1

Basic concepts, units, and laws of circuit theory

1.1 Properties of the electrical circuit

An electrical circuit comprises an arrangement of elements for the conversion, transmission and storage of energy. Energy enters a circuit via one or more *sources* and leaves via one or more *sinks*. In the sources energy is converted from mechanical, thermal, chemical or electromagnetic form into electrical form; in the sinks the reverse process takes place. Sources and sinks are linked by elements capable of transmitting and storing electrical energy. The familiar battery-operated flashlamp serves as a reminder of the energy flow processes in a circuit. In this device, energy is converted from chemical to electrical form in the battery and transmitted along wires to the lamp where most of the energy is converted into heat. A small but useful portion is emitted in the form of electromagnetic radiation in the visible part of the spectrum.

In an electrical circuit energy is conveyed through the agency of electrical *charge* and through the medium of *electric* and *magnetic fields*. An essential feature of any circuit, therefore, is the provision of conducting paths for the conveyance of charge. As indicated in fig. 1.1, sources and sinks are operative only when charge flows through them. The *rate* at which charge flows is referred to as the *current*; the greater the current the greater the energy transmitted between sources and sinks.

Charge is set in motion by the action of the electric field established throughout the circuit by the sources. This field provides the *electromotive force* (e.m.f.) which drives charge round the conducting paths in the circuit. Accompanying this flow of charge is the establishment of a magnetic field. Transmission of electrical energy is, therefore, manifest in a circuit by the presence of both electric and magnetic fields in addition to the movement of charge. The establishment of a field in a circuit is accompanied by an expenditure of energy, and this energy is stored within the region of space

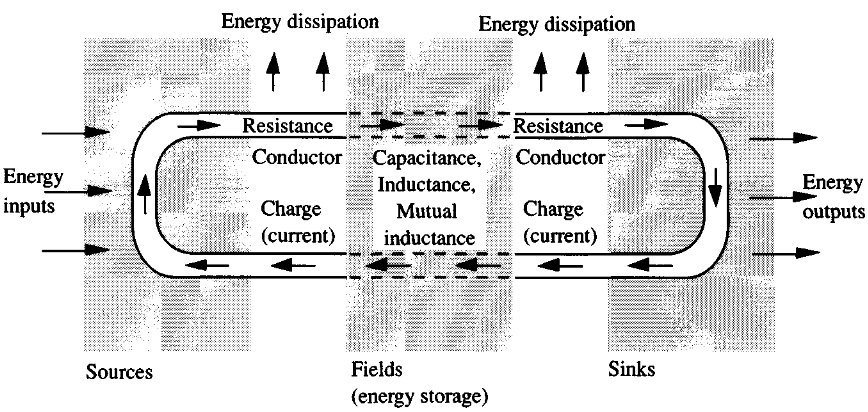
https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

occupied by the field. On subsequent decay of the field, energy is released to the circuit and is eventually absorbed by the sinks. Thus energy can be both stored and conveyed through the medium of a field. However, for the latter process to occur the field must vary with time. Referring again to fig. 1.1, if the sources produce a constant e.m.f., the resulting currents and fields will all be constant and, in this case, there must be a continuous conducting path between sources and sinks along which charge can flow (indicated by the dashed lines in the figure). If, on the other hand, the sources produce a time-varying e.m.f., currents and fields will be time-varying and the conducting path need not be continuous.

This distinction leads to two of the major classes of circuits dealt with in this book: (1) *direct current* (d.c.) circuits in which fields are static and currents are constant and unidirectional: (2) *alternating current* (a.c.) circuits in which the directions of currents and fields alternate in a regular, periodic fashion.

It will be apparent from the above discussion that the electrical behaviour of a circuit is characterized by the strength and distribution of the currents and fields which arise when it is connected to an electrical energy source. The electrical characteristics of a circuit may, therefore, be described generally by means of three elemental properties: *resistance*, *capacitance* and *inductance* (including *mutual inductance*). Resistance is a property associated with the current-carrying paths in a circuit. Capacitance and inductance are properties associated respectively with the parts of a circuit in which electric and magnetic fields arise. Capacitive and inductive elements are often referred to as *storage elements* because of the energy storage properties of a field. A knowledge of the three elemental properties, for a particular circuit, allows us to specify, at least in principle,

Fig. 1.1. Elements of the electrical circuit.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

the magnitudes and directions of the currents which will flow as a result of the application of a given distribution of e.m.f.

Circuits containing only the three basic elements, resistance, capacitance and inductance, are termed *passive* circuits. (*Active* circuits contain also devices such as transistors which, unlike passive elements, are capable of energy amplification.)

If the elemental properties of a passive circuit depend only on its geometry and the materials of which it is made, the circuit is described as being *linear*. If, however, these properties depend additionally on the current or e.m.f. existing in the circuit at any instant, the circuit is described as being *non-linear*. Special techniques are required for the analysis of non-linear circuits; these are dealt with in chapter 7.

Finally, it should be noted that as an inevitable consequence of the movement of charge along a conductor, electrical energy is converted into heat (we are here excluding the superconducting type of circuit), thus the circuit itself acts inherently as an energy sink.

1.2 The lumped circuit model

Practical circuits consist of interconnected assemblies of components: *resistors*, *capacitors* and *inductors*, each designed to exhibit one elemental property to the exclusion of the others.* It is, however, impossible to manufacture a component exhibiting a single property in pure form. Furthermore, all of the interconnections between components will themselves possess each of the three elemental properties to some degree. Consequently, the way in which the elemental properties are distributed in a circuit is often ill defined and, in order to render the circuit amenable to analysis, it is usually necessary to make certain simplifying assumptions and approximations. The most basic of these consists in treating the circuit as if it were composed of pure, discrete elements connected together by conductors possessing no significant properties in themselves. This approach results in the so-called *lumped circuit model*.

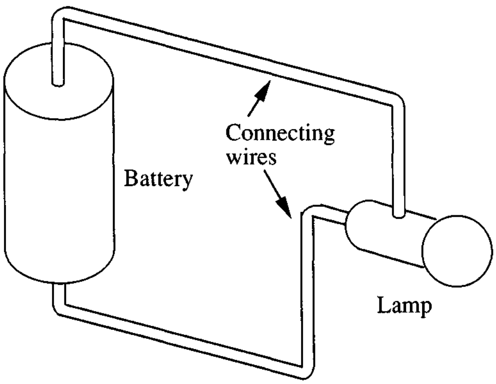
Consider again the flashlamp the component parts of which are depicted in fig. 1.2(*a*). Each part, comprising battery, connecting wires, and lamp, possesses resistance which is distributed in some fashion round the closed path forming the circuit. The circuit also contains distributed capacitance and inductance, but only a cursory knowledge of the principles upon which this device operates tells us that these properties can be safely neglected. The circuit model, therefore, need include only resistance as shown in fig. 1.2(*b*). In this model the battery is represented by an energy source together

* Note that the circuit *component* is distinguished from the circuit *property* by the terminators *-or* and *-ance* respectively.

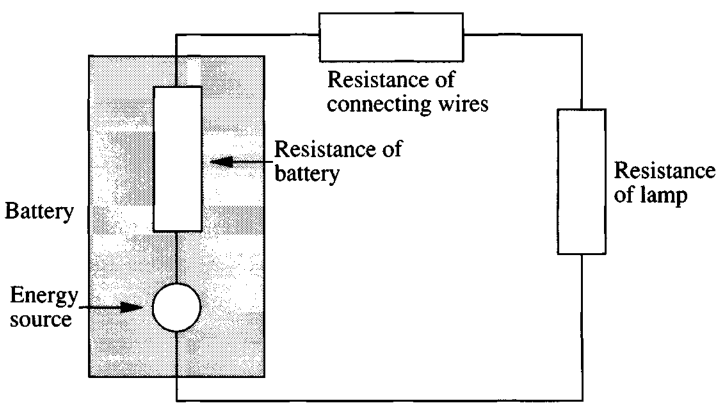
with a concentrated or lumped resistance which accounts for all distributed resistance within the battery. The distributed resistance of the connecting wires and the resistance of the lamp are similarly represented by separate lumped resistances. These lumped elements are joined by conductors which are assumed to be *perfect*, that is, by conductors having zero resistance.

The flashlamp exemplifies the simplest possible type of modelling in which there is a close correspondence between the component parts of the real circuit and the lumped elements of the model. Most of the circuits in this book fall into this category. It should be mentioned, however, that the process of devising suitable models for the type of circuit encountered in, for example, telecommunications systems which operate at high frequencies, is

Fig. 1.2. Circuit modelling.



(*a*) Flashlamp: physical components



(*b*) Lumped circuit model

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

often extremely difficult. Each component and interconnection may have to be represented by a combination of elemental properties and the designer may eventually have to select for analysis one among perhaps several possible lumped models, testing each against past experience or by means of actual circuit measurement.

The lumped circuit modelling technique is directly applicable only when the dimensions of the circuit under consideration are small compared with the wavelength corresponding to the frequency of the source excitation. Circuits not falling into this category, such as high-frequency transmission lines (characterized also by a continuous distribution of elemental properties), require special methods of analysis. The lumped modelling technique provides only a starting point for the development of the theory applicable to such circuits.

1.3 Charge and current

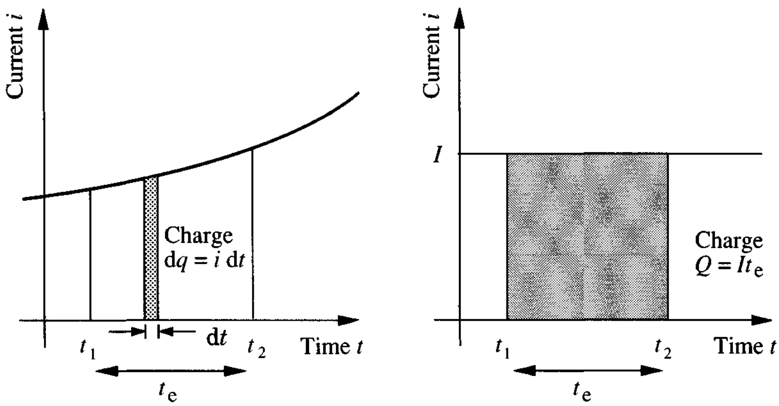
We have stated previously that current in a conductor is equal to the rate of flow of charge. If i is the instantaneous current, and a small quantity of charge $\mathrm{d}q$ flows in time $\mathrm{d}t$, then

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{1.1}$$

The instantaneous current will in general vary with time (fig. 1.3(*a*)). We can calculate the total amount of charge q which flows during a time interval $t_1 \leq t \leq t_2$ by integrating (1.1).

$$\int_0^q \mathrm{d}q = q = \int_{t_1}^{t_2} i\,\mathrm{d}t$$

Fig. 1.3. Relationship between charge and current.



(*a*) Time–varying current (*b*) Direct current

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

The graphical interpretation of this integral is also shown in the figure.

If the time interval commences at the origin, $t_1=0$ and $t_2=t$, and the above integral becomes

$$q=\int_0^t i\,\mathrm{d}t \tag{1.2}$$

For a direct current of magnitude I (fig. 1.3(b)), the charge Q which flows in a time interval $t_1 \leq t \leq t_2$ is

$$Q=I\int_{t_1}^{t_2} \mathrm{d}t=I(t_2-t_1)=It_\mathrm{e} \tag{1.3}$$

where $t_\mathrm{e}=t_2-t_1$ is the elapsed time.

The units of charge and current are respectively the *coulomb* and the *ampere**.

Although the concept of charge is basic to our understanding of the way in which energy flows in an electrical circuit, the ampere is chosen as the fundamental electrical unit in the SI system rather than the coulomb. The reason for this is that it is easier to detect and measure charge in motion than at rest. The former gives rise to a magnetic field which in turn can be detected by utilizing forces resulting from interaction with other magnetic fields. (See definition of the ampere, appendix A.) This is discussed more fully in reference 6.

So far we have not considered the physical nature and origin of electrical charge and indeed for the purposes of the theory contained in this book it is unnecessary to do so. The established physical picture (according to the Rutherford–Bohr model of the atom) conceives of charge as being carried by atomic particles each bearing a discrete amount of charge. But, even in the smallest currents encountered in practice, the number of charge carriers involved in the transport process is very great and the discrete nature of the flow is not normally detectable. A concept of current as consisting of a smooth fluid-like flow is, therefore, adequate for nearly all practical purposes.

Detailed experimental observation reveals that charge carriers can possess two kinds of charge: positive and negative. Under the action of the same electric field, charges of different kind move in opposite directions. A given amount of positive charge moving along a conductor in one direction is indistinguishable, so far as any observable external effect is concerned,

* Appendix A contains information on the International System of Units (SI), and an explanation of the symbols, abbreviations and nomenclature used throughout the text.

from the same quantity of negative charge moving in the opposite direction. By an internationally accepted convention, the direction of current flow is chosen to be that of the direction of motion of positive charge.

In metallic conductors the carriers are electrons which possess negative charge and move in a direction opposite to that of the defined direction of positive current. In semiconductors and electrolytes charge of both kinds exist (carried by electrons and positively charged holes, or ions) and the current is the net result of the movement of positive and negative charge in opposite directions. It must be emphasized, however, that in circuit analysis we are not normally concerned with the nature of charge flow from this microscopic point of view, and we, therefore, talk freely about positive charge moving in metallic conductors even though the charge is in reality carried by electrons.

The reference direction of positive charge flow or current in part of a circuit is indicated diagrammatically by means of an arrow placed on or alongside the conducting path in question. The direction of current between two points A and B in a circuit may also be indicated without ambiguity by means of a double subscript notation. Thus we may write I_{AB}, which is understood to mean a current of magnitude I amperes flowing in a conventional positive sense from A to B. A positive current flowing from B to A would be written I_{BA}; it follows therefore that $I_{BA} = -I_{AB}$. This notation will be valuable in our development of techniques for circuit analysis.

1.4 Potential difference, energy and power

Consider a current of constant magnitude flowing through a section of a metallic conductor AB as shown in fig. 1.4. It is observed experimentally that the passage of current through a conductor is accompanied by the release of energy in the form of heat. It follows that the potential energy of the charge entering the conductor at A must be greater than that of the charge leaving at B since the evolution of heat implies that work is done by the charge during its passage from A to B. A potential energy difference therefore exists between the points A and B. The SI unit of potential energy difference (or simply *potential difference* (p.d.)) is the *volt*, and we say that a voltage exists between A and B. The end of the conductor at the higher potential is indicated conventionally by a (+) sign and that at the lower potential by a (−) sign. A double subscript notation may also be used with advantage to express the magnitude and direction (or *polarity*) of a voltage existing between two points A and B in a circuit. We may write V_{AB} which is understood to mean a p.d. of constant magnitude V volts, A being *positive* with respect to B.

Referring again to fig. 1.4, if a potential difference of one volt exists between A and B, then one coulomb of charge passing between A and B will produce one *joule* of heat energy. Generalizing this statement; if between two points on a metallic conductor there exists a constant potential difference of V volts, and a total of Q coulombs of charge passes between them, the heat output J, in joules, is given by

$$J = VQ \tag{1.4}$$

In terms of current this becomes, using (1.3),

$$J = VIt_e \tag{1.5}$$

where I is a current of constant magnitude and t_e is the elapsed time.

From (1.5) the power P (*watts*) is given by the energy dissipated in the conductor per unit time, that is,

$$P = \frac{J}{t_e} = VI \tag{1.6}$$

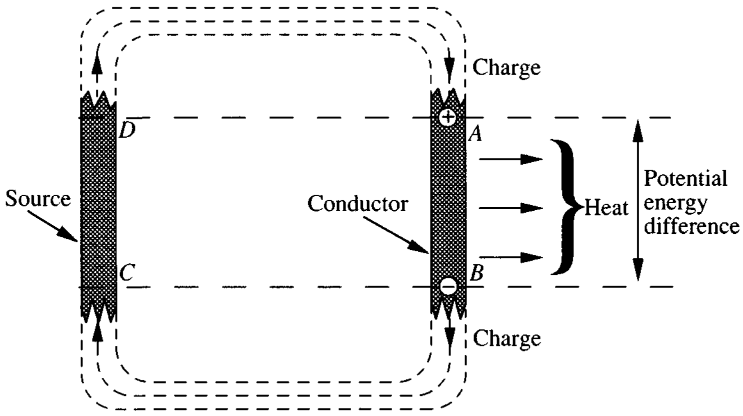
For the general case where both voltage and current vary with time, the energy is, at any instant of time t,

$$J = \int_0^t vi \,\mathrm{d}t \tag{1.7}$$

and the instantaneous power is

$$p = vi \tag{1.8}$$

Fig. 1.4. Potential difference. The potential energy lost by the charge as it flows from A to B is recovered as it flows from C to D.



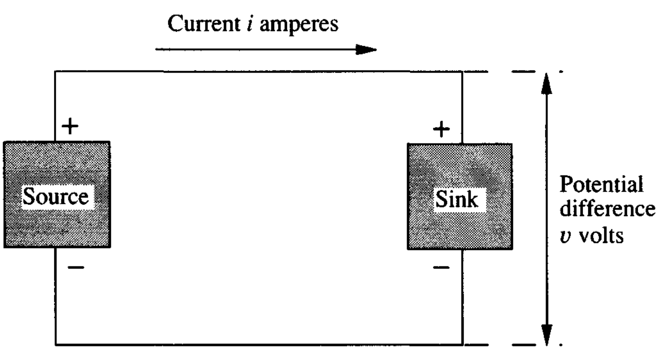
https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

It follows from the principle of conservation of energy that if heat is to be dissipated continuously in the section AB, the potential energy lost by the charge in passing from A to B must be made up by a corresponding gain in potential energy elsewhere. In the system shown in fig. 1.4 this occurs as the charge passes through a section CD of a source. The magnitude of the potential difference across CD is, of course, identical to that across AB. For obvious reasons, the latter is often referred to as a voltage *drop* (or *fall*), and the former as a voltage *rise*.

Although the relationships shown in (1.7) and (1.8) have been established by considering the particular case of a metallic conductor, they apply generally to any sink in which electrical energy is converted to some other form. Consider the circuit shown in fig. 1.5. Source and sink are joined by perfect conductors so that the p.d. across both is the same and equal to v. The polarity of this voltage is, according to our convention, indicated by the $(+)$ and $(-)$ signs. Unit positive charge, on passing through the sink from the positive terminal to the negative terminal, loses a total potential energy of v volts, and on passing through the source from the negative terminal to the positive terminal this potential energy is completely regained. The instantaneous power flow from source to sink is given by the product vi.

For circuits containing a multiplicity of elements the magnitude and direction of power flow at any particular element or in any part of the circuit may be ascertained by considering the associated directions of the voltage and current at the terminals concerned. In fig. 1.6, P is any element or part of a circuit at which the instantaneous values of voltage and current are defined. If the product vi is positive, power is being delivered to P while if the product is negative, P is supplying power to the external circuit. In terms of our double subscript notation, power is delivered to P if the product $v_{AB}i_{AB}$ is positive. (Note carefully the order of the subscripts in this product.)

Fig. 1.5. Energy flow between source and sink. Instantaneous power $p = vi$ watts.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

If we apply this convention to fig. 1.5, we see that the direction of power flow is in accordance with the meaning which has so far been attached to the terms source and sink. That this is not always the case may be seen by comparing the two circuits shown in fig. 1.7.

In these circuits we assume that the voltage of source P is greater than that of Q and that, as a consequence, there will be a net e.m.f. acting in such a direction as to cause current to flow clockwise round the circuit as shown. Examination of the direction of this current in relation to the polarities of the two sources connected as in fig. 1.7(a), confirms that both sources are delivering power to the sink. However, if the polarity of Q is reversed, as in fig. 1.7(b), current enters its positive terminal, the product vi is positive, and we conclude that energy is being delivered to Q. In other words, what has hitherto been called a source is now effectively acting as a sink.

Many practical sources exhibit this property of reversibility. One common example is the battery which can be recharged by connecting it to

Fig. 1.6. Power in a circuit element: P receives power if product vi is positive; P delivers power if product vi is negative.

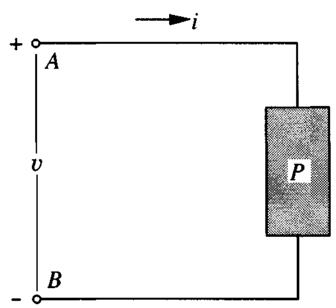
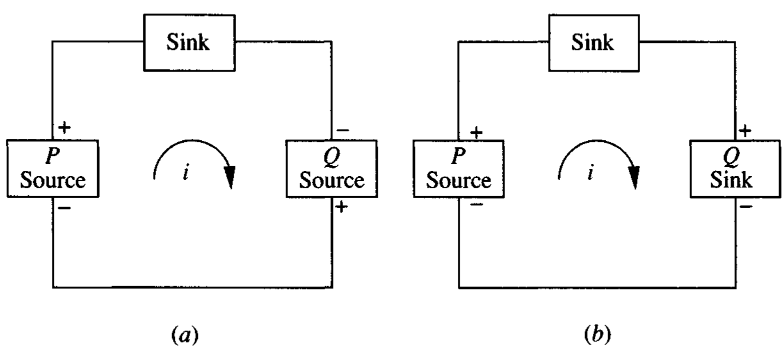


Fig. 1.7. (a) Sources P and Q deliver energy to sink. (b) Polarity of Q reversed: Q receives energy from P.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

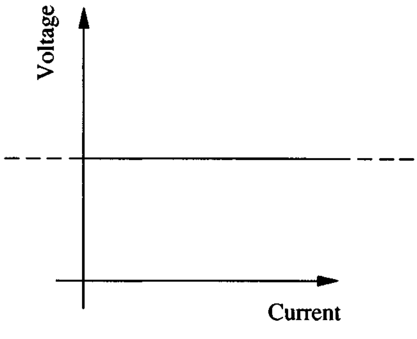
a supply capable of forcing current through it in a reverse direction. Electrical energy is thereby converted and stored in chemical form. In a resistive conductor the energy conversion is, of course, irreversible.

1.5 Ideal voltage and current sources

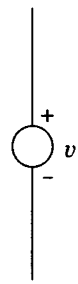
The function of a source is to deliver energy to the circuit to which it is connected. It does this, as we have seen, by imparting potential energy to the charges which pass through it, the energy gained by each unit of charge being equal to the p.d. across its terminals. Many practical engineering applications require a source capable of maintaining a substantially constant p.d. across its terminals irrespective of the current which flows through it. No source can be made which does this perfectly and usually, with a source such as a battery for example, the terminal voltage falls as the load current increases. This leads us to the concept of the *ideal voltage source* (also referred to as an ideal voltage generator) defined as one for which the terminal p.d. is independent of the load current. The utility of this concept lies in the fact that the electrical behaviour of a great many practical sources can be described by means of an ideal source in combination with one or more passive circuit elements. This will be discussed more fully in later chapters.

The relationship between terminal voltage and load current (called variously the *voltage–current*, *volt–ampere*, or *load characteristic*) for an ideal voltage source is shown in fig. 1.8(a). This is simply a straight line parallel to the current axis. If we are dealing with a source whose terminal voltage varies with time, then the voltage axis on this graph must be interpreted as indicating instantaneous values. Fig. 1.8(b) shows the conventional graphical symbol for the ideal voltage source.

Fig. 1.8. The ideal voltage source.



(a) Voltage–current characteristic



(b) Graphical symbol

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

Another type of source, of theoretical and practical importance, particularly in electronic circuits, is the *current source* (or current generator). An *ideal* current source may be defined as a source which delivers a specific current to a circuit irrespective of the voltage across its terminals. The voltage–current characteristic for such a source is shown in fig. 1.9. As with the ideal voltage source, the ideal current source can be used in combination with other circuit elements to describe the electrical characteristics of practical current sources.

While there is a universally accepted symbol for the ideal voltage source, there is no such corresponding symbol for an ideal current source. A selection of some of the more commonly used symbols is presented in fig. 1.10; in this book we shall adopt that shown in fig. 1.10(*d*). In all cases the direction of conventional positive current is specified by an arrow.

We should mention, finally, that the ideal sources which we have considered here are termed *independent* sources because the voltage or current, as the case may be, is maintained at its specified value in-

Fig. 1.9. The ideal current source: voltage–current characteristic.

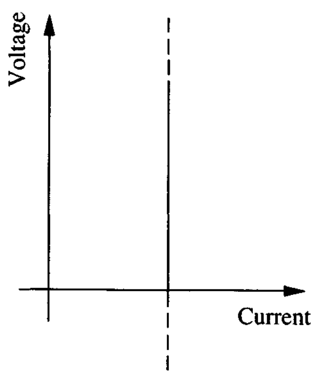
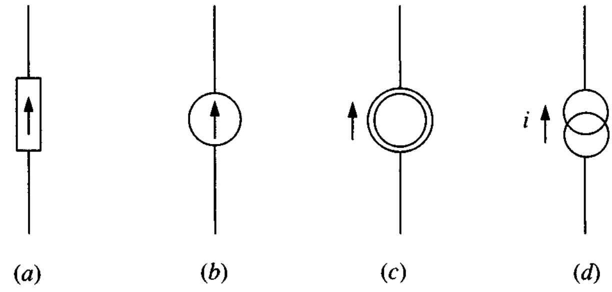


Fig. 1.10. Graphical symbols for the ideal current source.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

dependently of any constraints imposed by the external circuit. Later (section 2.13) we shall discuss another type of ideal source the output of which is controlled by a voltage or current parameter elsewhere in the circuit. This type of source is termed a *dependent* or *controlled* source.

1.6 Kirchhoff's laws

Kirchhoff's current and voltage laws are particular expressions of two of Maxwell's general electromagnetic equations; they apply only to circuits which can be represented by a lumped model. (Their derivation from Maxwell's equations can be found in many books on electromagnetism (see for example reference 7). Within the constraints and limitations mentioned in section 1.2 concerning the lumped circuit model, these two laws provide the basis for *all* circuit analysis.

Kirchhoff's laws are basically conservation laws: the current law expresses the conservation of charge (or more explicitly the continuity of current), and the voltage law expresses the conservation of energy. Although the principles underlying Kirchhoff's laws present little conceptual difficulty, the application of the laws in circuit analysis requires a thorough appreciation of the sign conventions and rules which govern the algebraic combination of currents and voltages. In the two following sections, attention is given to these related aspects of Kirchhoff's laws as well as to the laws themselves.

1.6.1 The current law

The current law in its simplest form may be derived by considering the flow of current between two lumped elements connected together by a perfect conductor (fig. 1.11(*a*)). A junction such as this, between two or more elements, is called a *node*. (Kirchhoff's current law is also known as the *node law* because it relates to the currents at a node.)

Two currents are shown in fig. 1.11(*a*), both flowing in the conductor from left to right. Since there is no reservoir at the node in which charge can accumulate, it is obvious that the currents must be continuous through the node and we can write

$$i_1 = i_2 \tag{1.9}$$

This states simply that the currents flowing into and out of the node are equal. This relationship may, however, be expressed in another way. By transposing (1.9) we obtain

$$i_1 + (-i_2) = 0 \tag{1.10}$$

Diagrammatically this is equivalent to changing the reference direction of i_2

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

as shown in fig. 1.11(b). The law now states that the sum of the two currents flowing *into* the node, taking into account their algebraic signs, is zero.

Finally, a third expression of the law is obtained by reversing the direction of i_1 to give

$$i_2+(-i_1)=0 \tag{1.11}$$

The reference directions corresponding to this equation are shown in fig. 1.11(c), and we see that the sum of the currents flowing *out* of the node is zero.

The above arguments, based essentially on the principle of continuity of current, can be extended to include any number of elements connected together at a node, and in its general form Kirchhoff's current law can be summarized by the following relationships:

$$\sum i_{\text{in}}=\sum i_{\text{out}} \tag{1.12}$$

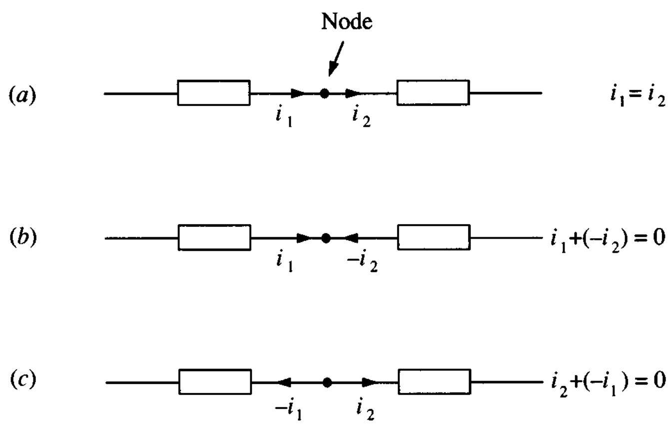
or

$$\sum i=0 \tag{1.13}$$

where the sums are taken over all i.

Equation (1.12) expresses the continuity relationship in its most direct form. Equation (1.13) states that the algebraic sum of the currents flowing into (or out of) a node is zero. These equations express precisely the same relationship between the currents at a node, but they each lead to a slightly different formulation of the complete circuit equations and it is a matter of convenience which of them one chooses. We consider this further in chapter 2.

Fig. 1.11. Alternative expressions of Kirchhoff's current law at a node.



Returning for a moment to the circuit of fig. 1.11(*a*), and to the directions assigned to the currents i_1 and i_2, it will be clear from the intervening discussion that these directions were chosen quite arbitrarily, we could equally well have shown them flowing from right to left, both flowing into or both flowing out of the node. In general, it does not matter how the currents at a node are assigned since their purpose is simply to provide a frame of reference on which to base an explicit expression of the current law. The following example will help to make this clear.

1.6.2 Worked example on the current law

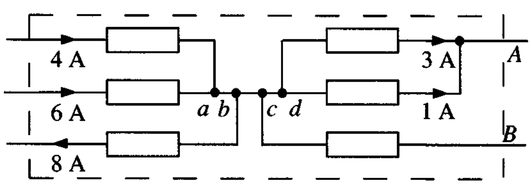
Fig. 1.12(*a*) shows part of a circuit containing six elements. The magnitudes and directions of the currents (referred to conventional positive current) through five of the elements are indicated. Find the two currents in the conductors at A and B.

To apply Kirchhoff's current law one must first identify the nodes that are relevant to the problem. It may appear at first sight that we have to apply the law at each of the nodes a, b, c, and d separately, but if we recall that the connections between elements of our lumped circuit are perfect, and that there is no potential difference between any two points on a perfect conductor, we see immediately that the connections in the central region of this circuit can be rearranged as shown in fig. 1.12(*b*). There are now two clearly identifiable nodes, P and Q.

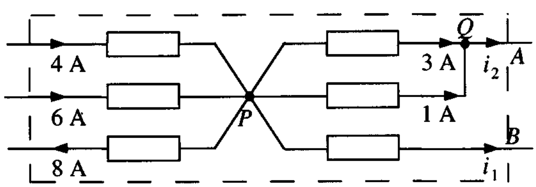
Let the two required currents be i_1 and i_2 and assign their reference directions outward from the circuit as indicated. For this problem it is immaterial which of the forms of the current law is used, so choosing (1.13),

Fig. 1.12. (*a*) Circuit for worked example on the current law. (*b*) Rearrangement to identify node at *P*.

(*a*)



(*b*)



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

and equating the algebraic sum of the currents flowing *out* of the node to zero, we obtain:

At node P $\quad -4-6+8+3+1+i_1=0$

whence $\quad i_1=-2\text{A}$

At node Q $\quad -3-1+i_2=0$

whence $\quad i_2=4\text{A}$

We see that the current i_1 is negative. This simply means that our original choice of reference direction for the current was (as it transpires) opposite to the direction of conventional positive current.

This example may be used to illustrate a further consequence of the current law. If we calculate the algebraic sum of all the currents entering (or leaving) the complete circuit inside the boundary indicated by the dashed line, we find that this sum also is zero. Thus currents flowing *inwards* across the boundary are:

$$4+6-8-(-2)-4=0$$

As may be readily ascertained, a similar result is obtained for any part of the circuit defined by a closed boundary cutting through two or more of its conducting paths. Kirchhoff's current law, therefore, applies in its most general form to any region of a circuit defined by a closed boundary. This follows immediately from the fact that, within such a boundary, charge can neither be created nor destroyed, neither can it be stored. (A more complete discussion of this aspect of Kirchhoff's current law will be found in reference 4.)

1.6.3 The voltage law

The potential difference across an element has been defined in section (1.4) as the energy gained or lost by unit charge as it passes through the element. It was also shown with reference to fig. 1.5 that the potential energy gained by the charge passing through the source was equal to the potential energy lost in passing through the sink. These ideas may be expressed formally by writing $v_1=v_2$ where v_1 is the potential rise and v_2 is the potential fall.

Following the same line of argument as used in our approach to the current law, the above equation can be rearranged to give either $v_1+(-v_2)=0$ or $v_2+(-v_1)=0$, and we can interpret these as equivalent to stating that the algebraic sum of the potential differences in the circuit is zero. The two alternative forms arise because we can choose to define positive potential difference acting in either a clockwise sense or a counterclockwise sense round the circuit.

The same arguments can be applied to circuits containing any number of elements and conducting paths if we take the voltage v_1 to mean the sum of all contributions to the potential rise and v_2 the sum of all contributions to the potential fall as any closed path in the circuit is traversed. Thus Kirchhoff's voltage law may be stated in either of two forms:

$$\sum v_{\mathrm{rise}} = \sum v_{\mathrm{fall}} \tag{1.14}$$

or

$$\sum v = 0 \tag{1.15}$$

where the sums are taken over all v and, in the latter equation, the algebraic sum is intended. The interpretation of these equations is made clear in the example given below.

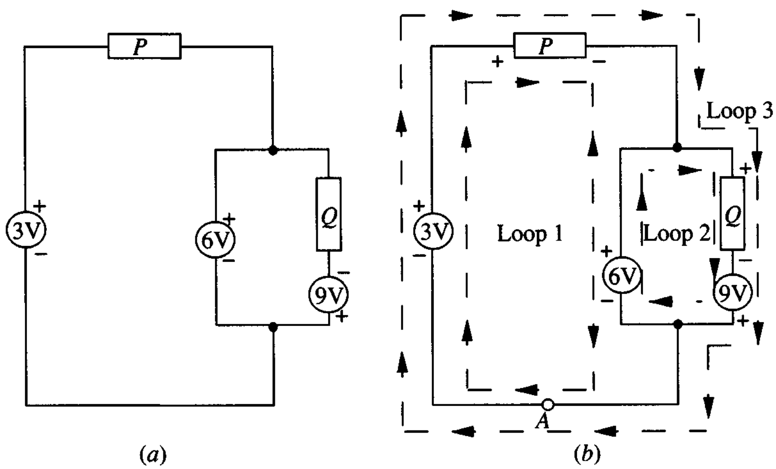
1.6.4 Worked example on the voltage law

In the circuit of fig. 1.13(a), find the magnitudes and directions of the voltages across the elements P and Q.

Since there are two unknown voltages in this problem, the application of Kirchhoff's voltage law consists in choosing two paths or loops in the circuit from which two equations can be set up; these can then be solved for the two unknowns. Three possible loops are indicated in fig. 1.13(b) any two of which are sufficient to set up the required equations. For this problem we choose loops (1) and (3).

Let the magnitudes of the required voltages be V_P and V_Q, and assign

Fig. 1.13. Circuit for worked example on the voltage law.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

their directions (arbitrarily) as indicated by the (+) and (−) signs in fig. 1.13(*b*).

Method (1): using equation (1.14)
Starting at point *A*, and traversing loop (1) in a clockwise direction, we obtain a voltage rise in the 3 volt source, a fall in *P*, and a fall in the 6 volt source. Hence

$$3 = V_P + 6$$
$$V_P = -3\text{V}$$

Similarly for loop (3) we obtain a rise in the 3 volt source, a fall in *P*, a fall in *Q*, and a rise in the 9 volt source. Hence,

$$3 + 9 = V_P + V_Q$$

but

$$V_P = -3\text{V}$$

therefore

$$V_Q = 15\text{V}$$

Method (2): using equation (1.15).
Taking voltages acting in a clockwise sense we obtain:
for loop (1)

$$3 + (-V_P) + (-6) = 0$$

hence,

$$V_P = -3\text{V}$$

and for loop (3)

$$3 + (-V_P) + (-V_Q) + 9 = 0$$

hence,

$$V_Q = 15\text{V}$$

The application of Kirchhoff's Laws in practical circuit analysis is considered in greater detail in chapter 2.

1.7 Resistance

1.7.1 Ohm's Law

The relationship between p.d. across a conductor and the current flowing through it depends on the shape of the conductor and the materials

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

of which it is made. For some materials, for instance semi-conducting compounds, this relationship may be of the general non-linear form shown in fig. 1.14(*a*). For metals, carbon, and many other materials the voltage–current relationship is linear, as shown in fig. 1.14(*b*). The ratio of voltage to current is constant, and the relationship takes the simple form known as Ohm's Law viz:

$$\frac{v}{i} = R \tag{1.16}$$

where R is the resistance in *ohms*.

It is often convenient for the purpose of circuit analysis to express Ohm's law in the alternative form:

$$\frac{i}{v} = \frac{1}{R} = G \tag{1.17}$$

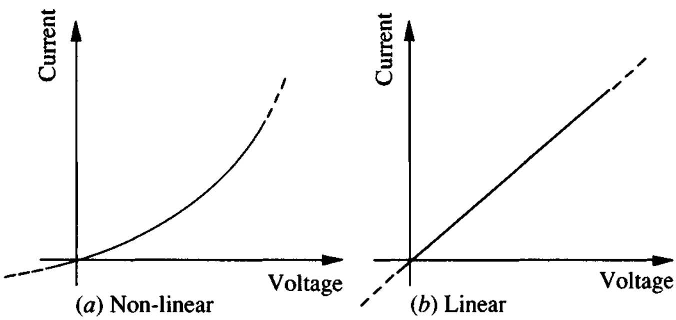
where G is the conductance in *siemens*.

A conducting element for which Ohm's law is obeyed is called a linear resistance. Two commonly used graphical symbols for a linear resistance are shown in fig. 1.15. In this book the symbol of fig. 1.15(*a*) is preferred.*

Also indicated in this figure are the associated directions of voltage and current according to the sign conventions described previously. With conventional positive current flowing in the direction shown, and with v a positive number, the polarity of the voltage is as indicated by the (+) and (−) signs.

Materials which obey Ohm's law are called *ohmic* materials. The resistance of a bar of such material, if it is homogeneous and has a uniform

Fig. 1.14. Voltage–current relationships for resistance.



* The British and European standard symbol for resistance is that shown in fig. 1.15(*b*); in this book it is used, for pedagogical reasons, to denote immitance.

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

cross section, is found to be proportional to the length of the bar l and inversely proportional to its cross-sectional area A. Thus we may write

$$R=\rho\frac{l}{A} \tag{1.18}$$

where ρ is a constant of the material known as the *resistivity*. The unit of resistivity is the *ohm metre*. The relationship (1.18) may also be written in terms of the conductance:

$$G=\sigma\frac{A}{l} \tag{1.19}$$

where $\sigma=1/\rho$ is a constant known as the *conductivity* of the material; this is expressed in *siemens per metre*

1.7.2 Power dissipation in resistance

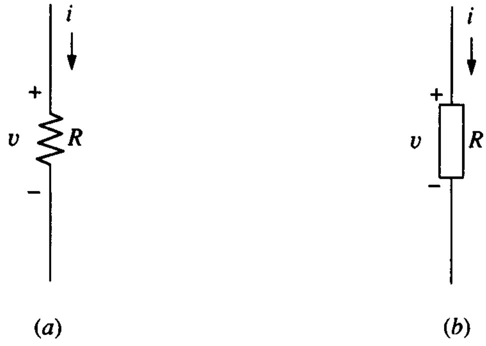
We have mentioned previously that current flowing through a resistance results in the irreversible conversion of electrical energy into heat. The power dissipated is given, according to (1.8), by the product vi, and this is always positive. This expression for the power dissipation is true whether the resistance is linear or non-linear. For the linear case, however, alternative and often more convenient expressions may be obtained by eliminating either v or i from (1.8) using Ohm's law. Thus, using $v=iR$ gives:

$$\text{power}=vi=i^2R \tag{1.20}$$

and using $i=v/R$ gives:

$$\text{power}=vi=\frac{v^2}{R} \tag{1.21}$$

Fig. 1.15. Graphical symbols used for resistance.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

The energy converted into heat in a conductor as a result of the passage of a current through it is often referred to as the 'i^2R' *loss* or, when the conductor is of copper, the *copper loss.*

1.7.3 Resistances in combination

(*a*) *Series connection.* With reference to fig. 1.16(*a*), if the single linear resistance R_s is to be equivalent to the series combination of the two linear resistances R_1 and R_2, a voltage v_{AB} applied to either circuit must cause the same current to flow. Let this current be i then, by Ohm's law, the voltage across R_1 is $v_1 = iR_1$, and the voltage across R_2 is $v_2 = iR_2$. Therefore, by Kirchhoff's voltage law,

$$v_{AB} = v_1 + v_2 = i(R_1 + R_2)$$

But $v_{AB} = iR_s$, hence

$$iR_s = i(R_1 + R_2)$$

The equivalent resistance of two resistances connected in series is, therefore, given by

$$R_s = R_1 + R_2 \tag{1.22}$$

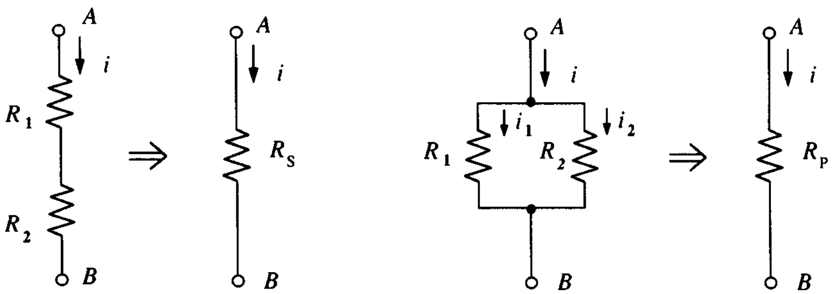
Now consider the series combination of any number of resistances $R_1, R_2 \ldots R_n$. The equivalent resistance R can be found by repeated application of (1.22), taking resistances two at a time, to give

$$R = R_1 + R_2 + \ldots + R_n \tag{1.23}$$

In terms of conductances $G = 1/R$, $G_1 = 1/R_1$ etc., (1.23) becomes

$$\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2} + \ldots + \frac{1}{G_n} \tag{1.24}$$

Fig. 1.16. Equivalent resistance for series and parallel combinations of resistances.



(*a*) Series (*b*) Parallel

(*b*) *Parallel connection.* As for the series connection, the criterion for equivalence between the two circuits shown in fig. 1.16(*b*) is that, under the action of the same applied voltage, the resulting currents must be identical. Thus:

$$\text{Current through } R_1 = i_1 = \frac{v_{AB}}{R_1}$$

$$\text{Current through } R_2 = i_2 = \frac{v_{AB}}{R_2}$$

Therefore the total current i is, by Kirchhoff's current law,

$$i = i_1 + i_2 = v_{AB}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

But

$$i = v_{AB}/R_{\mathrm{p}}$$

hence

$$\frac{v_{AB}}{R_{\mathrm{p}}} = v_{AB}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

or

$$\frac{1}{R_{\mathrm{p}}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{1.25}$$

The parallel combination of two resistances is encountered very frequently in circuit analysis and it is often denoted symbolically by $R_1//R_2$.

Equation (1.25) is then often more conveniently expressed as

$$R_1//R_2 \equiv R_{\mathrm{p}} = \frac{R_1 R_2}{R_1 + R_2} \tag{1.26}$$

We refer to this expression as the 'product over sum rule'. Note that this rule is not applicable to combinations of *more* than two resistances in parallel.

By repeated application of (1.25) we obtain the equivalent resistance of any number of resistances connected in parallel viz.:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \tag{1.27}$$

In terms of conductances $G = 1/R$, $G_1 = 1/R_1$, etc., (1.27) becomes:

$$G = G_1 + G_2 + \ldots + G_n \tag{1.28}$$

This formula is useful for calculating the resistance of many paths in parallel.

1.8 Capacitance

Capacitance is that property of the circuit which defines the distribution of the electric field within the circuit when it is energized. In the following sections we establish the voltage–current relationships for a linear capacitance and from this an expression for the energy stored in terms of the voltage across the *plates* of the capacitance. Relationships are also derived which allow us to represent any arrangement of interconnected lumped capacitances as a single lumped capacitance.

1.8.1 The voltage-current relationship for capacitance

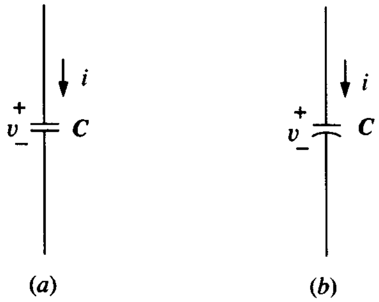
When a source of voltage is applied to two conductors forming the plates of a capacitance, positive charge is transferred from the conductor connected to the negative terminal of the source to the conductor connected to the positive terminal. The quantity of charge transferred is found to depend on the size and shape of the conductors and on the *dielectric* medium between them. For vacuum, air and many other dielectric materials the charge transferred is proportional to the applied voltage. Thus we may write

$$q = Cv \tag{1.29}$$

where q is the instantaneous value of the charge, and C is a constant of proportionality known as the capacitance. The unit of capacitance is the *farad*, and the graphical symbol for a linear capacitance used in this book is shown in fig. 1.17(a).

The relationship (1.29) defines a linear capacitance, and it is true only if the applied voltage is not so high as to cause breakdown (electrical conduction) or other changes in the dielectric.

Fig. 1.17. Graphical symbols used for capacitance.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

The current i which flows as a result of the application of a voltage is found by differentiating (1.29):

$$i=\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}(Cv)$$

Provided we are dealing with a conductor–dielectric system of fixed geometry, this relationship becomes:

$$i=C\frac{\mathrm{d}v}{\mathrm{d}t} \tag{1.30}$$

By integrating (1.30) we obtain an explicit expression for the voltage in terms of current. Thus,

$$v=\frac{1}{C}\int_0^t i\,\mathrm{d}t+v_0 \tag{1.31}$$

in which v_0 is the voltage across the capacitance at $t=0$.

1.8.2 Energy storage in capacitance

Consider a capacitance through which a current i flows as a result of an applied voltage v. The instantaneous power flow to the capacitance is given by the product vi which becomes, using (1.30), $vC\mathrm{d}v/\mathrm{d}t$. During the interval $\mathrm{d}t$ the flow of energy to the capacitance is, therefore,

$$v\left(C\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}t=Cv\,\mathrm{d}v$$

and the total energy delivered to the capacitance when its voltage is v is given by:

$$\text{Energy}=C\int_0^v v\,\mathrm{d}v=\tfrac{1}{2}Cv^2 \tag{1.32}$$

This stored energy is released by the capacitance when the voltage is reduced to zero. (During the release of energy the product vi will be negative according to the sign convention established in section 1.4.) Because the energy of any closed system cannot change instantaneously (instantaneous change of energy implying infinite power), it follows that the voltage across a capacitance cannot change instantaneously.

1.8.3 Capacitances in combination

(*a*) *Series connection.* Referring to fig. 1.18(*a*), the two circuits between A and B are equivalent if, when the same voltage is applied to each,

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

the resulting currents are equal. Let the current be i and the voltage be v_{AB}. If we assume zero initial voltage, then the voltage v_1 across C_1 will be, from (1.31),

$$v_1 = \frac{1}{C_1}\int_0^t i\,\mathrm{d}t$$

and the voltage across C_2 will be

$$v_2 = \frac{1}{C_2}\int_0^t i\,\mathrm{d}t$$

Therefore, using Kirchhoff's voltage law, we have

$$v_{AB} = v_1 + v_2 = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_0^t i\,\mathrm{d}t$$

but

$$v_{AB} = \frac{1}{C_s}\int_0^t i\,\mathrm{d}t$$

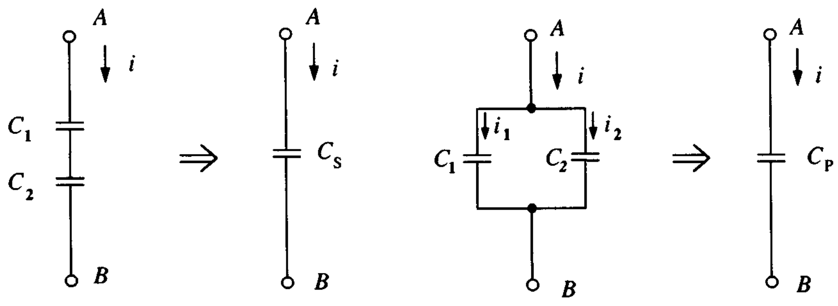
hence

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} \tag{1.33}$$

An expression for the equivalent capacitance of any number of capacitances in series may be found by repeated application of (1.33), viz.:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n} \tag{1.34}$$

Fig. 1.18. Equivalent capacitance for series and parallel combinations of capacitances.



(*a*) Series (*b*) Parallel

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

(*b*) *Parallel connection.* For the parallel circuit of fig. 1.18(*b*), the current through C_1 is, from (1.30),

$$i_1 = C_1 \frac{\mathrm{d}v_{AB}}{\mathrm{d}t}$$

and the current through C_2 is

$$i_2 = C_2 \frac{\mathrm{d}v_{AB}}{\mathrm{d}t}$$

The total current is given by

$$i = i_1 + i_2 = (C_1 + C_2)\frac{\mathrm{d}v_{AB}}{\mathrm{d}t}$$

But, for the equivalent capacitance C_p,

$$i = C_\mathrm{p} \frac{\mathrm{d}v_{AB}}{\mathrm{d}t}$$

hence

$$C_\mathrm{p} = C_1 + C_2 \tag{1.35}$$

By repeated application of (1.35), we may extend this result to include any number of capacitances connected in parallel, thus,

$$C = C_1 + C_2 + \ldots + C_n \tag{1.36}$$

1.9 Inductance

Inductance is that elemental property of a circuit which defines the magnetic field distribution when the circuit is energized. In the following sections we derive voltage–current and other relationships for inductance corresponding to those derived for capacitance.

1.9.1 The voltage–current relationship for inductance

Fig. 1.19(*a*) shows part of a current carrying circuit. The magnetic flux created by the current links with the circuit itself, the amount of flux linkage being a function of the circuit geometry. By forming the circuit into a coil, as represented schematically in fig. 1.19(*b*) the flux linkage is enhanced. Provided there is no iron or other magnetic material present, the flux, and therefore the flux linkage, is found to be proportional to the current. If we denote the flux linkage by ϕ, then

$$\phi = Li \tag{1.37}$$

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

where L is a constant of proportionality, dependent upon the circuit geometry, which is called the *self-inductance* (or simply *inductance*). The unit of inductance is the *henry*, and the graphical symbol preferred in this book is shown in fig. 1.20(*a*).

The relationship between voltage and current is derived using Faraday's law of induced e.m.f. This states that the e.m.f. induced in a circuit is equal to the rate of change of flux linkage, that is,

$$\text{e.m.f.} = -\frac{\mathrm{d}}{\mathrm{d}t}(\phi) \tag{1.38}$$

In this equation the negative sign indicates that the induced e.m.f. acts in a direction such as to oppose the cause of the change of flux linkage (Lenz's law).

Fig. 1.19. Flux linkage.

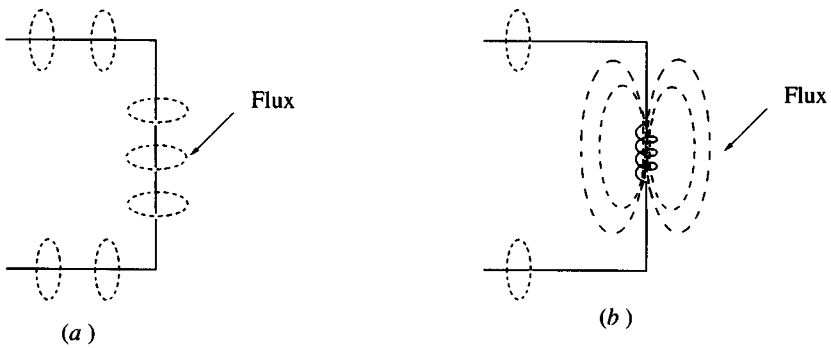
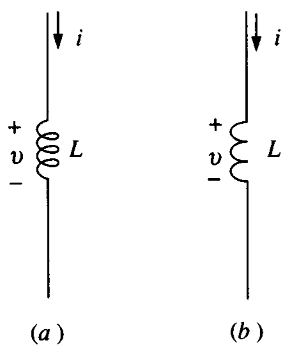


Fig. 1.20. Graphical symbols used for inductance.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

Combining (1.37) and (1.38) we have

$$\text{e.m.f.} = -\frac{\mathrm{d}}{\mathrm{d}t}(Li)$$

which becomes, for a conductor system of fixed geometry,

$$\text{e.m.f.} = -L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{1.39}$$

Now consider the situation shown in fig. 1.21 in which a time varying voltage v is applied to part of a circuit which has no properties other than pure inductance. This will cause a current i to flow whose rate of change is such that the induced e.m.f. will exactly counterbalance the applied voltage, hence we may write

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{1.40}$$

The current i which flows as a result of applying v is found explicitly by integrating (1.40), thus,

$$i = \frac{1}{L}\int_0^t v\,\mathrm{d}t + i_0 \tag{1.41}$$

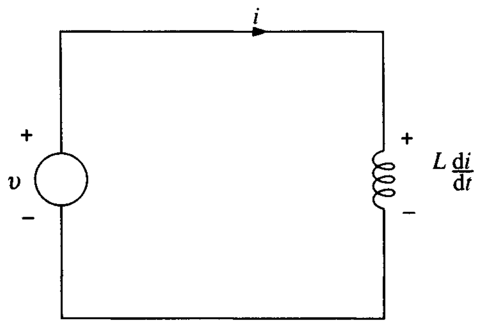
where i_0 is the current flowing through the inductance at $t = 0$.

1.9.2 Energy storage in inductance

Referring to fig. 1.21, the instantaneous power flow to the inductance is vi and the energy in the time interval $\mathrm{d}t$ is given by

$$iL\frac{\mathrm{d}i}{\mathrm{d}t}\mathrm{d}t = Li\,\mathrm{d}i$$

Fig. 1.21. Voltage–current relationship for inductance.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

hence, the total energy storage at any current i is:

$$\text{Energy} = L\int_0^i i\,\mathrm{d}i = \tfrac{1}{2}Li^2 \tag{1.42}$$

Remarks similar to those made in connection with the storage and release of energy in a capacitance (section 1.8.2) apply also in the case of inductance. During the acquisition of energy the product vi is positive. During the release of energy the current diminishes, the polarity of the voltage reverses and the product vi becomes negative. The energy, and therefore the current, cannot change instantaneously.

1.9.3 Inductances in combination

(*a*) *Series connection.* The circuits shown in fig. 1.22(*a*) are equivalent if, upon application of identical voltages, the resulting currents are equal. From (1.40) the voltage across L_1 will be $v_1 = L_1\mathrm{d}i/\mathrm{d}t$, and that across L_2 will be $v_2 = L_2\mathrm{d}i/\mathrm{d}t$. The total voltage will, therefore, be $(v_1 + v_2)$, and this must be equal to v_{AB}. Hence,

$$v_{AB} = (v_1 + v_2) = L_1\frac{\mathrm{d}i}{\mathrm{d}t} + L_2\frac{\mathrm{d}i}{\mathrm{d}t} = L_s\frac{\mathrm{d}i}{\mathrm{d}t}$$

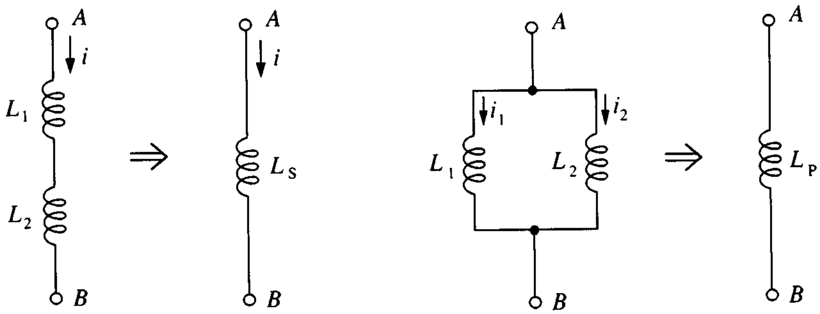
or

$$L_s = L_1 + L_2 \tag{1.43}$$

Extending this to any number of elements by repeated application we have

$$L = L_1 + L_2 + \ldots + L_n \tag{1.44}$$

Fig. 1.22. Equivalent inductance for series and parallel combinations of inductances.



(*a*) Series (*b*) Parallel

(*b*) *Parallel connection.* Adopting the same criterion for equivalence for the circuits of fig. 1.22(*b*), we have for the sum of the currents i_1 and i_2 flowing in L_1 and L_2 respectively (assuming zero initial currents),

$$i = i_1 + i_2 = \frac{1}{L_1}\int_0^t v_{AB}\,\mathrm{d}t + \frac{1}{L_2}\int_0^t v_{AB}\,\mathrm{d}t$$

But this sum must be equal to the current through L_p, hence,

$$\frac{1}{L_\mathrm{p}}\int_0^t v_{AB}\mathrm{d}t = \frac{1}{L_1}\int_0^t v_{AB}\,\mathrm{d}t + \frac{1}{L_2}\int_0^t v_{AB}\,\mathrm{d}t$$

or

$$\frac{1}{L_\mathrm{p}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{1.45}$$

Again, by repeated application we may extend this result to include any number of elements:

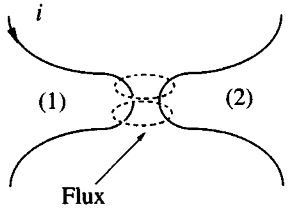
$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n} \tag{1.46}$$

1.10 Inductively coupled circuits

When two circuits are brought into close proximity, the flux produced by current flowing in one circuit can link with the other circuit as shown schematically in fig. 1.23. Faraday's law applies regardless of the source of flux linkage so that if the current varies so will the flux linkage, and an e.m.f. will be induced in the second circuit. We say that the two circuits are inductively coupled.

The inductive coupling effect can be enhanced by shaping the two circuits in the form of coils wound closely together (fig. 1.24), as in the transformer. Sometimes the inductive coupling effect is the cause of unwanted interference between adjacent circuits, and steps have to be taken to reduce the coupling by the provision of magnetic screens.

Fig. 1.23. Inductive coupling between two circuits.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

In this section the concept of mutual inductance is introduced, and from this the voltage–current relationships are established for inductively coupled circuits.

1.10.1 Mutual inductance

The amount of flux linked with the second circuit in fig. 1.23 (or fig. 1.24) depends on the geometry of the two circuits and, in the absence of any non-linear magnetic materials, it will be proportional to the current in the first circuit. Let us designate the flux linkage by ϕ_{21} (read as flux linkage in circuit (2) resulting from a current in circuit (1)), and let the current be i_1. We may then write

$$\phi_{21} = M_{21} i_1 \tag{1.47}$$

where M_{21} is a constant known as the mutual inductance of the two circuits. As for self-inductance, this is expressed in units of the henry. The mutual inductance is conventionally taken as being always positive.

By the same arguments we can derive a similar equation for the situation where circuit 2 carries a current i_2, that is,

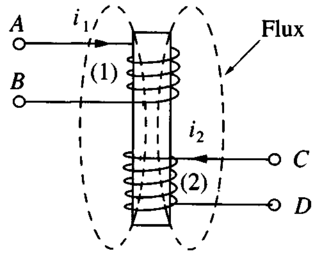
$$\phi_{12} = M_{12} i_2 \tag{1.48}$$

Applying Faraday's law to (1.47) and (1.48) we obtain for the voltage induced in (2), $M_{21}\mathrm{d}i_1/\mathrm{d}t$; and for the voltage induced in (1), $M_{12}\mathrm{d}i_2/\mathrm{d}t$. (It is assumed here that the two circuits are fixed geometrically.)

It may be shown that for *iron-free* circuits, $M_{21} = M_{12}$ (see for example reference 6). Therefore, in what follows the mutual inductance will be designated simply as M.

We can now establish the complete voltage–current relationships for mutual inductance with reference to fig. 1.25, which shows the circuit model for the inductively coupled coil arrangement of fig. 1.24. The coils possess pure self-inductances L_1 and L_2, and the mutual inductance between them

Fig. 1.24. Inductive coupling between two coils: fluxes reinforce with currents and winding directions shown.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

is M; resistance is assumed to be negligible. Both coils are wound in the same direction so that, with the currents i_1 and i_2 assigned as shown (both flowing clockwise when viewed from above in fig. 1.24), the fluxes produced are in the same direction through the two coils. The self-induced voltage in coil (1) will be, from (1.40), $L_1 \mathrm{d}i_1/\mathrm{d}t$ and to this we must add the voltage induced in coil (1) due to current in coil (2), namely, $M\mathrm{d}i_2/\mathrm{d}t$. Hence, the total voltage across AB is

$$v_1 = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + M\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{1.49}$$

The two voltages are in the same sense and therefore add because the fluxes are in the same direction.

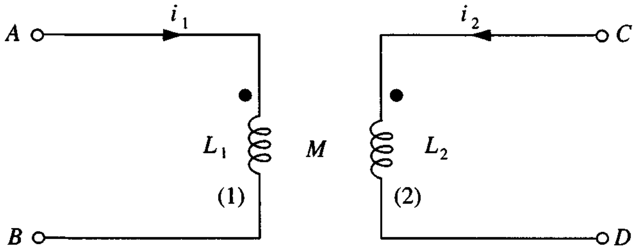
Similarly, for the second coil, the voltage across CD is

$$v_2 = L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + M\frac{\mathrm{d}i_1}{\mathrm{d}t} \tag{1.50}$$

If the direction of either current were reversed, the signs of the second terms in these equations would be negative.

It is sometimes important in circuit diagrams (and on the coils themselves) to indicate the relative directions of the windings. This is done by polarity markings. The conventional scheme is as follows: place a dot on one terminal of coil (1) and imagine the current i_1 to enter the dotted terminal. Now determine the direction of current i_2 in coil (2) that will give flux in the same direction as that produced by i_1. Place a dot on the terminal of coil (2) that i_2 enters. (If there are more than two coils, then any one of them is used as reference and the dots are placed on all others with respect to the reference coil.) According to this convention, the top terminal of each coil of fig. 1.24, and each inductance in fig. 1.25, should have a dot. (Alternatively, of course, the bottom terminal of each coil may be marked.) The ends of the coils so marked are called *corresponding ends.*

Fig. 1.25. Circuit model of the inductively coupled coil arrangement shown in fig. 1.24. Dots indicate ‘corresponding ends’.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

Note that with assigned currents *both* entering (or both leaving) corresponding ends, the terms containing M in the circuit equations (1.49) and (1.50) are positive. If currents are assigned such that one current enters and the other leaves a corresponding end, the terms containing M are negative. If no information is provided concerning corresponding ends in a particular circuit containing inductively coupled coils, the mutual inductance terms must be written: $\pm M\mathrm{d}i/\mathrm{d}t$.

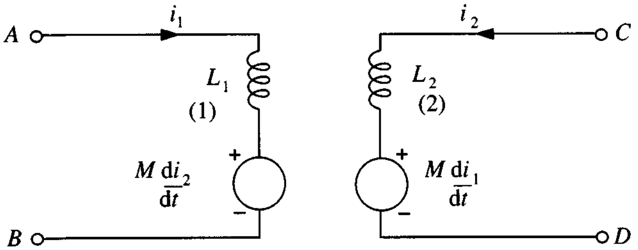
The scheme described above implies that the corresponding ends of coils will all move in potential together when a voltage is applied to one of them. This provides a simple method of identifying, by means of a voltage measurement, the corresponding ends of a system of inductively coupled coils.

Sometimes, a circuit model, by separating out effects that cannot actually be separated physically, provides help in understanding how the circuit works. Fig. 1.26 is an alternative circuit model for the inductively coupled coil arrangement shown in fig. 1.24 in which the induced voltages due to mutual inductance have been represented by ideal sources. These sources cannot be isolated from L_1 and L_2, and so this model is not physically realizable, however, the circuit equations are the same as those for fig. 1.25 and as far as external connections are concerned, the circuits of figs. 1.25 and 1.26 exhibit identical behaviour. It will be noted that the two sources in fig. 1.26 are examples of dependent sources, that is, their values depend on currents flowing in the circuit itself.

1.10.2 The coefficient of coupling

Consider the circuit of fig. 1.26 with its terminals AB connected to a voltage source and terminals CD short circuited. The circuit for this situation is shown in fig. 1.27. Applying Kirchhoff's voltage law to the circuits (1) and (2) we obtain

Fig. 1.26. Alternative circuit model of Fig. 1.24. L_1 and L_2 are separate inductances with no mutual inductance between them.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M\frac{\mathrm{d}i_2}{\mathrm{d}t}$$

and

$$0 = M\frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t}$$

since $v_2 = 0$.

Upon rearrangement and elimination of $\mathrm{d}i_2/\mathrm{d}t$, these equations yield

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{v_1 L_2}{L_1 L_2 - M^2}$$

The equivalent input inductance (that is, the effective inductance between the terminals AB) is

$$L_{AB} = \frac{v_1}{\mathrm{d}i_1/\mathrm{d}t} = \frac{L_1 L_2 - M^2}{L_2}$$

We can now calculate the energy stored in the system using (1.42), thus,

$$\text{Energy} = \tfrac{1}{2}L_{AB}i_1^2 = \tfrac{1}{2}\frac{(L_1 L_2 - M^2)i_1^2}{L_2} \tag{1.51}$$

The stored energy must be positive, otherwise, the inductance could act as a source of energy indefinitely, and this is not possible for a passive circuit element. Therefore, $(L_1 L_2 - M^2)$ must be greater than or equal to zero and so

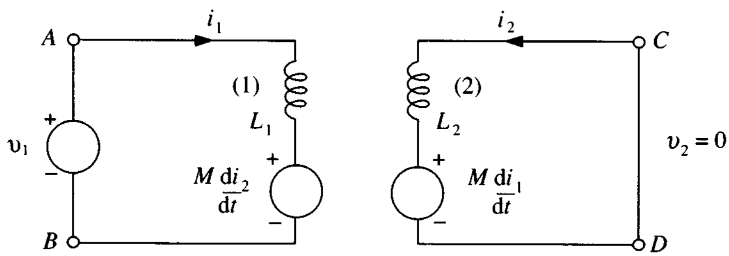
$$M^2 \leqslant L_1 L_2$$

We usually write

$$M = k(L_1 L_2)^{\frac{1}{2}} \quad (0 \leq k \leq 1) \tag{1.52}$$

where k is the *coefficient of coupling*. When k has a value near unity, the

Fig. 1.27. Circuit for deriving the relationship: $M = k(L_1 L_2)^{\frac{1}{2}}$.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

coupling is said to be close, or tight. For $k=1$, all the flux produced by a current in one coil links all the turns of the other coil.

1.10.3 The effective inductance of two series-connected coupled coils

In the circuit of fig. 1.28 the two coils each have self-inductances L_1 and L_2 and mutual inductance M. Since there are two possible ways of arranging the series connection we have two possible circuit models (derived from fig. 1.26) as shown in fig. 1.29. In both arrangements the same current exists in both coils, however, in fig. 1.29(*a*) the coils are connected in series aiding while in fig. 1.29(*b*) they are connected in series opposing. The effective inductance in each case may be found by applying Kirchhoff's voltage law. For fig. 1.29(*a*) we have

$$v=L_1\frac{\mathrm{d}i}{\mathrm{d}t}+M\frac{\mathrm{d}i}{\mathrm{d}t}+L_2\frac{\mathrm{d}i}{\mathrm{d}t}+M\frac{\mathrm{d}i}{\mathrm{d}t}$$

or

$$v=(L_1+L_2+2M)\frac{\mathrm{d}i}{\mathrm{d}t}$$

and so the effective inductance for circuit (*a*) is

$$L_3=L_1+L_2+2M \tag{1.53}$$

Fig. 1.28. Series connected coupled coils.

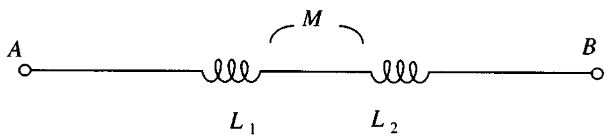
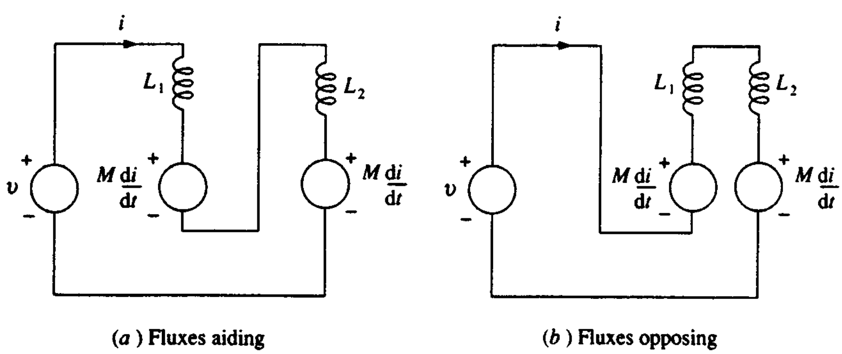


Fig. 1.29. Circuit models for series connected coupled coils.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

When the same procedure is applied to circuit (b) the result is

$$L_4 = L_1 + L_2 - 2M \tag{1.54}$$

1.11 Passive circuit components

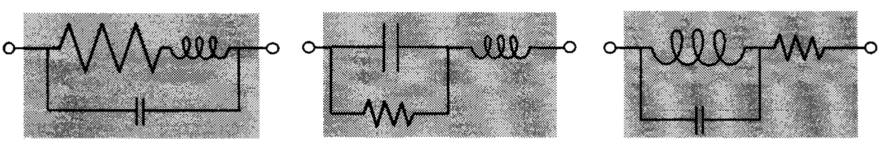
Passive circuits are made up of components – resistors, capacitors, inductors and mutual inductors – each designed, ideally, to exhibit one of the elemental properties to the exclusion of the others. These components are usually, although not always, required to be time-invariant and linear, that is, their properties should not be affected by changes in operating conditions, nor should their values depend in any way on the applied currents and voltages.

Real components fall short of these ideal requirements in several respects. Generally, all three elemental properties will be associated with any given component so that it will exhibit its named property over a limited range of frequencies only. A resistor, for example, is often constructed from a length of high-resistivity wire wound upon a suitable former to form a compact coil. Such a coil will possess inductance; there will also be capacitance between the turns of the coil.* It is sometimes necessary, therefore, to devise circuit models like those shown in fig. 1.30, which will represent the characteristic behaviour of the component over a particular range of operating frequencies.

The 'purity' of a component refers to the extent to which it is free from extraneous (or stray or parasitic) elements. Most modern components possess a high degree of purity, and it is usually permissible – for example, in the case of a resistor – to neglect stray inductance and capacitance and to represent it simply by a resistance in the circuit model, but the designer must always take care to ensure that the model accurately reflects the properties of the circuit.

The properties of real components also vary, to some extent, with time

Fig. 1.30. Models for passive circuit components.



Resistor | Capacitor | Inductor

* By folding back the length of wire on itself before winding it into a coil, the self flux linkage and therefore the inductance may be greatly reduced – a technique known as *bifilar winding.*

https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

and temperature, the latter effect being dependent both on the ambient temperature and on the power dissipated in the component itself, that is, upon applied voltages and currents. Components also suffer from long-term changes in the values of their properties – an effect known as 'ageing'.

1.12 Summary of basic circuit relations

Table 1.1 contains a summary of the basic equations and laws of circuit theory. In this table v and i denote, respectively, the voltage and current across and through the circuit element concerned at a particular instant of time.

The reader will observe that there exists a striking symmetry in these circuit relations. Kirchhoff's two laws, for instance, are of identical mathematical form with v in one replacing i in the other: the same applies to Ohm's law expressed in its resistive and conductive forms. We see also that the formula for the series combinations of resistances is of the same form as the parallel combination of conductances. The complete sets of equations applying to capacitance and inductance evince this underlying symmetry also, i interchanging with v and C interchanging with L. The equations relating to combinations of L and combinations of C reveal a reciprocal relationship of the same type as that existing between R and G.

These symmetries are a manifestation of a general principle of circuit theory which we call *duality*. Expressed in general terms this principle states that, for any linear circuit whose behaviour is described by a certain set of equations, a dual circuit can be found for which the circuit equations are of the same mathematical form. However, in the equations for the dual circuit, current and voltage are interchanged and each element is replaced by its dual element: R for G, L for C, etc. For example, a circuit comprising two resistances connected in series to an ideal voltage source would have as its dual two conductances connected in parallel to an ideal current source. This principle sometimes provides alternative and illuminating ways of approaching circuit analysis and synthesis; we refer to it at several points throughout this text. (There are certain restrictions to this general principle: for instance, it cannot be applied directly to circuits containing mutual inductance. See for example references 1 and 4.)

1.13 Problems

1. A circuit element is shown in fig. 1.31 for which the reference directions of voltage and current are defined.

(*a*) If $v = -3$ V and $i = -2$ A (both constant), is the element acting as a source or as a sink? What is the power delivered or received by the element?

(*b*) If $v = 3$ V (constant) and $i = (2t + 1)$A, what is the total amount of

Table 1.1. *Summary of basic equations and laws of circuit theory*

Description	Law or relationship	Unit	Equation
Charge and current	$i=\frac{dq}{dt}$	ampere	1.1
	$q=\int_0^t i\,dt$	coulomb	1.2
Energy	$J=\int_0^t vi\,dt$	joule	1.7
Power	$p=vi$	watt	1.8
Kirchhoff's laws			
Current	$\sum i_{\text{in}}=\sum i_{\text{out}}$		1.12
	$\sum i=0$		1.13
Voltage	$\sum v_{\text{rise}}=\sum v_{\text{fall}}$		1.14
	$\sum v=0$		1.15
Resistance (linear)			
Ohm's law	$R=\frac{v}{i}$	ohm	1.16
	or		
	$G=\frac{i}{v}$	siemen	1.17
Power	i^2R or $\frac{v^2}{R}$	watt	1.20, 1.21
Series combination	$R=R_1+R_2+\ldots+R_n$	ohm	1.23
	$\frac{1}{G}=\frac{1}{G_1}+\frac{1}{G_2}+\ldots+\frac{1}{G_n}$	ohm	1.24
Parallel combination	$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\ldots+\frac{1}{R_n}$	siemen	1.27
	$G=G_1+G_2+\ldots+G_n$	siemen	1.28
Capacitance (linear)			
Charge and voltage	$q=Cv$	coulomb	1.29
Current and voltage	$i=C\frac{dv}{dt}$	ampere	1.30
	$v=\frac{1}{C}\int_0^t i\,dt+v_0$	volt	1.31
Stored energy	$\frac{1}{2}Cv^2$	joule	1.32
Series combination	$\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}+\ldots+\frac{1}{C_n}$	farad^{-1}	1.34

Table 1.1. (*cont.*)

Description	Law or relationship	Unit	Equation
Parallel combination	$C=C_1+C_2+\ldots+C_n$	farad	1.36
Inductance (linear)			
Flux linkage and current	$\phi=Li$	weber	1.37
Voltage and current	$v=L\dfrac{\mathrm{d}i}{\mathrm{d}t}$	volt	1.40
	$i=\dfrac{1}{L}\displaystyle\int_0^t v\,\mathrm{d}t+i_0$	ampere	1.41
Stored energy	$\frac{1}{2}Li^2$	joule	1.42
Series combination	$L=L_1+L_2+\ldots+L_n$	henry	1.44
Parallel combination	$\dfrac{1}{L}=\dfrac{1}{L_1}+\dfrac{1}{L_2}+\ldots+\dfrac{1}{L_n}$	henry^{-1}	1.46
Mutual inductance (linear)			
Flux linkage and current	$\phi_{21}=Mi_1$	weber	1.47
Voltage and current	$v_1=L_1\dfrac{\mathrm{d}i_1}{\mathrm{d}t}\pm M\dfrac{\mathrm{d}i_2}{\mathrm{d}t}$	volt	1.49
Coefficient of coupling	$k=\dfrac{M}{(L_1L_2)^{\frac{1}{2}}}$		1.52

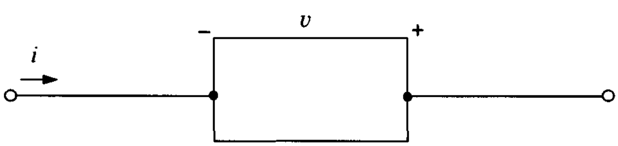
charge that flows during the interval $0\leqslant t\leqslant 10$ seconds? Calculate the total energy delivered or received by the element during this interval.

2. State which of the elements A, B, C, D and E in the circuit of fig. 1.32 are sources and which are sinks. Find the total power transfer from sources to sinks.

3. What is the resistance looking into the terminals AB of the circuit shown in fig. 1.33? If the terminals AB are connected together, what is the conductance between points C and D?

4. In the circuit of fig. 1.34, find the voltages V_{AE}, V_{BE} and V_{CE}, and the currents I_{AB} and I_{CB}.

Fig. 1.31. Circuit for problem 1.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

Fig. 1.32. Circuit for problem 2.

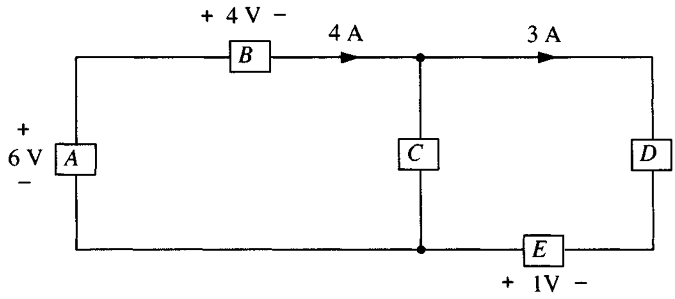


Fig. 1.33. Circuit for problem 3.

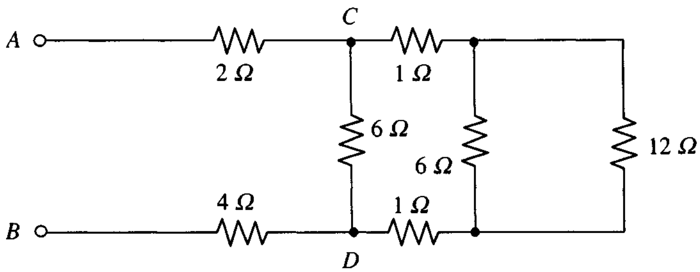


Fig. 1.34. Circuit for problem 4.

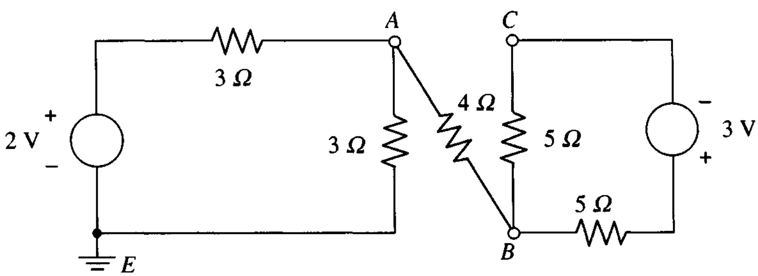
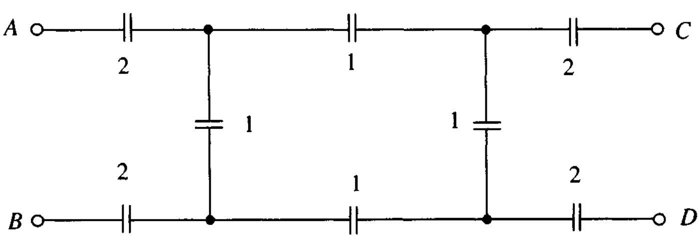


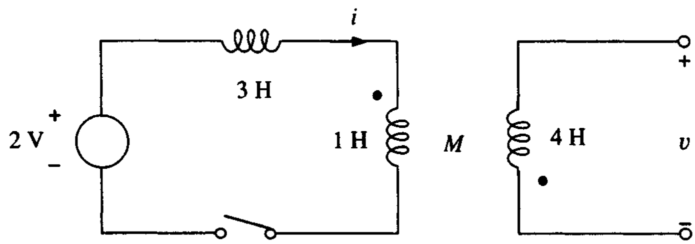
Fig. 1.35. Circuit for problem 6.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press

5. A 1 MΩ resistor and a 2 μF capacitor are connected in series across an ideal current source which delivers a current $i = 5e^{-2t}$ mA. The capacitor is uncharged at $t = 0$. Determine the energy stored in the capacitor and the voltage across the current source at the instant $t = 0.5$ s.
6. Find the change in the capacitance measured at AB in fig. 1.35 when terminals CD are connected together (capacitance in μF).
7. The mutual inductor in the circuit of fig. 1.36 has a coupling coefficient of 0.5; what is its mutual inductance? Determine the current i, the voltage v, and the total energy stored in the circuit at an instant 4 seconds after closure of the switch.

Fig. 1.36. Circuit for problem 7.



https://doi.org/10.1017/CBO9781139170093.003 Published online by Cambridge University Press