will draw a secondary current of 12 A. The corresponding primary current is 6 A. The primary current is downward and the secondary current is upward as indicated by the arrows.

Now let point C and C' be connected together and let point B' be connected to point B, which is located midway between points A and B (so that the number of turns between A and B is equal to the number of turns

Fig. 4.23. Currents in a two-winding transformer, and an autotransformer with identical input and load conditions.



(*a*) Two-winding transformer

(*b*) Autotransformer

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between B and C). These connections will cause no change in the circuit because $V_{BC} = V_{B'C'}$ and the two voltages are in phase. Let winding $B'C'$ be removed. The new situation is shown in fig. 4.23(b) with currents as indicated; the net upward current in BC is now 6 A. By using the connections of fig. 4.23(b), we have replaced two windings carrying, respectively, 6 A and 12 A by a single winding carrying 6 A. We have eliminated the weight and cost of the secondary winding without affecting the transfer of energy from primary to secondary. (In addition, we have eliminated the loss of energy resulting from the resistance of the secondary winding, although this was assumed to be negligibly small in this case.) The transformer shown in fig. 4.23(b) is an *autotransformer.*

Because the currents are in opposite directions in the two halves of winding AC, it is the traditional approach to call section AB the primary and section BC the secondary. With these definitions it follows that for fig. 4.23(b)

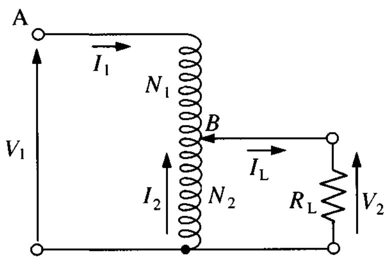
$$\text{Power in primary} = (120\,\text{V})(6\text{A}) = 720\,\text{W}$$
$$\text{Power in secondary} = (120\,\text{V})(6\,\text{A}) = 720\,\text{W}$$
$$\text{Load power} = (120\,\text{V})(12\,\text{A}) = 1440\,\text{W}$$

Thus we may say that for this transformer ratio half the power is *transformed* and half the power is supplied *conductively* from the input directly to the load.

Let us now generalize our approach by considering the autotransformer of fig. 4.24, where the two segments of the winding have, respectively, N_1 and N_2 turns. Let $N_2/(N_1 + N_2) = m$. The two segments of the winding carry the same flux, hence by the arguments leading to (4.13) the voltage ratio will be

$$\frac{V_2}{V_1} = \frac{N_2}{N_1 + N_2} = m \tag{4.31}$$

Fig. 4.24. Circuit model for an autotransformer with variable output voltage.



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Also, from the arguments leading to (4.14) we have

$$I_1N_1 = I_2N_2 = (I_L - I_1)N_2$$

or

$$I_1(N_1 + N_2) = I_LN_2$$

Hence the current ratio is:

$$\frac{I_L}{I_1} = \frac{N_1 + N_2}{N_2} = \frac{1}{m} \tag{4.32}$$

The current I_2 in the N_2 turns may be written

$$I_2 = I_L - I_1 = I_L - mI_L = (1 - m)I_L \tag{4.33}$$

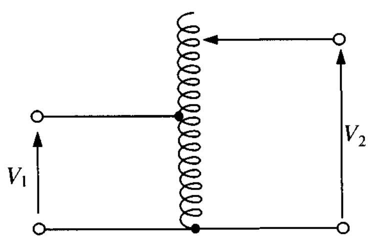
As $m \to 1$ the current I_2 becomes smaller and is zero when $m = 1$, corresponding to a direct connection between terminals A and B. As m gets smaller, $I_2 \to I_L$, and the saving resulting from the use of the autotransformer becomes negligible.

If the point B in fig. 4.24 is moveable, then the autotransformer is a convenient source of variable alternating voltage. A common type of autotransformer designed for use in electronics laboratories has the winding configuration shown in fig. 4.25. By adjustment of the moveable tap the user may obtain output voltages from zero to a few volts above the line voltage.

Although the circuit diagram of the autotransformer resembles that of the resistance-type voltage divider, the principles of operation of the two circuits are quite different. In the voltage divider a substantial fraction of the input power appears as heat in the resistor. Except for small losses all the input power to the autotransformer appears in the load.

Care must be exercised in using the autotransformer because there is a direct connection between input and output. When they are used in the

Fig. 4.25. Autotransformer connections to provide output voltage greater than input voltage.



laboratory, autotransformers often are supplied from a unity turns ratio isolating transformer as a safety measure.

Autotransformers find applications in power systems where small voltage changes are required. For example, if it is desired to get 2000 volts from a 2400 volt line, one may use an autotransformer with $m = 2000/2400 = 5/6$. Such a transformer is considerably cheaper in first cost and in operation than a conventional two-winding device. The lack of isolation between primary and secondary is of no concern in this application.

4.15 Maximum power transfer

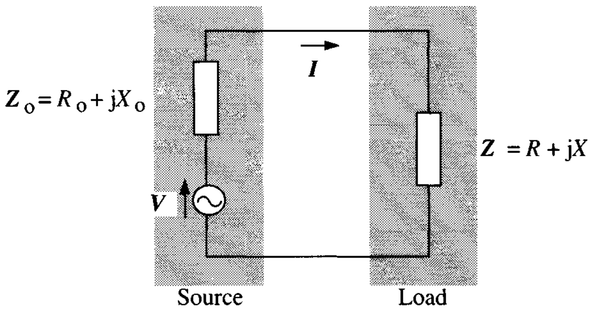
The transmission of an a.c. signal through an electrical network is accompanied inevitably by a loss of signal power, and it is often important to ensure that the loss is as small as possible. We now consider the factors affecting the transfer of power from one section of a network to another, and derive the conditions for which the power transfer is a maximum.

In fig. 4.26 a practical voltage source is shown connected to a load. The source may represent an actual signal source, for example, some form of inductive or capacitive transducer, or it may be the Thévenin equivalent of a section of a complex network. Likewise, the load $\boldsymbol{Z}$ may be the equivalent impedance at the terminals of a complex network to which the transducer or first network is connected. It is shown below that the efficiency of power transfer depends only on the relative values (magnitudes and angles) of the source and load impedances.

For the circuit of fig. 4.26

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}_0 + \boldsymbol{Z}} = \frac{\boldsymbol{V}}{(R_0 + R) + \mathrm{j}(X_0 + X)}$$

Fig. 4.26. Basic circuit for derivation of the maximum power transfer theorem.



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so,

$$I=\frac{V}{\sqrt{[(R_0+R)^2+(X_0+X)^2]}}$$

Therefore, power to the load is

$$P=I^2R=\frac{V^2R}{(R_0+R)^2+(X_0+X)^2} \tag{4.34}$$

We now assume that $\boldsymbol{Z}_0$ is fixed while $\boldsymbol{Z}$ is variable. There are three cases to consider appertaining to the way in which the load $\boldsymbol{Z}$ is constrained to vary.

Case 1: R and X are independently variable.

In this case, since X appears only in the denominator of (4.34), the value of X that maximizes P is

$$X=-X_0 \tag{4.35}$$

This means simply that the circuit must be brought to a state of resonance to maximize power transfer.

The power is then

$$P=\frac{V^2R}{(R_0+R)^2}$$

To find the optimum value of R, we differentiate this expression, set $\mathrm{d}P/\mathrm{d}R=0$ and solve for R. This gives

$$R=R_0 \tag{4.36}$$

So, if both the resistance and reactance of the load are adjustable, maximum power is delivered to the load when the load impedance $\boldsymbol{Z}$ is the complex conjugate of the source impedance $\boldsymbol{Z}_0$, that is,

$$\boldsymbol{Z}=\boldsymbol{Z}_0^* \tag{4.37}$$

The relationship (4.37) is referred to as the *maximum power theorem*, and the load is said to be *matched* to the source. For this condition, equal amounts of power are absorbed by the load and the internal resistance of the source, and the efficiency of the system is at best only 50%. It will be obvious from (4.35) that, if the source contains reactance, matching can be achieved only at one particular frequency.

Case 2: R is variable, X is fixed.

To find the optimum value of R we may differentiate the expression (4.34) directly, however, it is somewhat easier to rearrange this expression so that R appears in the denominator only; thus

$$P = \frac{V^2}{\frac{1}{R}[(R_0+R)^2+(X_0+X)^2]} \tag{4.38}$$

The value of R that maximizes the power is then found from

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{R_0{}^2}{R}+2R_0+R+\frac{1}{R}(X_0+X)^2\right]=0$$

or

$$R=\sqrt{[R_0{}^2+(X_0+X)^2]} \tag{4.39}$$

In this case the efficiency of power transfer is less than 50%.

Case 3: magnitude of $\boldsymbol{Z}$ is variable, angle of $\boldsymbol{Z}$ is fixed.
The angle of $\boldsymbol{Z}$ is $\tan^{-1}(X/R)=$ constant, therefore $X/R=$ constant $=a$, say, or $X=aR$. Substitution in (4.38) gives

$$P = \frac{V^2}{\frac{1}{R}[(R_0+R)^2+(X_0+aR)^2]}$$

Setting the differential of the denominator of the above expression to zero (as in Case 2 above) yields

$$R^2+X^2=R_0{}^2+X_0{}^2$$

or, taking the square root of both sides of the expression,

$$Z=Z_0 \tag{4.40}$$

For this case we see that we must set the *magnitudes* of the source and load impedances equal to obtain maximum power transfer. Again the efficiency is less than 50%.

The impedance transforming property of the transformer (equation (4.15)) may be utilized to achieve the condition specified by (4.40). For example, if the generator impedance is $8+\mathrm{j}6$ so that $Z=10\,\Omega$, and if the load is a pure resistance of $2100\,\Omega$, then maximum power will be delivered to the load if a transformer is employed such that:

$$\left(\frac{N_2}{N_1}\right)^2=\frac{2100}{10}, \qquad \text{or} \qquad \frac{N_2}{N_1}=\sqrt{210}=14.49$$

In practice the turns ratio must be a whole number so we choose a ratio of either 14 or 15. The curve of P versus Z has a broad maximum rather than a sharp peak, therefore, either ratio would be equally satisfactory. For the same reason the assumption of an ideal transformer (implicit in the use of (4.15)) will not in practice give rise to any significant error.

Matched conditions may also be achieved by inserting between source and load a two-port network the elements of which are arranged to provide the requisite impedance transformation. If the elements are purely reactive, no power will be absorbed by the network itself. An 'L-section' combination of inductance and capacitance may, for example, be used to match a resistive load to a resistive source, as indicated in fig. 4.27. Values of L and C may be found such that the impedance Z_{AB} looking into terminals AB will equal the source resistance R_0 when the two port is terminated by load resistance R. For this matched condition we have

$$Z_{AB}=R_0=\mathrm{j}\omega L+\frac{R(1/j\omega C)}{R+1/\mathrm{j}\omega C}$$

which gives

$$R_0+\mathrm{j}\omega CRR_0=R-\omega^2LCR+\mathrm{j}\omega L$$

Equating real and imaginary parts:

$$R_0=R-\omega^2LCR \qquad \text{or} \qquad LC=\frac{R-R_0}{\omega^2R}$$

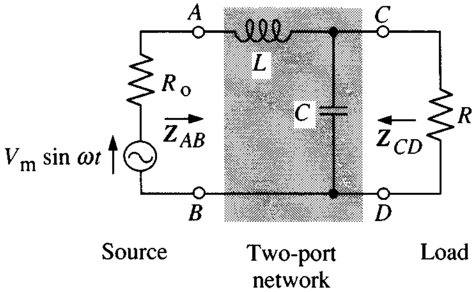
and

$$\omega CRR_0=\omega L \qquad \text{or} \qquad \frac{L}{C}=RR_0$$

Combining these expressions we obtain

$$\begin{aligned} C&=\frac{1}{\omega R}\sqrt{\left(\frac{R-R_0}{R}\right)} \\ L&=\frac{R_0}{\omega}\sqrt{\left(\frac{R-R_0}{R}\right)} \end{aligned} \tag{4.41}$$

Fig. 4.27. Matching source and load by means of a reactive two-port network.



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Note that these expressions are functions of ω; matching is achieved, therefore, at one particular frequency only. A result identical to (4.41) is obtained by putting $Z_{CD}=R$.

†4.16 The transformer bridge

Impedances of *like* kind may be compared by means of a conventional four-arm, a.c. bridge (section 3.10) using the circuit shown in fig. 4.28(a). Resistances R_1 and R_2 form a voltage divider across the a.c. source and the standard impedances $\boldsymbol{Z}_3$ and the unknown impedance $\boldsymbol{Z}_4$ are placed in opposite arms. If the impedances can each be represented by a series combination of resistance and reactance, the balance conditions are:

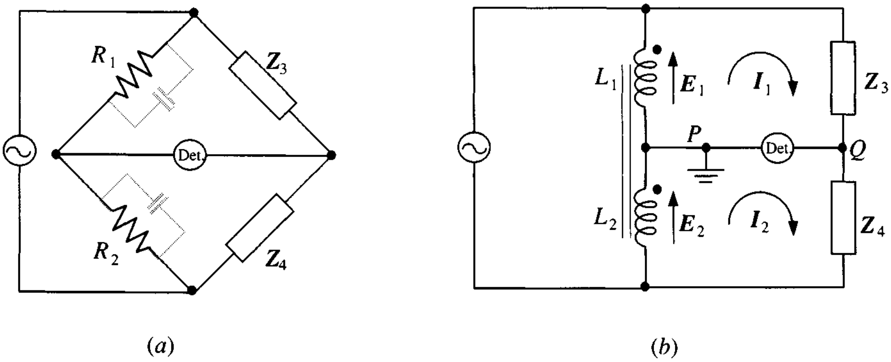
$$\frac{R_1}{R_2}=\frac{\boldsymbol{Z}_3}{\boldsymbol{Z}_4}=\frac{R_3+\mathrm{j}X_3}{R_4+\mathrm{j}X_4}$$

which gives

$$\frac{R_1}{R_2}=\frac{R_3}{R_4}=\frac{X_3}{X_4} \tag{4.42}$$

Two difficulties arise in connection with the use of such a bridge in practice. Firstly, from (4.42) we see that in order to evaluate the unknown reactance X_4 and resistance R_4, the ratio R_1/R_2 must be known accurately. If, however, the bridge is to be used for comparing a wide range of values of impedance against a single standard impedance this ratio must be adjustable over a correspondingly large range. But it is technically difficult and expensive to manufacture precision resistance ratio arms that can be made variable over a wide range, and in practice the range of the bridge

Fig. 4.28. A.C. bridges using: (a) resistance ratio arms; (b) inductively coupled ratio arms.



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must be extended by providing a large number of different standard impedances.

The second difficulty that arises in connection with the circuit of fig. 4.28(a) is the presence of stray capacitances across the two resistance arms (indicated by the dotted components) which will have the effect of altering the balance conditions in an unpredictable way.

Both of these difficulties may, to a very large extent, be circumvented by the arrangement shown in fig. 4.28(b) in which R_1 and R_2 have been replaced by a pair of coupled coils. The coils are closely wound on a core of high permeability so that the coupling coefficient closely approaches unity. If for the moment we assume that the windings have negligible resistance, the two coils will behave like an ideal transformer and the voltages E_1 and E_2 across the coils will be in direct ratio to their turns, that is,

$$\frac{E_1}{E_2}=\frac{N_1}{N_2} \tag{4.43}$$

Now with corresponding ends of the coils arranged as shown, and taking E_1 and E_2 as phase reference, we may write

$$\begin{aligned} \boldsymbol{I}_1\boldsymbol{Z}_3+(\boldsymbol{I}_1-\boldsymbol{I}_2)R_{\mathrm{D}}&=E_1 \\ (\boldsymbol{I}_2-\boldsymbol{I}_1)R_{\mathrm{D}}+\boldsymbol{I}_2\boldsymbol{Z}_4&=E_2 \end{aligned} \tag{4.44}$$

Where R_{D} is the resistance of the detector.

At balance the current in the detector is zero, that is, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ must be equal in magnitude and phase, hence from (4.44) and (4.43) we obtain

$$\frac{\boldsymbol{Z}_3}{\boldsymbol{Z}_4}=\frac{E_1}{E_2}=\frac{N_1}{N_2}$$

or

$$\frac{N_1}{N_2}=\frac{R_3}{R_4}=\frac{X_3}{X_4} \tag{4.45}$$

We see from this expression that the balance condition is dependent only upon the ratio of the turns on the two coils, which can be fixed in manufacture to very high precision and which, unlike resistance ratio arms, is not subject to the influence of temperature changes or ageing of the components. Moreover, by providing tapping on the two windings at suitably arranged intervals the bridge ratio may be changed in precisely defined steps over a very wide range.

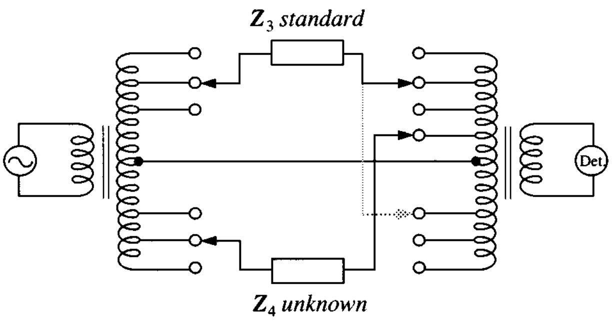
An impedance connected across either of the windings will draw current, but because the ratio of the voltages across the two windings is fixed by the turns ratio, the bridge balance will remain unchanged. Stray capacitances across the windings will, therefore, have no effect on the operation of the bridge. Also, at balance, both points P and Q will be at the same potential so if P is connected to earth, stray capacitances from these points to earth will also have no effect.

So far we have assumed that the resistances of the two coils are zero and that the coupling between them is perfect. In practice there will be resistance and leakage reactance associated with both windings so that current drawn by an additional impedance connected across one coil will produce a voltage drop. This drop however will, by transformer action, cause a drop in the other coil and by proper design the two can be made to compensate so that the ratio of the voltages across the coils remains virtually unchanged.

A practical form of the transformer ratio-arm bridge is shown in fig. 4.29. In this circuit two transformers are used each having an additional winding, to which source and detector are connected. This arrangement avoids earth loops which might affect bridge balance, and allows the various sections of the bridge to be more efficiently screened from one another.

At balance the two currents flowing through Z_3 and Z_4 create equal and opposite ampere turns in the transformer on the detector side of the bridge. Thus, by arranging a series of tappings on this transformer the bridge ratio may be further multiplied, and an overall ratio ranging from 1 to 10^6 may be readily achieved. A single adjustable standard usually suffices for this type of bridge. A further advantage of this arrangement is that by connecting the standard to the opposite side of the transformer on the detector side, as indicated by the dotted line, it is possible to compare unlike impedances; thus, an inductive impedance may be measured using a capacitance standard. The parameters of three-terminal networks, both active and passive, may also be conveniently measured with this type of bridge.

Fig. 4.29. Practical form of transformer bridge.



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The transformer bridge allows impedances to be compared in the frequency range 1–10 kHz with a precision typically of 1 part in 10^4, and a precision of better than 1 in 10^6 is achievable. By the use of ferrite cores in the construction of the transformer, measurements may be made at frequencies of up to 250 MHz with a precision of a few per cent.

4.17 Summary

In a network containing reactive as well as resistive elements the voltage V and the current I at a pair of terminals will, in general, differ in phase by some angle ϕ. The current may be resolved into two components: a component $I\cos\phi$, called the in-phase current, and a component $I\sin\phi$, called the quadrature current. The product of voltage and in-phase current, gives the power at the terminals:

$$\text{Power} = VI\cos\phi \qquad \text{watts} \tag{4.3}$$

This quantity is called the real (or active) power to distinguish it from two other related quantities:

$$\text{Reactive power} = VI\sin\phi \qquad \text{vars} \tag{4.4}$$

and

$$\text{Apparent power} = VI \qquad \text{volt-amperes} \tag{4.5}$$

The real power is simply the apparent power multiplied by $\cos\phi$ – the power factor. For a purely resistive network the power factor is unity while for a purely reactive network it is zero. The relationship between real, reactive and apparent powers may be shown diagramatically by means of the power diagram or power triangle (fig. 4.4). It is also sometimes convenient to express the relationship in complex form:

$$S = P + \mathrm{j}Q \tag{4.10}$$

where S is the apparent power, P the real power, and Q the reactive power.

The principle of conservation of energy applies to both real and reactive powers, which implies that the total real power flowing into a network must equal the sum of the powers dissipated in the individual resistances within the network; likewise the total reactive power must equal the sum of the reactive power associated with the reactances within the network. (Principle of conservation of watts and vars, equation 4.9.) Reactive powers associated with inductance and capacitance carry opposite sign for the purposes of calculating total reactive power.

When energy is drawn from a distribution system, it is desirable to operate at unity power factor because the current requirement for a given power delivered is then a minimum. Industrial loads are predominantly inductive and, therefore, draw a lagging current. Connection of a capacitor, drawing a leading current from the supply, will reduce the total reactive power thereby improving the power factor. Such improvement is often economically justifiable in the case of large industrial power consumers.

The transformer is of fundamental importance as a component in power distribution systems; it is also used in electronic and communications equipment of all kinds. The analysis of the transformer as a circuit element is greatly simplified by reference to the loss-free *ideal* transformer in which the voltage ratio is in direct proportion to the turns ratio between primary and secondary windings and the current ratio is in indirect proportion to the turns ratio (equations (4.13) and (4.14)). Real transformers may then be characterized by circuit models based on the ideal transformer with components added to account for winding resistances, leakage inductances and an exciting current. By employing the impedance transforming properties of the transformer, a simplified circuit model may be derived in which all elements are referred to either the primary circuit or the secondary circuit. This results in a circuit model consisting of just two series elements and two shunt elements (fig. 4.18). These elements may be determined experimentally from simple short-circuit and open-circuit tests.

In electrical and electronic circuits intended for signal transmission it is important to ensure that the loss of signal power is as small as possible. Optimum power transmission between a source of impedance $\boldsymbol{Z}_0$ and a load impedance $\boldsymbol{Z}$ is achieved by matching source to load according to the maximum power theorem:

$$\boldsymbol{Z} = \boldsymbol{Z}_0^* \tag{4.37}$$

where $\boldsymbol{Z}_0^*$ is the complex conjugate of $\boldsymbol{Z}_0$. For purely resistive circuits this condition implies that source and load resistances should be the same. The impedance transforming properties of the transformer or combinations of reactive elements can be utilized to achieve matched conditions.

4.18 Problems

1. For the circuit of fig. 4.30 determine.

(a) the power dissipated in each branch of the circuit.

(b) the watts and vars to the whole circuit.

(c) the power factor of the circuit.

2. Find in the circuit of fig. 4.31 the pure reactance or reactances X that will make the overall power factor 0.8.

3. A load which takes 3 MW at 0.6 power factor lagging is fed by a line whose inductive reactance is five times its resistance. In order to provide a

load voltage of 75 kV it is found that the input to the line must be 90 kV. Find the line current.

If a capacitor is connected in parallel with the load to bring the power factor to 0.9 lagging, what must then be the input voltage to the line?

4. Two impedances, Z_1 and Z_2, are connected in parallel. The resistive component of the first branch is 5 Ω. When the parallel combination is connected to a supply voltage of 240 V, the first branch takes a lagging current of 21.5 A and the second branch takes a leading current at power factor 0.6. The total power supplied is 3.69 kW.

Determine:

(a) the branch and total currents.

(b) the impedances of the two branches.

(c) the impedance of the parallel combination.

(London University)

5. If the alternating voltage across a certain load is represented by the complex number V and the alternating current through the load is given by I, demonstrate that the power is the real part of either VI^* or V^*I.

A voltage of $(100\sin\omega t + 20\sin(2\omega t + \pi/2))$ is applied to a circuit consisting of a resistor and a capacitor in series. The impedance of the circuit at angular frequency ω is $(10 - j20)$ ohms. Calculate the r.m.s. current, the power dissipated and the reactive power.

(Oxford University)

Fig. 4.30. Circuit for problem 1.

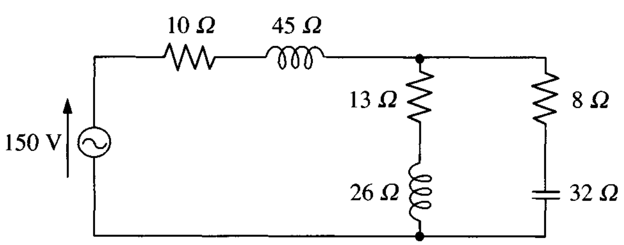
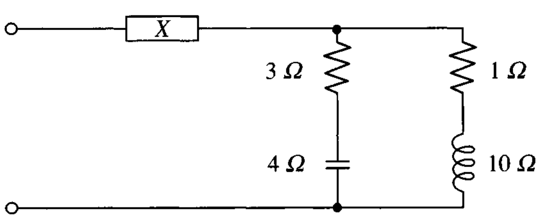


Fig. 4.31. Circuit for problem 2.



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6. In the circuit shown in fig. 4.32 the transformer is to be assumed ideal with turns ratio 1 : 2.

(a) Show that at angular frequency $\omega = 10^2$ rad/s the current I_2 is in phase with the voltage across the transformer secondary winding.

(b) Given that $I_0 = 2$A at 100 rad/s determine the magnitudes of the currents I_1, I_2 and I_3 and their phase angles relative to the current I_0.
(London University)

7. Describe briefly how an equivalent circuit for a power transformer can be derived from measurements made in open- and short-circuit tests.

A 415/240 V, 50 Hz, single-phase transformer has winding resistance 0.15 Ω and leakage reactance j1.0 Ω, both referred to the high-voltage winding.

Estimate the terminal voltage on the low-voltage winding when the transformer supplies the following load from a 415 V source:

(a) a resistance of 6 Ω.

(b) a resistance of 6 Ω in parallel with a 500 μF capacitor.

(Cambridge University: First year)

8. The maximum efficiency for a single-phase, 50 Hz transformer rated at 1000 kVA, 2000/250 V is obtained when it is supplying, at the secondary side, 70% of full load at unity power factor and 250 V. The following data are available for the transformer:

Turns ratio, 8 : 1.

Primary winding resistance, 0.04 Ω.

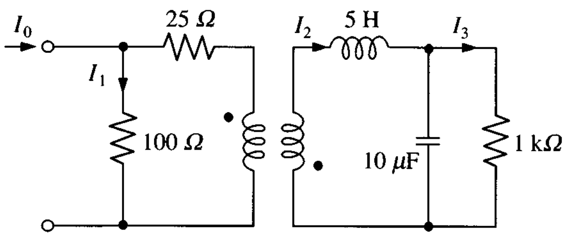
Secondary winding resistance, 0.001 Ω.

Leakage reactance referred to the primary winding, 1.04 Ω.

Estimate the maximum efficiency and calculate the magnitude of the in-phase component of the current when the secondary winding is on open circuit. Estimate also the readings on the measuring instruments used in a test on the transformer with full-load current in the primary and secondary short circuited.

(Cambridge University: Second year)

Fig. 4.32. Circuit for problem 6.



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9. How are the Thévenin and Norton equivalents of a two-terminal network related to each other? Show that if a load is connected at the two terminals, the calculated load current is the same, whichever equivalent circuit is used.

In the circuit of fig. 4.33 the resistor R is adjusted until it dissipates maximum power. Find:

(a) the ohmic value of R.

(b) the current and power in R.

(c) the total power drawn from the 8 V and 2 A sources.

(Newcastle University)

10. A vibration measuring instrument is equivalent to a 10 mV source in series with an impedance of $(900+j1200)\,\Omega$. What is the maximum power it can supply to an amplifier whose input impedance is an adjustable resistor?

What would be the maximum power if a suitable capacitor were connected in series with the input? What should be the value of this capacitor if the vibration frequency is 1 kHz?

11. A practical voltage source has an output voltage of E on open circuit and an internal impedance of $Z_1=R_1+jX_1$. It is connected to a load impedance $Z_2=R_2+jX_2$, whose magnitude may be changed without

Fig. 4.33. Circuit for problem 9.

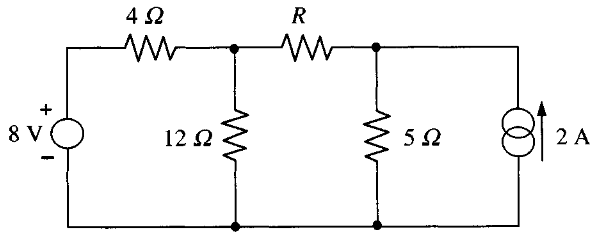
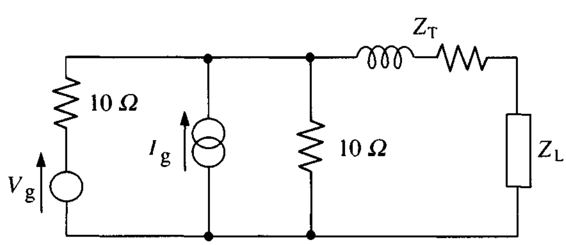


Fig. 4.34. Circuit for problem 12.



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change of angle, i.e. $\theta = \tan^{-1}(X_2/R_2) = \text{constant}$. Show that maximum power transfer from source to load occurs for the condition:

$$|Z_2| = |Z_1|.$$

An electromechanical vibration transducer has an impedance of $(500+j600)\,\Omega$ and an output of 0.2 mV on open circuit. It is to be transformer-coupled to an amplifier having an input resistance of 100 kΩ. Determine the transformer turns ratio required to establish the maximum voltage at the input of the amplifier. What is the magnitude of this voltage? It may be assumed that the transformer is ideal.
(Cambridge University: Second year)

12. In the circuit shown in fig. 4.34, a variable load impedance Z_L is supplied through a transmission line $Z_T = (5+j12)\,\Omega$ from a voltage generator $V_g = 10\underline{/60^\circ}$ and a current generator $I_g = 5\underline{/0}$.

(a) Determine the value of impedance Z_L to absorb maximum power.

(b) Calculate the power absorbed by the load calculated in part (a).

(Sheffield University: First year)

13. A 1.0 MHz generator having an open-circuit voltage of 10 V and an output resistance of 50 Ω supplies a variable impedance $R + jX$, as shown in fig. 4.35(a). Derive an expression for the maximum power P_m which can be supplied to the load, and find the corresponding values of R and X.

Fig. 4.35. Circuit for problem 13.

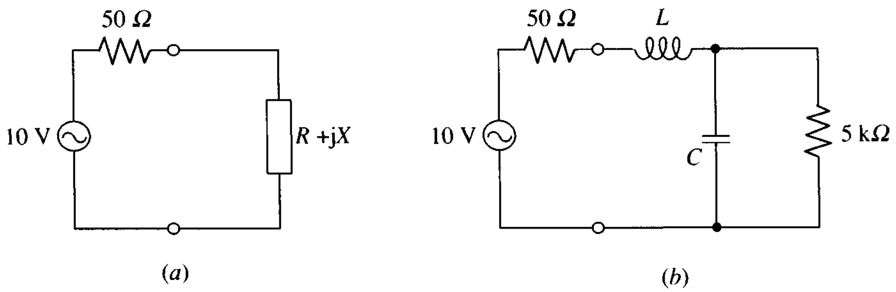
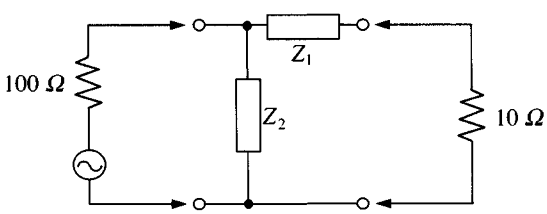


Fig. 4.36. Circuit for problem 14.



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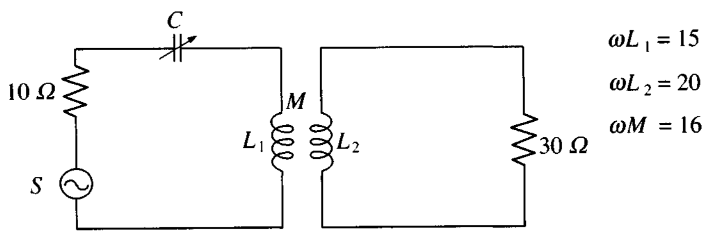
Show that the circuit shown in fig. 4.35(*b*) can be so designed that the power delivered to the 5.0 kΩ resistor has the same value P_m, and determine the appropriate values of L and C.
(London University)

14. In the circuit of fig. 4.36 the network of pure reactors Z_1 and Z_2 is to be used to transfer maximum power from the signal generator of output resistance 100 Ω on the load of resistance 10 Ω at an angular frequency of 10^5 rad/s. Find the components required for Z_1 and Z_2.

In what ratio is the power supplied to the load reduced if the Z_1, Z_2 network is omitted?
(Cambridge University: Second year)

15. In the circuit of fig. 4.37 impedances are given in ohms at the operating angular frequency ω of the signal source S. Find the reactance of the capacitor C which will give maximum power transfer to the 30 Ω load.
(Cambridge University: Second year)

Fig. 4.37. Circuit for problem 15.



https://doi.org/10.1017/CBO9781139170093.006 Published online by Cambridge University Press