7

Non-linear circuit analysis

7.1 Introduction: linear and non-linear elements

In the preceding chapters, discussions have been confined to circuits that could be modelled by linear elements. The property that characterizes a linear element is independent of current or voltage. For example, the voltage–current characteristic of a resistor, modelled by a pure, linear resistance R is a straight line passing through the origin and having a slope equal to $1/R$ (fig. 7.1(*a*)). A linear resistor is also *bilateral*, that is, its voltage–current relationship is the same in the first and third quadrants of the characteristic. This property implies that the component can be connected into a circuit without regard to the polarity of the voltage to which it is subjected.

An example of a non-linear resistor is the incandescent lamp whose v–i characteristic is shown in fig. 7.1(*b*). The non-linearity in this case results from the great increase in temperature of the filament as it becomes incandescent. The characteristics of the lamp are the same whichever way round it is connected to the supply, and the device, although non-linear, is bilateral. On the other hand, the diode, whose v–i characteristic is shown in fig. 7.1(*c*), is neither linear nor bilateral.

Linear inductors and linear capacitors present a behaviour similar to that of the linear resistor. The response, $v_L = L\dfrac{\mathrm{d}}{\mathrm{d}t}[i(t)]$, of a linear inductor to a changing current is independent of the magnitude of the current in the inductor. Likewise, $i_C = C\dfrac{\mathrm{d}}{\mathrm{d}t}[v(t)]$ is independent of the magnitude of the voltage across the capacitor. An ordinary capacitor rarely exhibits appreciable non-linear behaviour unless it is driven beyond the voltage range for which it was designed. An inductor that contains no ferromagnetic material, or which has an appreciable air gap in its ferromagnetic core if it is present, exhibits linear behaviour. If, however, the path for the flux is entirely contained within ferromagnetic material, then, except for vanishingly small amplitudes of currents, the device will be non-linear and will exhibit the phenomena of saturation and hysteresis that were discussed in chapter 4.

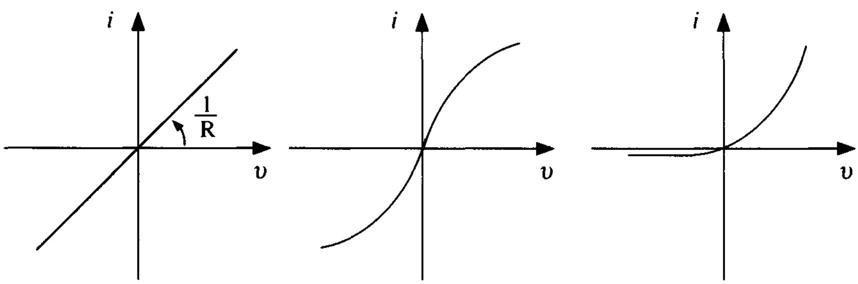
In chapter 2 theorems were developed that depended upon the property of linearity, and we saw in later chapters how one of the most important of these – superposition – allowed us to find the total response of a circuit to an excitation by adding or superposing the transient and steady-state responses found separately. This technique cannot be applied to circuits containing non-linear devices, such as the diode, since in this case the incremental response will no longer be proportional to the excitation. Similar remarks may be made concerning all of the linear circuit theorems. Consequently the theory and techniques appertaining to the analysis of non-linear circuits are very much more restricted than is the case for linear circuits.

In contrast with the linear circuit theorems, Kirchhoff's Laws, being essentially expressions of the laws of conservation of energy and charge, are universally applicable.

7.2 Graphical analysis

A non-linear device may always be represented by its experimentally determined v–i characteristic. A single v–i (or i–v) plot is sufficient to completely characterize a simple two-terminal device such as a resistor. When a device has more than two terminals, its behaviour must be described by either a family of curves or, more generally, several families of curves.

Fig. 7.1. Voltage–current characteristics of: (*a*) linear and, ((*b*) and (*c*)), non-linear devices.



(*a*) Resistor (*b*) Lamp (*c*) Diode

The graphical method is very often used to find the currents and/or voltages when two circuit elements whose characteristics are given are connected in series or in parallel. The graphical construction solves whichever of Kirchhoff's laws is appropriate to the given problem. The method will be introduced with the example, shown in fig. 7.2(*a*), of a series circuit consisting of a non-linear resistor, a linear resistor, and a voltage source. We wish to find the current I and the voltage V_2 across the non-linear resistor. The characteristic of the non-linear resistor, shown in fig. 7.2(*b*), gives one relation between i and v_2. A second relation is obtained by application of KVL to the circuit of fig. 7.2(*a*).

$$V_0 = v_1 + v_2 = iR + v_2$$

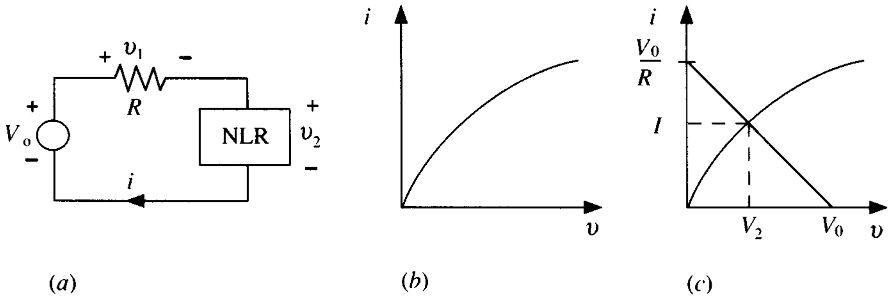
So

$$i = \frac{V_0}{R} - \frac{v_2}{R} \tag{7.1}$$

This is the equation of a straight line with slope $-1/R$ and intercept V_0/R. Also, when $i=0$, $v_2 = V_0$. In fig. 7.2(*c*) (7.1) is plotted on the same axes as the characteristic of the non-linear resistor. The intersection of the two lines provides the required combination of I and V_2. The voltage across R is $(V_0 - V_2)$. The construction of the straight line is simple. One locates V_0 on the voltage axis and V_0/R on the current axis and joins these points by a straight line.

Examination of fig. 7.2(*c*) shows that the straight line represents the characteristic of the linear resistor R in a coordinate system having its origin at V_0 and having voltage increasing to the left. Use of this fact makes possible the extension of the graphical method to the case of two non-linear resistors in series as illustrated in fig. 7.3.

Fig. 7.2. Graphical solution for a circuit containing a non-linear resistor (NLR): (*a*) circuit; (*b*) non-linear resistor characteristic; (*c*) graphical construction for solution.



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The same graphical approach is applicable when circuit elements are connected in parallel. Fig. 7.4(*a*) shows a parallel combination of a non-linear resistor and a linear resistor supplied from a current source of magnitude I_0. The problem is to find the current in each circuit element and the voltage across the combination. Since voltage now is the common quantity, we draw the non-linear characteristic with v as the dependent variable. This is shown in fig. 7.4(*b*). Now KCL gives

$$I_0 = i_1 + i_2 = i_1 + \frac{v}{R}$$

So

$$v = I_0 R - i_1 R \tag{7.2}$$

This is the equation of a straight line with slope $-R$ and intercept $I_0 R$. Furthermore, when $v=0$, $i_1 = I_0$. In fig. 7.4(*c*) the line represented by (7.2) is drawn on the characteristic of the non-linear resistor. The intersection of

Fig. 7.3. Graphical solution for two non-linear resistors in series.

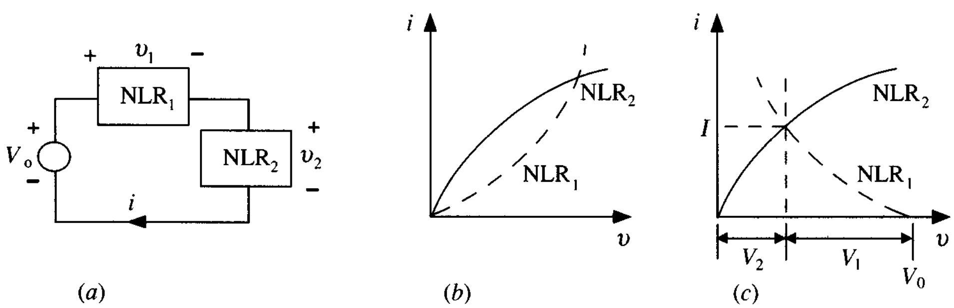
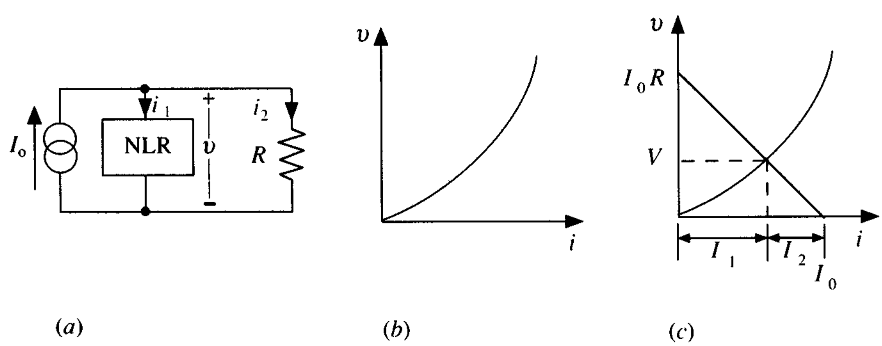


Fig. 7.4. Graphical solution for linear and non-linear resistors in parallel.



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the two lines gives the combination I_1 and V, representing the current in and the voltage across the non-linear resistor. The current in R is $I_2 = (I_0 - I_1)$.

In section 1.7.3 formulae were derived for the equivalent resistance of a series/parallel combination of several linear resistances. Non-linear resistances cannot be combined using these formulae and it is necessary to adopt other means; a graphical approach is often appropriate. In fig. 7.5 are drawn the characteristics of two non-linear resistors. If these elements are connected in series, then the current must be the same in both. For example, a current I_1 requires a voltage V_a across device number 1 and a voltage V_b across device number 2. The total voltage required to maintain I_1 in the series combination is therefore $V_a + V_b$. We thus determine one point on a new non-linear (composite) characteristic that represents the series combination of the two non-linear resistors. By assuming other currents, we may obtain other points on the composite characteristic.

If the circuit elements are connected in parallel then the composite curve is found by finding the total current required to maintain a given voltage across the combination. Details are shown in fig. 7.6.

Fig. 7.5. Composite v–i characteristic for two non-linear resistors in series.

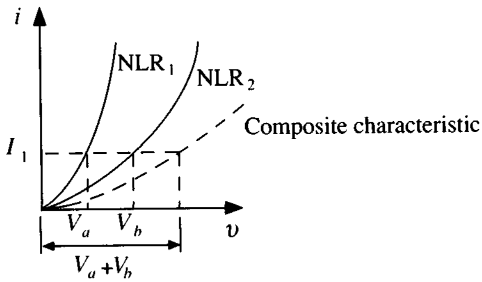
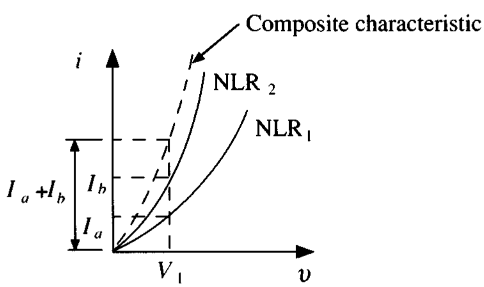


Fig. 7.6. Composite v–i characteristic for two non-linear resistors in parallel.



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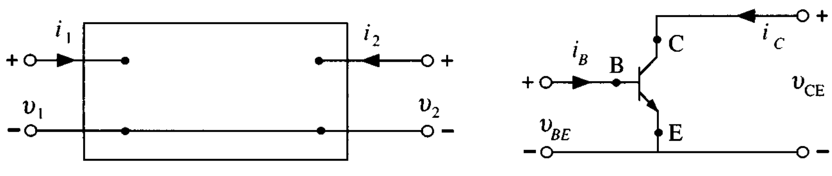
We examine next a three-terminal device as shown schematically in fig. 7.7(*a*). (This obviously is the special case of a two-terminal pair device (or two-port) in which there is a common connection between input and output.) Now there are two current–voltage pairs to be considered. Subscript 1 refers to the input and subscript 2 refers to the output. If the device is of any practical use, there will be interaction between input and output and two sets of characteristic curves will be required to describe the device behaviour.

Fig. 7.7(*a*) is an appropriate representation of a transistor. In what follows we shall assume that we are dealing with an npn transistor in the common-emitter connection.* The notation generally employed with transistors is shown in fig. 7.7(*b*).

The characteristics of a typical silicon transistor are shown in fig. 7.8. It is important to observe that i_B is almost independent of v_{CE} while i_C is strongly dependent upon i_B.

Suppose now that we make external connections to the transistor as shown in fig. 7.9. We wish to determine the currents I_B and I_C. Following the procedure already described, we construct *load lines* on the input and output characteristics as shown in fig. 7.10. (Note that the spacing between the curves in fig. 7.10(*a*) has been exaggerated for the sake of clarity.) From the intersections of the input load line with the family of input characteristics, we obtain pairs of values of I_B and V_{CE} that satisfy the constraints imposed by the combination of supply voltage V_1 and resistance R_1. When transferred to the output characteristic, these pairs of values determine points that establish a *transfer characteristic* indicated by the broken line in fig. 7.10(*b*). The intersection of the transfer characteristic with the output

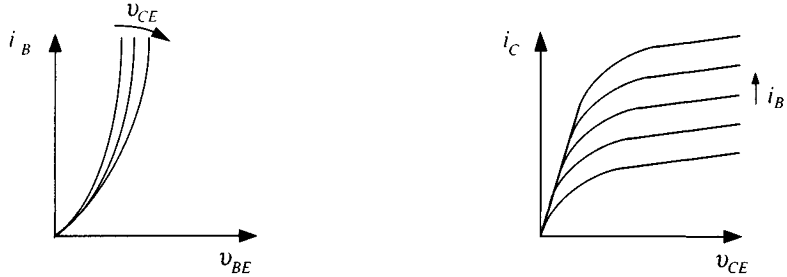
Fig. 7.7. Schematic representation of three-terminal devices.



(*a*) General device (*b*) Transistor (npn)

* The three terminals of a transistor are called, respectively, the base, the emitter and the collector. Any one of these may be the terminal that is common to input and output. The common-emitter connection is most often employed. Schematically, the transistor in the common-emitter connection is shown in fig. 7.7(*b*). For an npn device, the collector and the base normally are maintained positive with respect to the emitter. For further details see reference 5.

Fig. 7.8. Silicon transistor characteristics for common-emitter connection.



(*a*) Input characteristics (*b*) Output characteristics

Fig. 7.9. Transistor common-emitter connections to input and output.

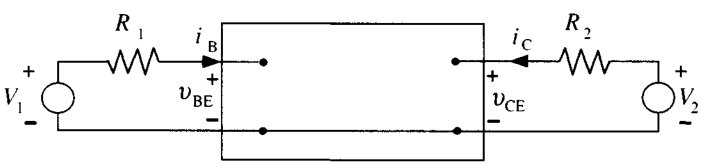
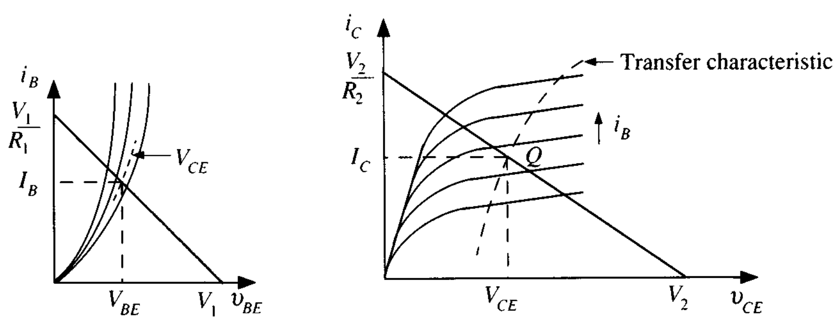


Fig. 7.10. Load-line method for determining operating point of a transistor.



(*a*) (*b*)

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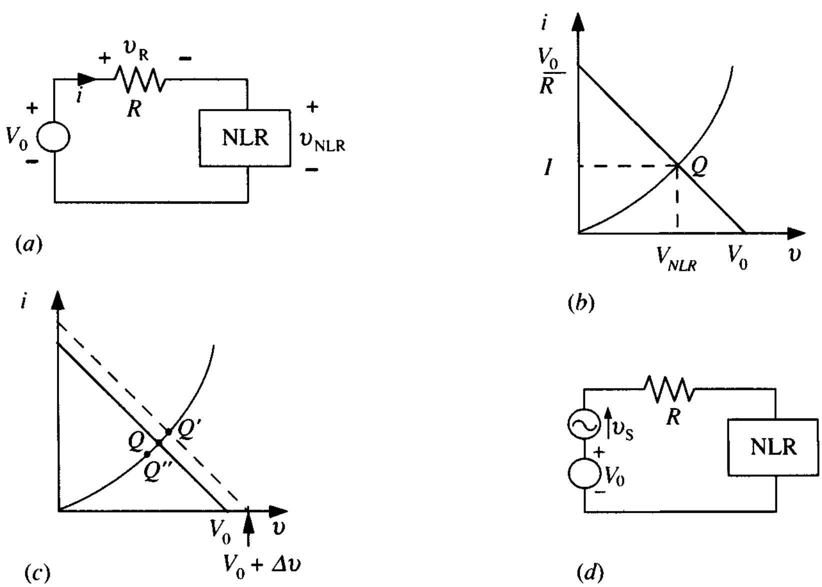
load line (determined by V_2 and R_2) represents the *operating point Q* and determines the current I_C and the corresponding voltage V_{CE}. We then use V_{CE} and the input characteristic to determine I_B and V_{BE}. The application of this method to bipolar transistors is simplified by the fact that the curves on the input characteristic are very close together. Consequently, the effect of v_{CE} upon i_B may be ignored. One simply finds I_B from the input characteristic (assuming that the device may be represented by a single line) and then uses the appropriate line on the output characteristic to establish the operating point.

7.3 Small-signal models

7.3.1 Non-linear resistor model

In Section 7.2 we described how to find the current in a series circuit consisting of a non-linear resistor (NLR), a linear resistor R and a voltage source V_0. The circuit and the graphical construction are repeated in fig. 7.11. The current I and the voltage V_{NLR} determine the *quiescent* or *operating* point, commonly designated by the letter Q. Now assume that V_0 increases by a small amount ΔV. There will then be a new quiescent point Q', found by a new construction as shown in fig. 7.11(c). A small decrease in V_0 will result in a shift of the operating point to Q''.

Fig. 7.11. Development of the small-signal model for a non-linear resistor.



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Let us now revise the circuit of fig. 7.11(a) to include a sinusoidal voltage source $v_s = \Delta V \sin\omega t$, as shown in fig. 7.11($d$). Now the operating point in fig. 7.11(c) will move periodically along the characteristic curve between limits Q' and Q''. As ΔV decreases, the relevant portion of the characteristic curve decreases in length. In the limit, as $\Delta V \to 0$, this portion of the curve may be approximated by a straight line whose slope is $\mathrm{d}i/\mathrm{d}v$ at point Q. Let $\mathrm{d}i/\mathrm{d}v = 1/r_n$. Then, *as far as the voltage* v_s *is concerned*, the circuit is equivalent to the series combination of two linear resistors and the source v_s as shown in fig. 7.12. Then the alternating component of the voltage across the NLR is

$$v_n = \frac{r_n}{r_n + R} v_s \tag{7.3}$$

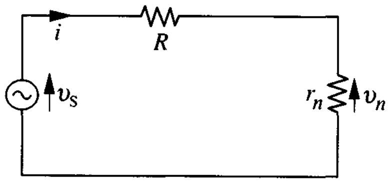
In fig. 7.11(d), the circuit is said to be *biased* by the source V_0 at the point Q whose coordinates are I and V_{NLR}. I is the *bias current* and the resistance r_n in fig. 7.12 is determined by the slope of the NLR characteristic at the Q point. The resistance r_n is referred to variously as the *slope*, *dynamic*, or *incremental* resistance, and once it has been determined, the circuit of fig. 7.12 is sufficient for calculating the alternating current behaviour of the circuit. Fig. 7.12 is truly valid only for alternating voltages of vanishingly small amplitudes. It is called the *small-signal* model. It is apparent that this model utilizes a linear representation of the NLR – a representation whose validity depends upon the amplitude of the driving alternating voltage and upon the shape of the NLR characteristic.

7.3.2 Transistor model

Let us now examine the possibility of obtaining a small-signal model of the transistor that may be used in circuit calculations without recourse to the characteristic curves.

Referring again to fig. 7.10 we assume that the transistor is biased at an appropriate point by means of batteries and resistors. Consider first the input characteristic, a change in v_{BE} can be caused *either* by a change in i_B or by a change in v_{CE}. If v_{CE} is held constant, then the combination (i_B, v_{BE})

Fig. 7.12. Small signal model for a non-linear resistor.



must lie on the appropriate v_{CE} line. Let $i_b = I_m \sin\omega t$ represent an alternating current with $I_m \ll I_B$. Then, just as in the case of the two-terminal device the characteristic may be represented by a straight line having the slope of the characteristic at the operating point. Then we may write

$$v_{be} = \left.\frac{\partial v_{be}}{\partial i_b}\right|_{v_{CE}} i_b = h_{ie} i_b \tag{7.4}$$

where $h_{ie} = \partial v_{be}/\partial i_b$ is the slope of the characteristic. Now let i_B remain constant at I_B and let v_{CE} change. Then the corresponding change in v_{BE} is

$$v_{be} = \frac{\Delta v_{be}}{\Delta v_{ce}} v_{ce}$$

In the limit then

$$v_{be} = \left.\frac{\partial v_{be}}{\partial v_{ce}}\right|_{i_B} v_{ce} = h_{re} v_{ce} \tag{7.5}$$

If both i_B and v_{CE} may change, (7.4) and (7.5) give

$$v_{be} = h_{ie} i_b + h_{re} v_{ce} \tag{7.6}$$

In fig. 7.10(*b*), i_C depends upon both v_{CE} and i_B. Suppose, with the transistor biased at the point Q, i_B is held constant and v_{CE} changes. Then the combination (i_C, v_{CE}) must move along the line that represents a constant value of I_B. So, again considering small amplitude a.c. quantities

$$i_c = \left.\frac{\partial i_c}{\partial v_{ce}}\right|_{i_B} v_{ce} = h_{oe} v_{ce} \tag{7.7}$$

Finally, if v_{CE} is held constant and if i_B changes the corresponding change in i_C is

$$i_c = \left.\frac{\partial i_c}{\partial i_b}\right|_{v_{CE}} i_b = h_{fe} i_b \tag{7.8}$$

So, if both i_B and v_{CE} change, (7.7) and (7.8) give

$$i_c = h_{fe} i_b + h_{oe} v_{ce} \tag{7.9}$$

It is a straightforward matter to devise a linear, two-port model that represents equations 7.6 and 7.9. This *hybrid parameter* model (so called because the *h*s do not all have the same dimensions) is shown in fig. 7.13. Observe that this model contains two dependent sources. In many transistors the parameter h_{re} is so small as to be negligible in most applications. (This corresponds to the situation where the curves on the input characteristic are very close together.) Also, h_{oe} is frequently very small. (This corresponds to the curves in the output characteristic being almost horizontal and thus having zero slope.) So, in many applications the appropriate small-signal model of the transistor is as shown in fig. 7.14. It must be remembered that the models of figs 7.13 and 7.14 apply only to currents and voltages of small amplitude, and that both models are based upon the assumption that d.c. voltages have been applied to bias the transistor at the appropriate operating point.

7.4 Piecewise-linear circuits

7.4.1 Piecewise-linear approximation

The small-signal model of a device uses a linear approximation that is valid over a narrow region of the device characteristic. We examine next a model that may represent a non-linear device over an arbitrarily wide region of its characteristic.

In general, a curve describing the characteristic of a real device may be

Fig. 7.13. Hybrid-parameter model of a transistor in the common-emitter connection.

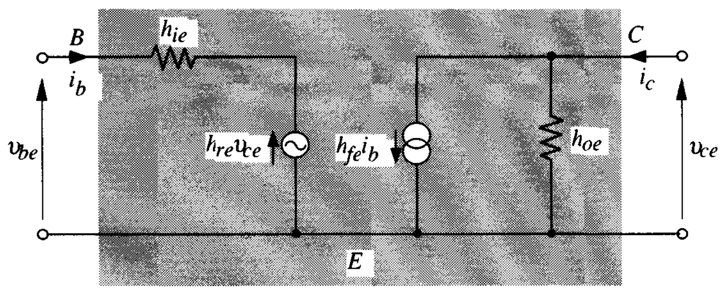
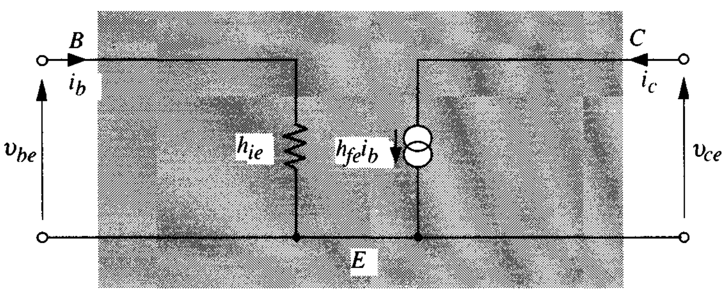


Fig. 7.14. Simplified hybrid-parameter model in the common emitter connection.



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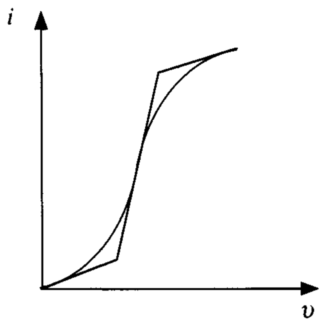
represented to any required accuracy by a broken line consisting of many short straight segments. Many real non-linear devices are represented adequately by two or three such segments. Fig. 7.15 shows one example of a characteristic and a three-segment approximation. If we can devise a model that has this piecewise-linear property, the device may be represented by the model and methods of linear circuit analysis may be used.

Two kinds of problems are of interest in applications of piecewise-linear models. The first type involves *synthesis* of a circuit that will reproduce, to whatever accuracy is required and over a specific range of operation, the non-linear behaviour of a device. The second type of problem is concerned with *analysis* of a given circuit to determine the slopes of the linear segments and the coordinates of the *break points* i.e. those points at which the slope of the characteristic changes. When slopes and break points are known, the piecewise-linear characteristic of the circuit may be drawn. In synthesis we are given a characteristic and we must find a circuit to represent it. In analysis we are given a circuit and are required to find the corresponding piecewise-linear representation of its characteristic. It is possible to construct piecewise-linear circuit models using resistances and *ideal diodes*.

7.4.2 The ideal diode

An ideal diode is a voltage-controlled two-terminal device that has the characteristics of a switch. If voltage of one polarity is applied, the diode is a short circuit (i.e., a closed switch) while if the voltage polarity is reversed the diode is an open circuit (i.e., an open switch). The diode is represented as shown in fig. 7.16(*a*) where the arrow represents the direction of current when the diode is conducting. The i–v characteristic of a diode is as shown in fig. 7.16(*b*), with positive values of i and v defined in fig. 7.16(*a*). Except where real diodes are specified, all diodes in the circuits that follow are ideal.

Fig. 7.15. Piecewise-linear approximation.



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7.4.3 Combinations of resistances and ideal diodes

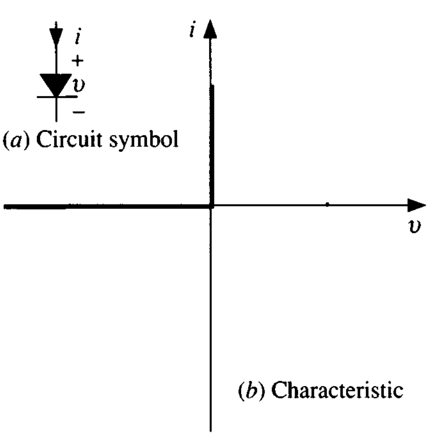
Before discussing synthesis and analysis, we examine the v–i characteristics of some resistance–diode combinations. These simple circuits may then be used as 'building blocks' in either the synthesis or the analysis of more complex circuits.

Fig. 7.17 shows four diode–resistance combinations and the v–i characteristic of each. For these circuits the characteristics are easily determined. For example, in fig. 7.17(a), when v is positive the diode conducts. Since a conducting diode has no resistance, there is no voltage drop across it and so the v–i characteristic is simply the straight line of slope $1/R$ that represents the resistance R. When v is negative, the diode does not conduct and so the current is zero.

The combinations shown in fig. 7.17 all have a single break point at the origin. By including a voltage source in the series circuits of figs. 7.17(a) and (b), we can shift the break point along the voltage axis (see fig. 7.18). A current source in parallel with the resistance–diode combinations of figs. 7.17(c) and (d) shifts the break point along the current axis (see fig. 7.19). Again, it is a simple matter to sketch the i–v characteristic. For example, in fig. 7.19(a), as long as the input current exceeds $(-I)$ there is a net forward current through the diode and so the voltage across it is zero. When i is less than $(-I)$ there is current through R producing a voltage drop that turns the diode off. As i decreases further, a current $(i-I)$ flows in R and the characteristic is a straight line having slope $1/R$ and passing through the point $(0, -I)$.

Consider next the effect of adding a second resistance to the circuit of fig. 7.17(a). The resulting circuit and its characteristic are shown in fig. 7.20(a).

Fig. 7.16. Ideal diode.



(a) Circuit symbol

(b) Characteristic

Fig. 7.17. Diode–resistance combinations with break point at the origin.

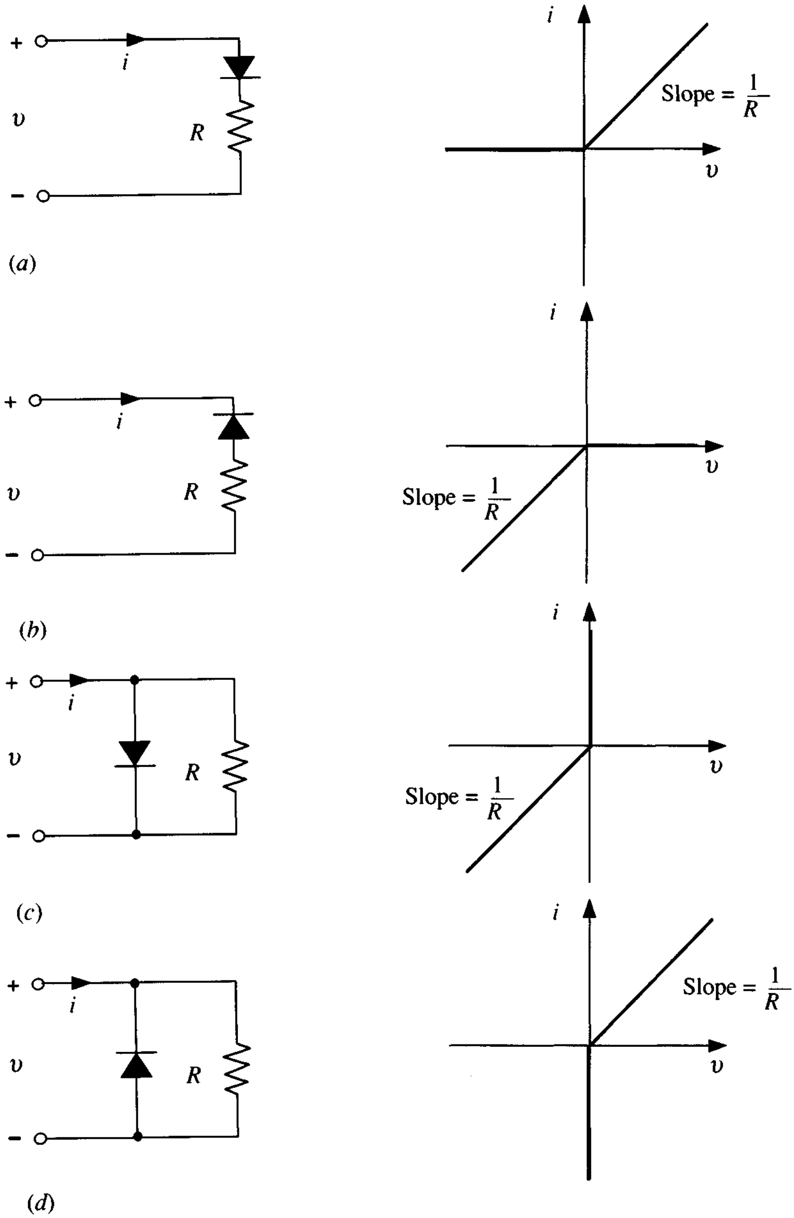


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Fig. 7.18. Diode–resistance combinations with the break point on the v-axis.

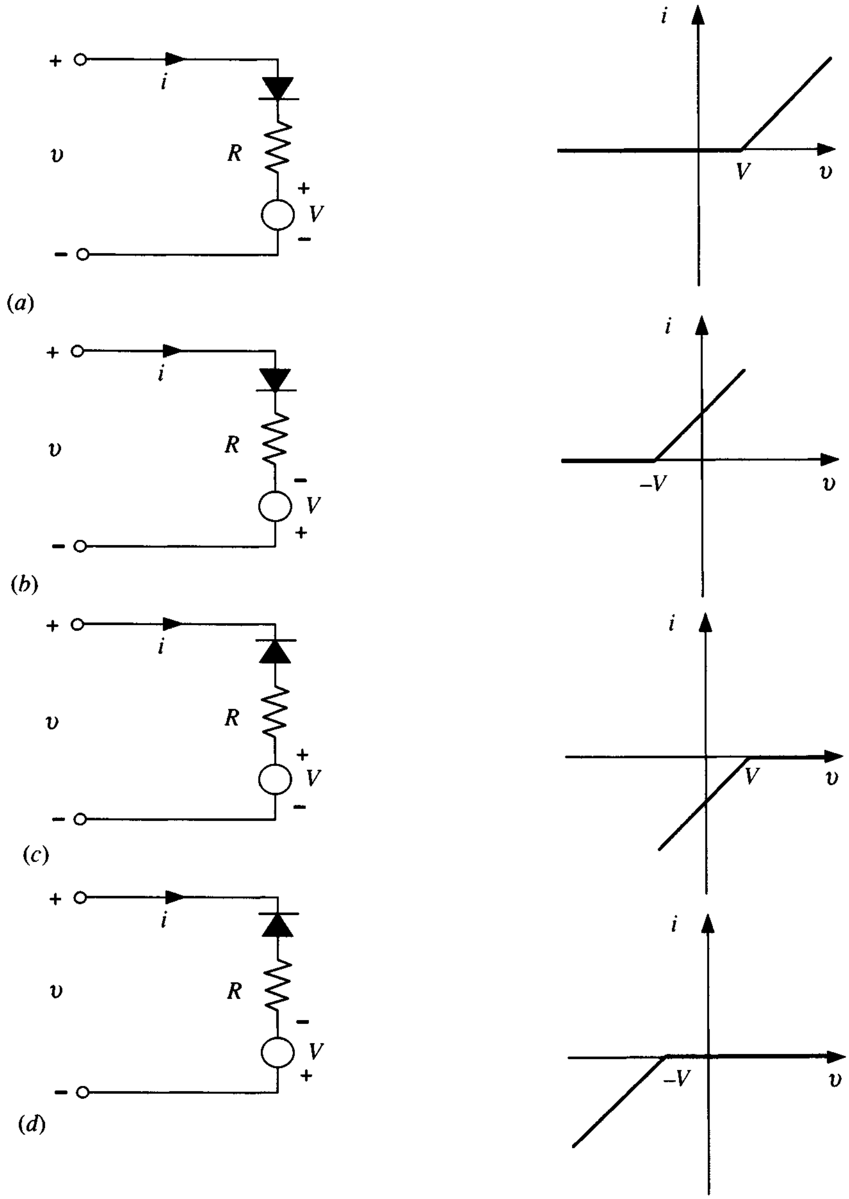

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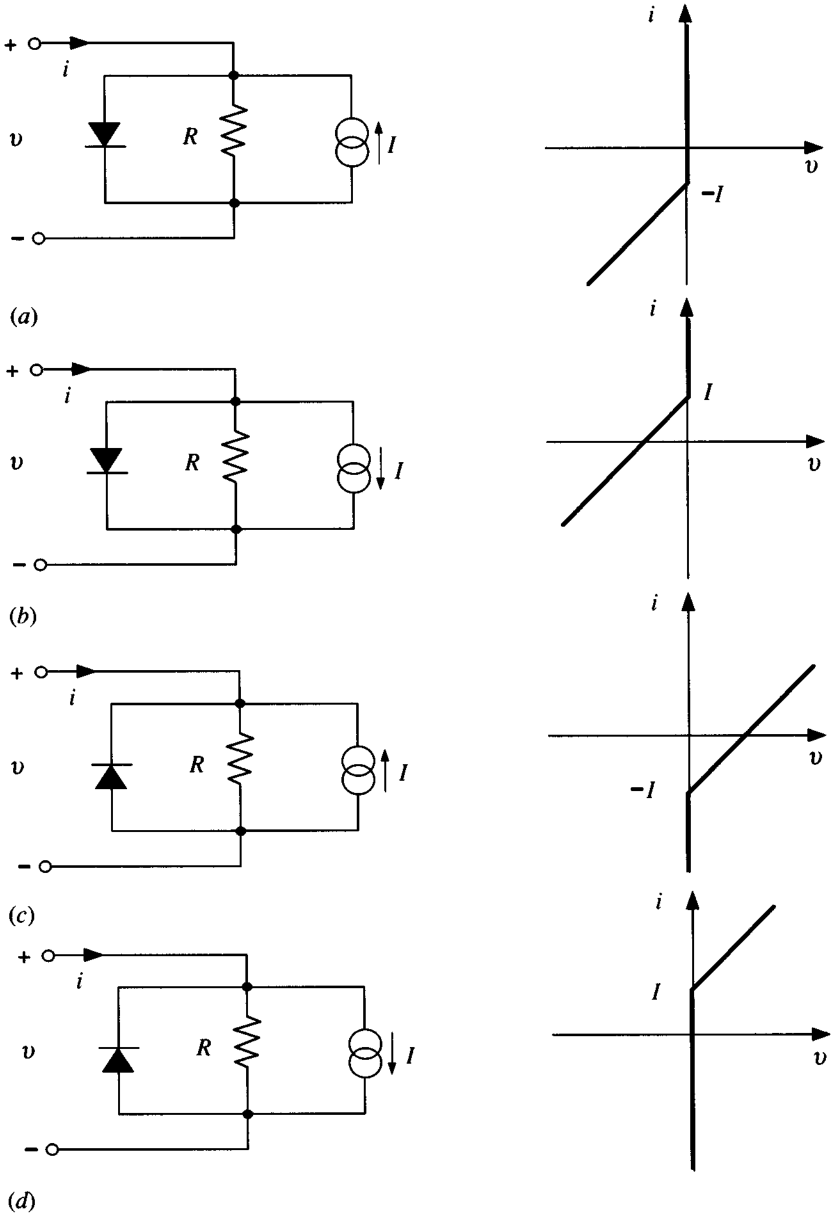
Fig. 7.19. Diode–resistance combinations with break point on i-axis.



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A resistance added to the circuit of fig. 7.17(c) yields the result shown in fig. 7.20(b).

Addition of a resistance to the circuits of figs. 7.18 and 7.19 causes the break point to move off the axis. Consider, for example, the circuit of fig. 7.18(a) shown again as fig. 7.21(a). Let R_2 be added as in fig. 7.21(b). We determine the break point by making use of the fact that *when the diode switches from the conducting state to the non-conducting state both the voltage v_d across the diode and the current i_d through the diode must be identically zero.* For the circuit of fig. 7.21(b)

$$v - v_d - i_d R_1 - V = 0 \tag{7.10}$$

and when $v_d = 0$ and $i_d = 0$, then $v = V$. The condition $i_d = 0$ means that there is no current in the diode branch. Therefore, at the break point, $i = V/R_2$.

Above the break point when the diode is conducting the slope of the characteristic is $1/(R_1//R_2) = (R_1 + R_2)/R_1R_2$, while below the break point (diode not conducting) the slope is $1/R_2$. Observe that at $v = 0$, $i = 0$, so the characteristic passes through the origin.

Consider next the circuit of fig. 7.19(b), redrawn in fig. 7.22(a). Let R_2 be added as in fig. 7.22(b). Now to find the break point we impose the condition, $i_d = 0$ and $v_d = 0$. When $v_d = 0$, the current through R_1 must be

Fig. 7.20. Diode resistance combinations with two different slopes.

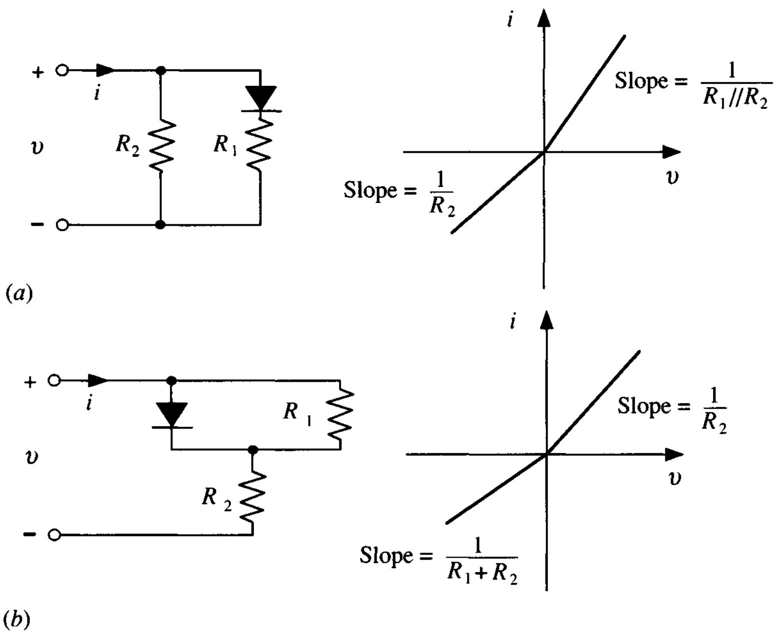

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Fig. 7.21. Effect of added parallel resistance to circuit of fig. 7.18(a). (a) Original circuit and characteristic; (b) Circuit and characteristic with addition of R_2.

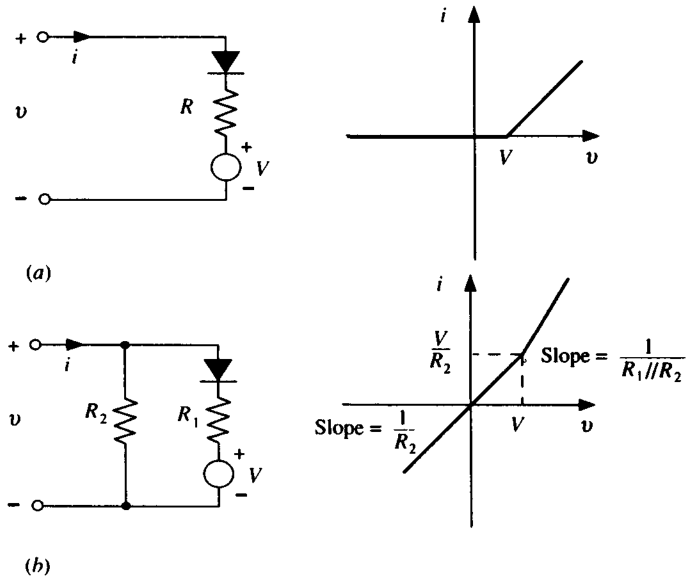
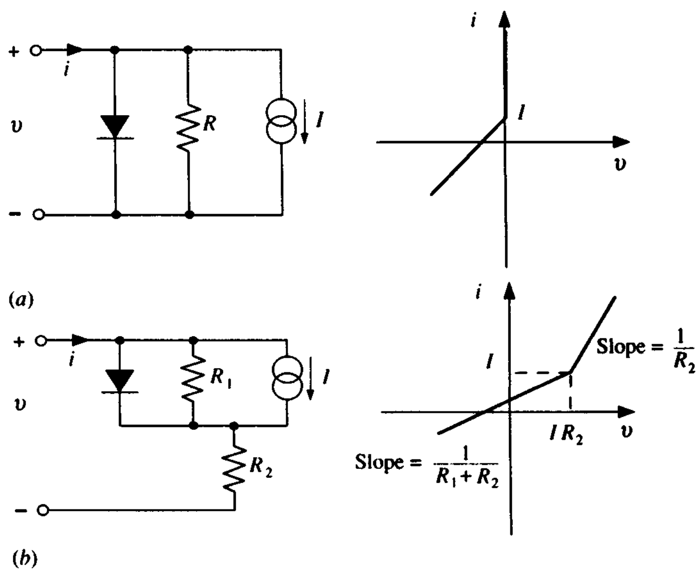


Fig. 7.22. Effect of added series resistance to circuit of fig. 7.19(b). (a) Original circuit characteristic; (b) Circuit and characteristic with addition of R_2.



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zero. Since at the same time $i_d = 0$, it follows that at the break point $i = I$. But if $i = I$ and the voltage across R_1 is zero, then at the break point $v = IR_2$.

Above the break point, with the diode conducting, the slope of the characteristic is simply $1/R_2$. When the diode is not conducting, KVL gives

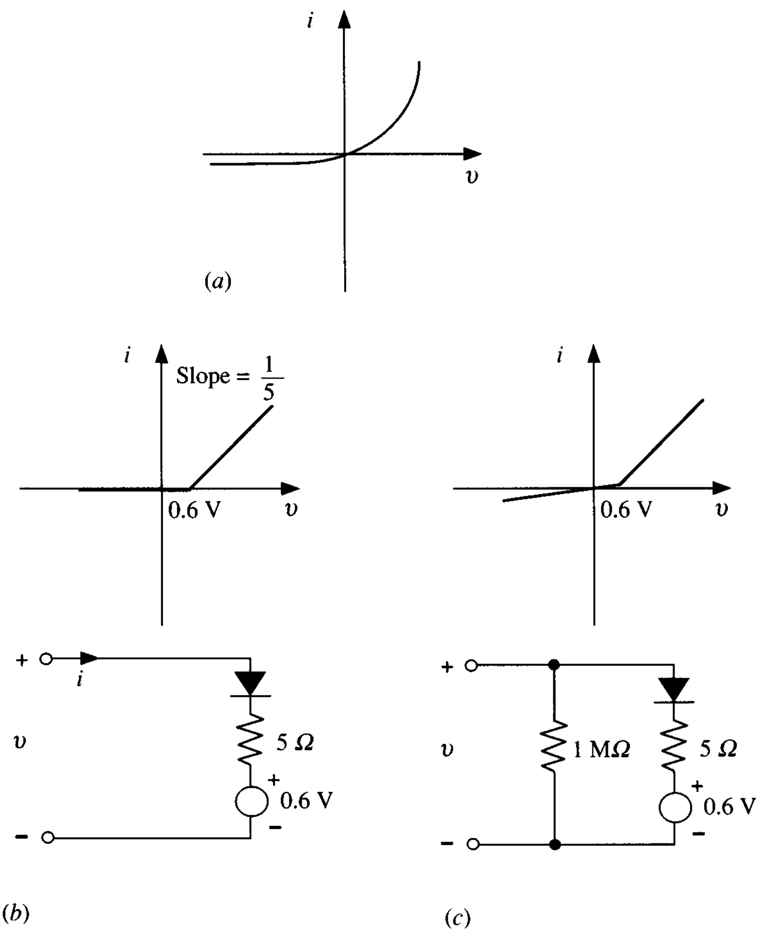
$$v - iR_1 + IR_1 - iR_2 = 0$$

and so

$$i = \frac{v + IR_1}{R_1 + R_2} = \frac{v}{R_1 + R_2} + \frac{R_1}{R_1 + R_2} I \tag{7.11}$$

From this expression we see that the slope of the characteristic below the

Fig. 7.23. Development of circuit model for the forward characteristic of a real diode.



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break point is $1/(R_1+R_2)$. For this characteristic to pass through the origin, it is necessary that $R_1=0$. Then the break point disappears because there is just the resistance R_2 across v.

7.4.4 The real diode

In contrast to the ideal diode that we have used so far, a real diode has a characteristic as shown in fig. 7.23(a). Typical semiconductor diodes have resistance of the order of a megohm for reverse voltage so the characteristic in the third quadrant is nearly horizontal. When forward voltage is applied, current rises exponentially with voltage.

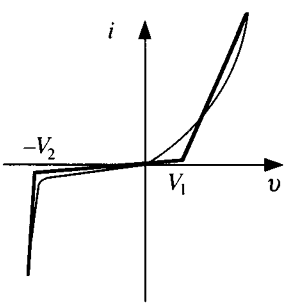
For many purposes the semiconductor diode may be represented by a piecewise-linear approximation having a single break point on the voltage axis as shown in fig. 7.23(b). An approximation that accounts for the small *reverse leakage current* that exists under reverse voltage conditions is shown in fig. 7.23(c).

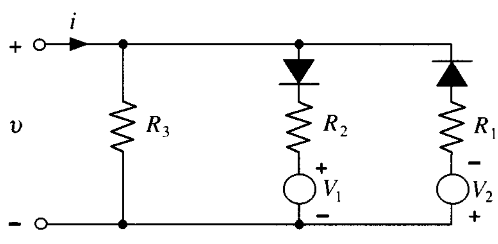
If the reverse voltage is made sufficiently large, a real diode exhibits *breakdown*, that is, it begins conducting heavily in the reverse direction. Its resistance then is of the order of a few ohms. Fig. 7.24 shows a model that accounts for breakdown. For most purposes R_3 may be omitted. Then both break points will lie on the voltage axis.

7.4.5 The Zener diode

The voltage at which reverse breakdown occurs in a semiconductor diode is termed the *Zener voltage*. In rectifier or other applications based upon the unidirectional conducting properties of the diode, the Zener voltage is of interest simply because it specifies the maximum peak inverse voltage that the diode will withstand. The *Zener diode*, on the other hand, is a device whose normal mode of operation lies within the reverse breakdown

Fig. 7.24. Circuit model for the complete characteristic of a real diode.





$V_1 = 0.7\text{ V}\quad V_2 = 10\text{ V}$

$R_1 = 10\,\Omega,\ R_2 = 20\,\Omega,\ R_3 = 1\text{ M}\Omega$

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region. Special care is taken in the manufacture of this device to ensure that the slope of the characteristic in this region is very steep, that is, that the effective resistance is very small (see fig. 7.25). Thus, the voltage remains substantially constant over a wide range of operating currents. Inspection of fig. 7.25 shows that there is a minimum reverse current, in the region of the knee, that must exist in the diode in order for it to operate in the constant voltage region. The maximum current is determined by the heat dissipating properties of the diode. Zener diodes are available with operating voltages between 2 V and 200 V and with power ratings up to the order of 50 W.

The primary application of the Zener diode is that of maintaining a constant voltage across a load regardless of fluctuations in supply voltage or load currents. A basic circuit used for this purpose is shown in fig. 7.26(*a*). The supply voltage V_0 is large in relation to the load voltage V_L so that the current through R_1 is substantially constant; thus, changes in load current I_L are reflected in equal and opposite changes in diode current I_Z. The diode current, therefore, swings over a range of values equal to the load current variation and, since the characteristic of the Zener diode has a finite slope, this produces small fluctuation in the load voltage. Variations in supply voltage will also cause the load voltage to vary. We now establish an expression, using piecewise-linear analysis, for the variation in V_L that results from small variations in I_L and V_0.

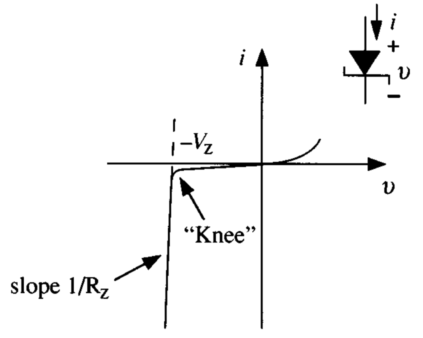
Consider the circuit of fig. 7.26(*b*) in which the Zener diode has been modelled by the circuit of fig. 7.18(*a*). By KVL we obtain

$$(I_L+I_Z)R_1+I_ZR_Z=V_0-V_Z$$

and

$$I_ZR_Z=V_L-V_Z$$

Fig. 7.25. Zener diode characteristic.



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Eliminating I_Z from these equations gives

$$\left(\frac{R_1}{R_Z}+1\right)V_L-\frac{R_1}{R_Z}V_Z+R_1I_L=V_0 \tag{7.12}$$

Now, from the total differential, the incremental change in V_L is given by:

$$\Delta V_L=\frac{\partial V_L}{\partial I_L}\Delta I_L+\frac{\partial V_L}{\partial V_0}\Delta V_0 \tag{7.13}$$

where ΔI_L and ΔV_0 are the incremental changes in load current and supply voltage.

Differentiating (7.12) we obtain

$$\left(\frac{R_1}{R_Z}+1\right)\frac{\partial V_L}{\partial I_L}+R_1=0$$

and

$$\left(\frac{R_1}{R_Z}+1\right)\frac{\partial V_L}{\partial V_0}=1$$

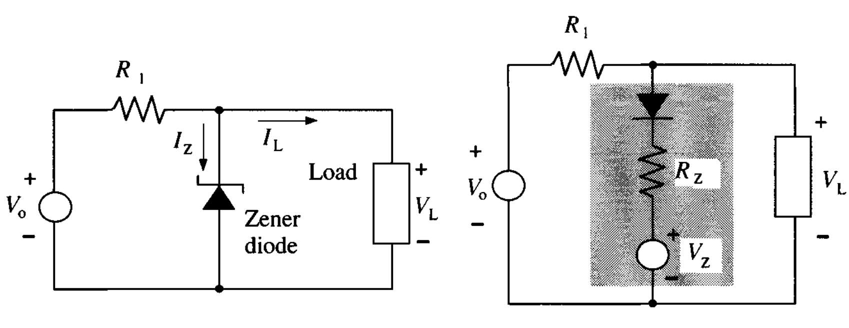
Hence, by substitution in (7.13)

$$\Delta V_L=-\left(\frac{R_1R_Z}{R_1+R_Z}\right)\Delta I_L+\frac{R_Z}{R_1+R_Z}\Delta V_0 \tag{7.14}$$

The coefficient of the first term in (7.14) is recognied as the parallel combination of R_1 *and* R_Z; the coefficient of the second term derives from the voltage-divider principle. Clearly, ΔV_L is reduced by making $R_1 \gg R_Z$, in which case (7.14) reduces to

$$\Delta V_L \simeq -R_Z\Delta I_L+\frac{R_Z}{R_1}\Delta V_0 \tag{7.15}$$

Fig. 7.26. Voltage stabilizer circuit incorporating a Zener diode.



(*a*) Circuit (*b*) Piecewise-linear circuit model

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Expressions (7.14) and (7.15) indicate that R_1 should be as large as possible in order to achieve good immunity from supply voltage variation. The value of R_1 determines, however, the quiescent operating point on the characteristic, and its upper limit will be set by the required load-current swing.

7.4.6 Analysis of piecewise linear circuits

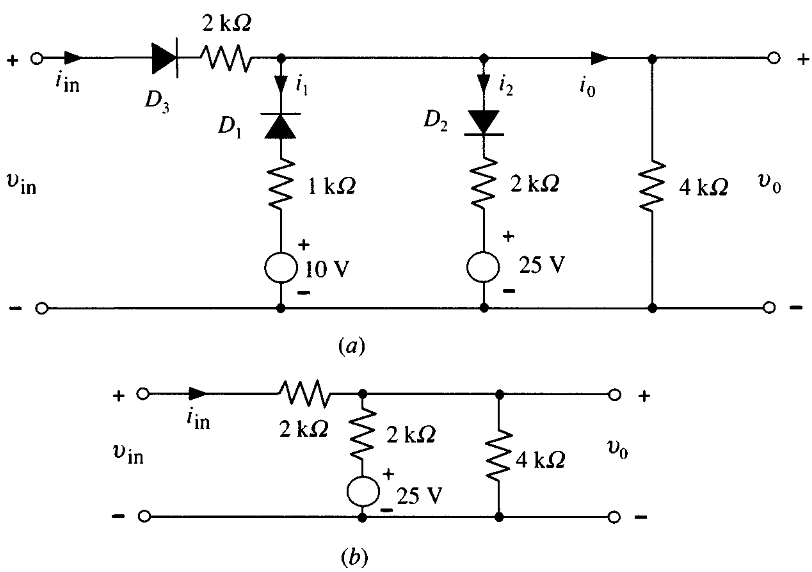
To analyze a circuit consisting of ideal diodes and resistors, we find break points by determining the input-voltage/input-current combination that exists as each diode changes from the non-conducting state to the conducting state. We then locate these break points on an i–v plane. By joining adjacent break points with straight lines we have the desired piecewise-linear characteristic of the given circuit.

Sometimes we may be interested in the output voltage v. input voltage characteristic. Once the break points have been determined, this characteristic may readily be found. The procedure is illustrated by the following example.

7.4.7 Worked example

For the circuit of fig. 7.27(a), find the break points and plot
(a) the i_{in} v. v_{in} characteristic;
(b) the v_{out} v. v_{in} characteristic.

Fig. 7.27. Circuits for worked example.



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Solution

Because there are three diodes, there are three break points. We find these by considering one diode at a time in the order of their subscripts.

Diode D_1: When $v_0 < 10$ V, D_1 conducts. When $v_0 > 10$ V, D_1 is reverse biased and so is not conducting. The first break point occurs at $v_0 = 10$ V. When $v_0 = 10$ V, D_2 is not conducting, so $i_2 = 0$. Also, at the first break point, $i_1 = 0$. Therefore, at the first break point, $v_0 = 10$ V, and

$$i_{in} = i_0 = 10/4000 = 2.5 \text{ mA}$$
$$v_{in} = 10 + (2.5 \times 10^{-3} \times 2000) = 15 \text{ V}$$

Diode D_2: When $v_0 < 25$ V, D_2 is reverse biased and does not conduct. When $v_0 > 25$ V, D_2 conducts. When $v_0 = 25$ V, D_1 is not conducting, so $i_1 = 0$. The second break point occurs then at $v_0 = 25$ V and $i_2 = 0$. So

$$i_{in} = i_0 = 25/4000 = 6.25 \text{ mA},$$
$$v_{in} = 25 + (6.25 \times 10^{-3} \times 2000) = 37.5 \text{ V}$$

Diode D_3: Assume D_3 is not conducting. Then v_0 is provided by the 10 V source in series with D_1 and $v_0 = 10(4/5) = 8$ V. For this value of v_0, D_2 is not conducting so $i_2 = 0$. If v_{in} exceeds 8 V, D_3 conducts, but if v_{in} is less than 8 V, D_3 cannot conduct. Therefore, the third break point is at $v_{in} = 8$ V, $i_{in} = 0$.

We now must determine what happens above the second break point, that is, when v_{in} exceeds 37.5 V. For this condition, D_1 is not conducting and D_2 conducts. We then have the circuit of fig. 7.27(*b*). By nodal analysis

$$\frac{v_0 - v_{in}}{2} + \frac{v_0}{4} + \frac{v_0 - 25}{2} = 0$$

which reduces to

$$v_0 = (0.4v_{in} + 10)$$

Then,

$$i_{in} = \frac{v_{in} - v_0}{2} = \frac{v_{in} - (0.4v_{in} + 10)}{2} = (0.3v_{in} - 5) \text{ mA}$$

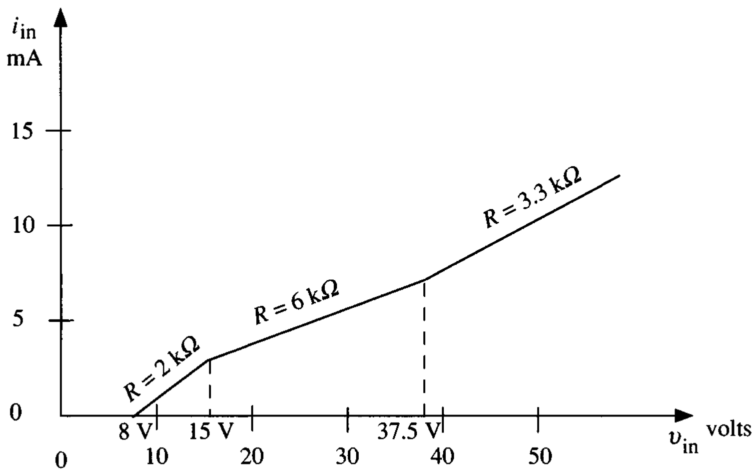
Therefore, for $v_{in} > 37.5$ V, the curve v_0 v. v_{in} has slope 0.4. The curve i_{in} v. v_{in} has slope 0.3×10^{-3}, corresponding to a resistance of 3.3. kΩ.

We now have information from which we may draw the two required characteristics. They are shown in fig. 7.28.

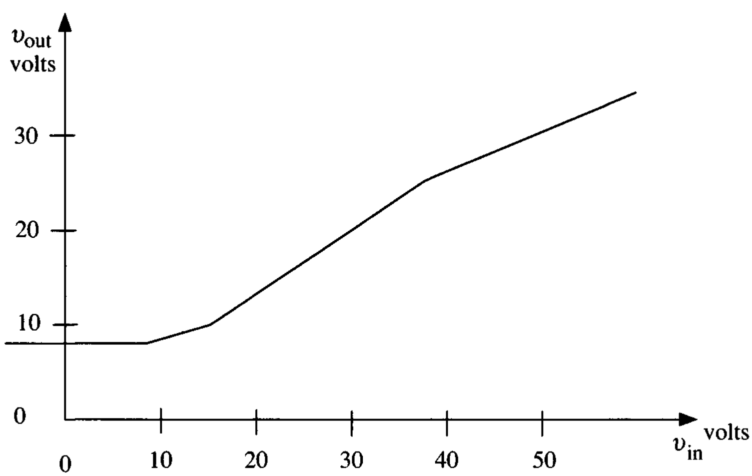
7.4.8 Synthesis of piecewise-linear circuits

The first requirement for synthesizing a piecewise-linear circuit is to decide upon an appropriate linear approximation for the given characteristic of the device or circuit. Then the slopes of the straight line segments may be found. The break points are the intersections of these segments. By referring to the 'building blocks' whose characteristics have already been found (figs. 7.17–7.22) we then choose the proper circuits and the appropriate values of resistors and sources to provide the required break points and slopes. The procedure is illustrated in the following example.

Fig. 7.28. Characteristics of the circuit of fig. 7.27.



(*a*) Input characteristic



(*b*) Transfer characteristic

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7.4.9 Worked example

Given the continuous characteristic of fig. 7.29(*a*), determine an appropriate three-segment piecewise-linear approximation and then design a circuit that has the desired characteristic.

Solution

The dashed line in fig. 7.29(*a*) indicates a three-segment linear approximation that may be taken as an adequate representation of the original characteristic. The coordinates of the break points are (0, 0), (V_a, I_a) and (V_b, I_b). The synthesis is accomplished by starting at the lowest values of v and i and working through successive break points.

In region I of fig. 7.29(*a*) the characteristic is a straight line through the origin. This is represented as in fig. 7.29(*b*) by a diode D_1 in series with a resistor R_1 of value $1/M_1\,\Omega$.

At break-point *a*, the slope increases, indicating that the total circuit resistance has decreased. Therefore, we add in parallel with our original D_1–R_1 combination a series combination of D_2, R_2 and V_2. The appropriate values are:

$$V_2 = V_a \text{ and } R_1R_2/(R_1+R_2) = 1/M_2$$

Since R_1 has already been found, R_2 may be calculated. The circuit now is as shown in fig. 7.29(*c*).

At break-point *b*, the slope decreases. This means that the circuit resistance in region III must be greater than in region II. To accomplish this resistance change we use a parallel diode/resistor/current-source combination as shown in fig. 7.29(*d*). Here, $I_3 = I_b$ and R_3 is chosen so that

$$R_3 + R_1R_2/(R_1+R_2) = 1/M_3$$

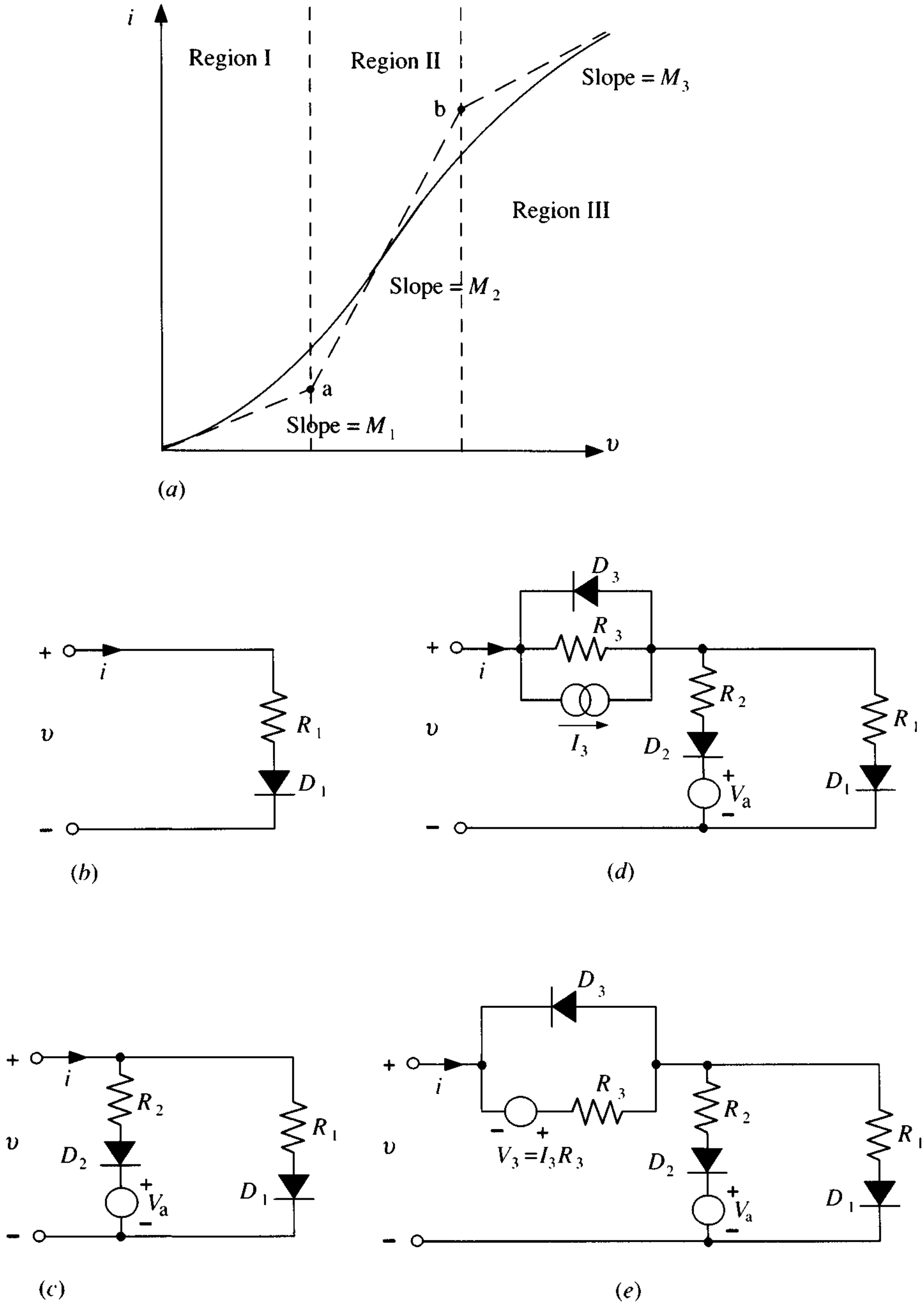
The justification for the parallel resistance combination is as follows. As long as $i < I_3$, there is no current through R_3, and diode D_3 carries current $(i - I_3)$. For $i > I_3$, some current must go through R_3. The resulting voltage drop across R_3 provides reverse bias for D_3 and so the diode does not conduct and its current is zero. Therefore, for $i > I_3$, R_3 is effectively added in series with the parallel combination of R_1 and R_2.

For design of a practical circuit, we would use the Thévenin–Norton transformation to convert the current source I_3 to a voltage source V_3, as shown in fig. 7.29(*e*).

7.5 Analytical methods

The characteristics of many non-linear devices may be expressed by means of analytical functions or approximated by power series. In such cases the circuit of which they form part may be solved either analytically or

Fig. 7.29. Diagrams for worked example: (*a*) original characteristic with three-segment linear approximation; (*b*) and (*c*) steps in circuit synthesis; (*d*) and (*e*) two forms of final circuit.



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numerically. For example, certain non-linear resistors, consisting of crystals of silicon carbide bonded together and fired at high temperature, have a voltage–current relationship of the form

$$i = kv^p \tag{7.16}$$

where k and p depend on the nature and physical state of the material. The index p usually lies in the range 3–5. If a device of this type is connected in a series circuit such as that shown in fig. 7.2, we have by KVL

$$V_0 = iR + v_2$$

But from (7.16) $v_2 = v = (i/k)^{1/p}$ hence,

$$V_0 = iR + \left(\frac{i}{k}\right)^{1/p}$$

This equation may be solved by numerical iteration using the recurrence relationship

$$i_{n+1} = \frac{V_0}{R} - \frac{1}{R}\left(\frac{i_n}{k}\right)^{1/p}$$

The voltage–current characteristic of a diode may also be expressed analytically by the equation

$$i = I_s(\mathrm{e}^{Kv} - 1) \tag{7.17}$$

where I_s and K are constants dependent upon temperature. (A similar equation (the Ebers–Moll equation) relates the collector current of a bipolar transistor to its base–emitter voltage.) By expanding the exponential term in (7.17) as a power series we obtain

$$i = I_s\left[\left(1 + Kv + \frac{(Kv)^2}{2!} + \frac{(Kv)^3}{3!} + \ldots\right) - 1\right]$$

or

$$\mathrm{i} = I_s\left[Kv + \frac{(Kv)^2}{2!} + \frac{(Kv)^3}{3!} + \ldots\right] \tag{7.18}$$

The polynomial formed by taking the first few terms of this expression may then be used to obtain an algebraic or numerical solution for a circuit incorporating the diode. The procedure is similar to that adopted above for the case of the non-linear resistor.

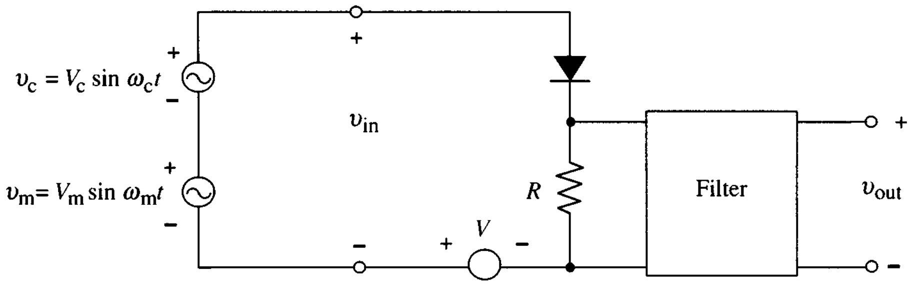
The characterization of a non-linear device by means of a power series provides a powerful tool for the analysis of modulators and frequency changer circuits used extensively in communication networks. We conclude this section with an examination of the simple circuit shown in fig. 7.30(*a*), which may be used both as a modulator and as a frequency changer.

When used as a modulator, two sinusoidal signals v_c and v_m, of widely differing frequencies ($\omega_c \gg \omega_m$), are applied to the input. The output is an *amplitude-modulated* wave of the form shown in fig. 7.30(*b*). The signal at the higher frequency is called the *carrier*, while that at the lower frequency is the *modulating* signal. From fig. 7.30(*b*) it is evident that the instantaneous value of the modulated wave is

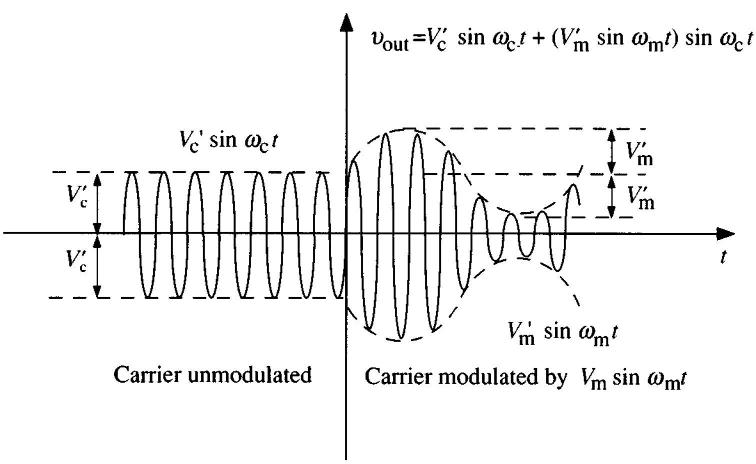
$$v_0 = V_c' \sin\omega_c t + (V_m' \sin\omega_m t)\sin\omega_c t \tag{7.19}$$

We now show that this form of output arises directly as a result of the non-linearity of the diode characteristic.

Fig. 7.30. Modulator (frequency changer) circuit.



(*a*) Diode modulator circuit and filter



(*b*) Output waveform of modulator

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The two input signals, together with a d.c. bias voltage V, which ensures that the diode operates in an appropriate region of its characteristic, are applied to the diode and resistance connected in series. If R is made sufficiently small, substantially the whole of the voltage $(V+v_{\mathrm{m}}+v_{\mathrm{c}})$ appears across the diode, and the current in the circuit may be described by an expression of the form (7.18). Thus, the voltage developed across R may be written:

$$v_R=a(V+v_{\mathrm{m}}+v_{\mathrm{c}})+b(V+v_{\mathrm{m}}+v_{\mathrm{c}})^2+\ldots \tag{7.20}$$

where a and b are constants. For our present purposes the first two terms of (7.20) are of interest; when these are expanded we obtain:

$$v_R= aV+av_{\mathrm{m}}+av_{\mathrm{c}}+bV^2+2bVv_{\mathrm{m}}+2bVv_{\mathrm{c}}+bv_{\mathrm{m}}^2+2bv_{\mathrm{m}}v_{\mathrm{c}}+bv_{\mathrm{c}}^2 \tag{7.21}$$

If

$$v_{\mathrm{m}}=V_{\mathrm{m}}\sin\omega_{\mathrm{m}}t \text{ and } v_{\mathrm{c}}=V_{\mathrm{c}}\sin\omega_{\mathrm{c}}t$$

then

$$v_{\mathrm{m}}^2=V_{\mathrm{m}}^2\sin^2\omega_{\mathrm{m}}t=\frac{V_{\mathrm{m}}^2}{2}(1-\cos 2\omega_{\mathrm{m}}t)$$

and

$$v_{\mathrm{c}}^2=V_{\mathrm{c}}^2\sin^2\omega_{\mathrm{c}}t=\frac{V_{\mathrm{c}}^2}{2}(1-\cos 2\omega_{\mathrm{c}}t)$$

Also

$$\begin{aligned} v_{\mathrm{m}}v_{\mathrm{c}}&=V_{\mathrm{m}}V_{\mathrm{c}}\sin\omega_{\mathrm{m}}t\sin\omega_{\mathrm{c}}t \\ &=\frac{V_{\mathrm{m}}V_{\mathrm{c}}}{2}[\cos(\omega_{\mathrm{c}}-\omega_{\mathrm{m}})t-\cos(\omega_{\mathrm{c}}+\omega_{\mathrm{m}})t] \end{aligned} \tag{7.22}$$

Substitution of these expressions in (7.21) followed by regrouping of terms gives

$$\begin{aligned} v_R=&\,aV+bV^2+\frac{bV_{\mathrm{m}}^2}{2}+\frac{bV_{\mathrm{c}}^2}{2} \\ &+(a+2bV)V_{\mathrm{m}}\sin\omega_{\mathrm{m}}t+(a+2bV)V_{\mathrm{c}}\sin\omega_{\mathrm{c}}t \\ &-\frac{bV_{\mathrm{m}}^2}{2}\cos 2\omega_{\mathrm{m}}t-\frac{bV_{\mathrm{c}}^2}{2}\cos 2\omega_{\mathrm{c}}t \\ &+bV_{\mathrm{m}}V_{\mathrm{c}}\cos(\omega_{\mathrm{c}}-\omega_{\mathrm{m}})t-bV_{\mathrm{m}}V_{\mathrm{c}}\cos(\omega_{\mathrm{c}}+\omega_{\mathrm{m}})t \end{aligned} \tag{7.23}$$

Now, the voltage v_R is applied to the input of a filter which takes the form of a resonant circuit tuned to the carrier frequency ω_c. This tuned circuit is designed to pass frequencies at, or close to, ω_c and reject all others. The first four terms of (7.23) constitute a d.c. component and are rejected, as are the terms in ω_m and $2\omega_m$, which lie well below the pass band. The terms in $2\omega_c$ lie above the pass band and are likewise rejected. The sum and difference frequencies, however, lie close to ω_c (because $\omega_c \gg \omega_m$) and are passed. The output voltage (assuming that amplitudes are unmodified by the filter) is then

$$v_{out} = (a+2bV)V_c\sin\omega_c t + bV_m V_c\cos(\omega_c-\omega_m)t - bV_m V_c\cos(\omega_c+\omega_m)t$$

But by (7.22) this may be written as

$$v_{out} = (a+2bV)V_c\sin\omega_c t + 2bV_m V_c\sin\omega_m t\sin\omega_c t$$

which is an amplitude modulated wave of the form (7.19).

As mentioned above, the circuit of fig. 7.30(a) may also be used as a frequency changer. A common application of frequency changing, is to be found in the superheterodyne receiver. In this case the received radio frequency signal is applied to the circuit together with a signal generated by a 'local' oscillator within the receiver. The frequency of the local oscillator is arranged to be close to that of the received signal so that their *difference* is much lower than the frequency of either. If in fig. 7.30(a), v_c is the received signal and v_m is the local oscillator signal, then $(\omega_c-\omega_m) \ll \omega_c$. The filter is tuned to the frequency $(\omega_c-\omega_m)$, and all terms in (7.23) other than the term in $(\omega_c-\omega_m)$ are rejected. Thus the frequency ω_c of the received signal is changed by the circuit to the lower frequency $(\omega_c-\omega_m)$. It may be shown that if the received signal is amplitude modulated, then the difference frequency is similarly amplitude modulated.

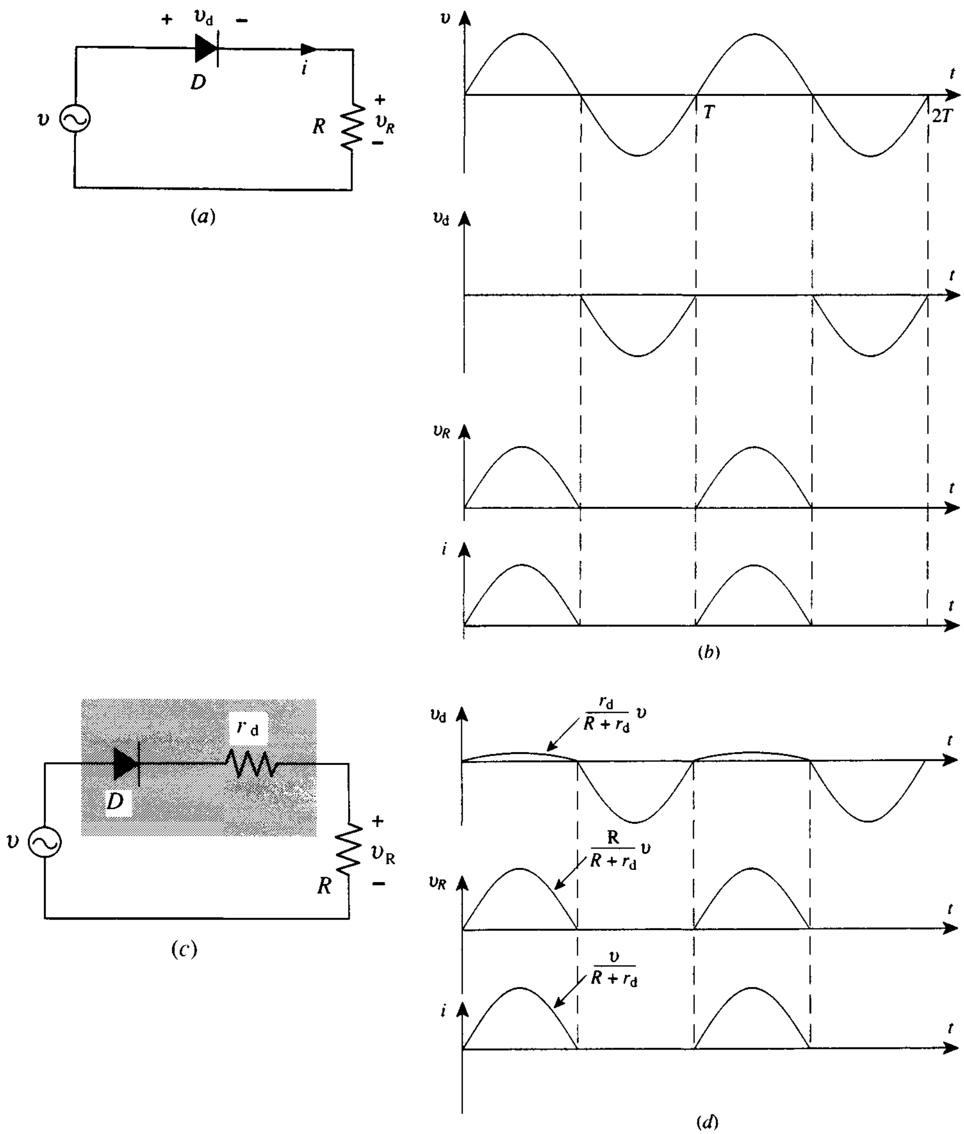
Practical modulator and frequency changing circuits utilize transistors rather than diodes, but the underlying principles of operation remain the same.

7.6 Rectifier circuits

7.6.1 Half-wave rectifier

A diode is often used as a *rectifier* to convert alternating voltage to unidirectional voltage. In fig. 7.31(a), the alternating voltage $v = V_m\sin\omega t$ is applied to the series combination of a diode and a resistor. If the diode is ideal, then during each positive half cycle of v the diode conducts and there is current in the circuit. The input voltage, the voltage across each element and the current are shown in fig. 7.31(b) as functions of time. The single

Fig. 7.31. Half-wave rectifier: (*a*) circuit using ideal diode; (*b*) waveforms for circuit (*a*); (*c*) circuit using real diode modelled by ideal diode and a resistance; (*d*) waveforms for circuit (*c*).



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diode is a *half-wave* rectifier, providing current in the load R that is always in the same direction and that exists for half of each cycle of the input voltage.

We have seen that a real diode may be modelled in several ways depending upon the importance of such factors as forward voltage drop, forward resistance, reverse current and reverse breakdown voltage. In most rectifier applications we need consider only the forward resistance and the reverse breakdown voltage. (If the amplitude V_m of the input voltage is very small, the forward voltage drop may cause a significant decrease in the fraction of a cycle during which the diode conducts, because the diode current will be small until the applied voltage exceeds the diode forward voltage drop.)

The forward resistance of the diode often must be included when calculating the load current. If the diode is to act effectively as a rectifier, its reverse breakdown voltage must exceed by a safe margin the maximum signal voltage that appears periodically as a reverse voltage across the diode. In specifying diodes for rectifier service it is appropriate to state the required *power dissipation* (which may be calculated from the r.m.s. current and the forward resistance) and the peak inverse voltage (PIV) that one expects to apply to the diode.

An appropriate model for the half-wave rectifier is as shown in fig. 7.31(*c*). Voltage and current waveforms are shown in fig. 7.31(*d*). The instantaneous current is

$$\begin{aligned} i &= I_m \sin\omega t \qquad && v > 0 \\ i &= 0 && v < 0 \end{aligned}$$

where

$$I_m = V_m/(r_d + R)$$

The average value of this current is of interest because it is what a d'Arsonval type ammeter would indicate if it were included in the circuit.

$$I_{av} = \frac{1}{T}\int_0^{T/2} I_m \sin\omega t \, dt = \frac{I_m}{\pi} \tag{7.24}$$

To calculate the power dissipated in the diode resistance and the power delivered to the load we must find the effective (r.m.s.) current

$$I = \left[\frac{1}{T}\int_0^{T/2} I_m{}^2 \sin^2\omega t \, dt\right]^{\frac{1}{2}} = \frac{I_m}{2} \tag{7.25}$$

The power rating of the diode then must be at least

$$P = I^2 r_d = \frac{I_m{}^2 r_d}{4} = \frac{V_m{}^2 r_d}{4(R + r_d)^2} \tag{7.26}$$

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A simple half-wave rectifier may be used to charge a storage battery. The fact that the current amplitude is not constant is immaterial; we are interested only in the total charge that passes through the battery. This total charge and the battery voltage together determine the amount of energy supplied and stored as chemical energy in the battery.

7.6.2 Worked example

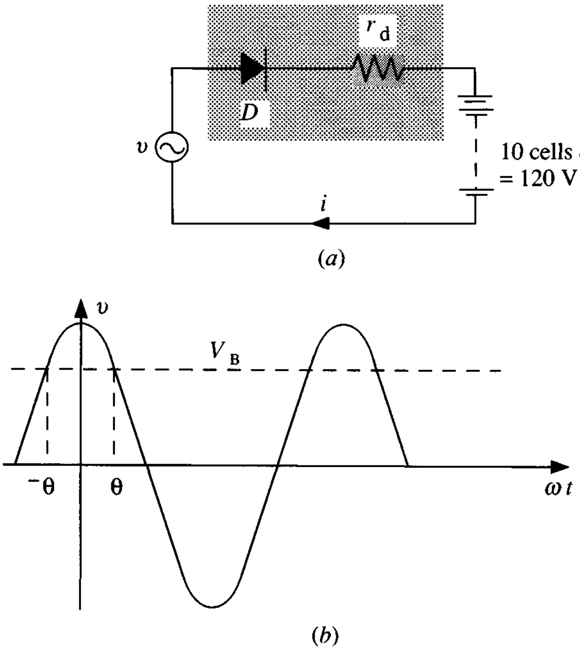
It is necessary to charge ten 12-volt automobile batteries. Design a half-wave circuit using a diode with 1 Ω forward resistance that will provide an average charging current of 1 A. (Assume the battery has negligible internal resistance.)

Solution: The batteries must be connected in series in order to ensure that all have the same charging current. We must determine:

(1) the amplitude of the supply voltage;
(2) the maximum current;
(3) the power dissipated in the diode;
(4) the peak inverse voltage (PIV) rating of the diode.

The circuit is shown in fig. 7.32(*a*). Calculations are simplified if we use $v = V_m \cos\omega t$. We saw that for a resistive load there is current during the

Fig. 7.32. Diagrams for worked example: (*a*) circuit; (*b*) definition of conduction angle.



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whole positive half cycle. Now, however, the bank of batteries provides a reverse bias of 120 volts and there will be current only when $v > V_B = 120$ V. So

$$i = \frac{1}{r_d}(V_m \cos\omega t - V_B) \tag{7.27}$$

It is convenient to define a *conduction angle* 2θ that represents the fraction of each positive half cycle during which the diode conducts; this is shown in fig. 7.32(*b*). We use the given information, $I_{av} = 1$ A, to calculate V_m. From (7.27), using $r_d = 1\,\Omega$,

$$I_{av} = \frac{1}{2\pi}\int_{-\theta}^{\theta}(V_m \cos\omega t - 120)\mathrm{d}\omega t$$

$$= \frac{1}{\pi}(V_m \sin\theta - 120\theta) = 1\ \mathrm{A} \tag{7.28}$$

Also from (7.27) the equation for the angle θ is

$$V_m \cos\theta - V_B = 0$$

or,

$$V_m = \frac{V_B}{\cos\theta}$$

Hence, (7.28) becomes

$$\frac{120}{\pi}(\tan\theta - \theta) = 1$$

To find θ we use the approximation, valid for small angles,

$$\tan\theta = \theta + (1/3)\theta^3$$

These two equations give,

$$\theta = 0.43 \text{ radians} = 24.6°$$

Then, $\cos\theta = 0.91$ and $V_m = 120/0.91 = 132$ V.
The maximum current is $I_m = (132 - 120)/1 = 12$ A.

To find the power dissipated in the diode we must find the effective current.

$$I = \left[\frac{1}{2\pi}\int_{-\theta}^{\theta}\left(\frac{V_m \cos\omega t - V_B}{r_d}\right)^2 \mathrm{d}\omega t\right]^{\frac{1}{2}}$$

This integral may be evaluated using the approximation $\cos\phi = 1 - \frac{1}{2}\phi^2$, and the relation $V_B = V_m \cos\theta$. Then

$$I = \left[\frac{V_m{}^2}{2\pi r_d{}^2} \int_{-\theta}^{\theta} \left(\frac{\theta^2}{2} - \frac{(\omega t)^2}{2} \right)^2 d\omega t \right]^{\frac{1}{2}}$$

When the integration is performed and the limits substituted, the result is

$$I = \left[\left(\frac{2}{15\pi} \right) \left(\frac{V_m \theta^5}{r_d{}^2} \right) \right]^{\frac{1}{2}}$$

The power dissipated in the diode is $I^2 r_d$, so,

$$P_{\text{diode}} = (2/15\pi)(132)^2(0.43)^5 = 10.87\,\text{W}$$

The maximum reverse voltage across the diode is $V_m + V_B = 252\,\text{V}$. Therefore, the PIV rating of the diode must be greater than 252 V.

7.6.3 Full-wave rectifier

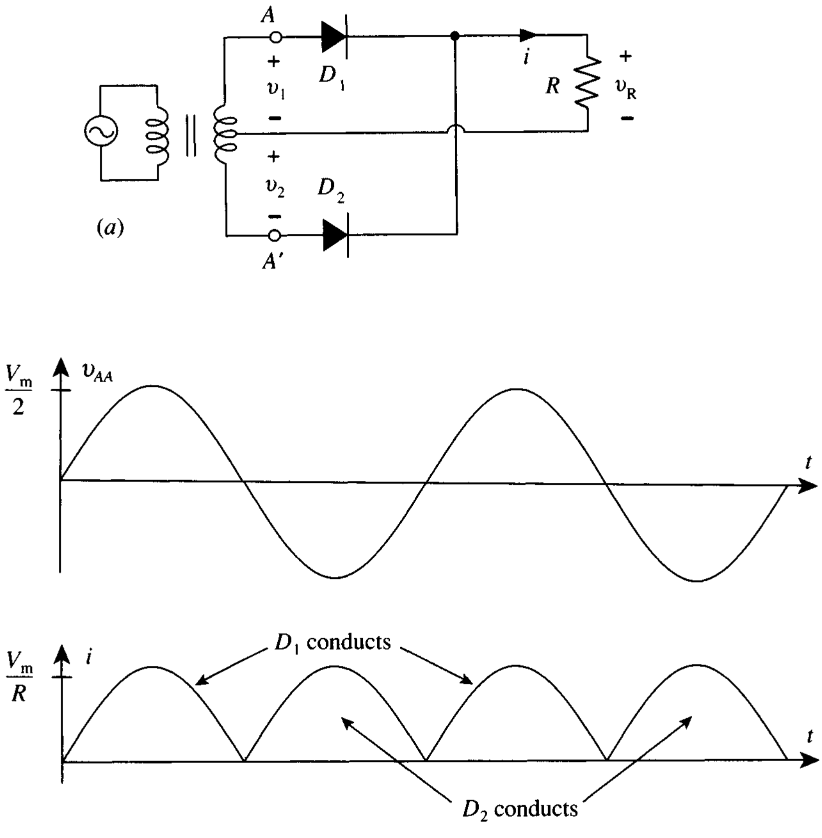
There are a few applications, such as the battery charger of the previous example, where the pulsating unidirectional current supplied by the half-wave rectifier is satisfactory. We shall see in a later section (7.9) how filters may be used to modify the output waveform of the half-wave rectifier and make the voltage more nearly constant. Filtering of the output voltage to get a constant value is easier if the output voltage does not remain zero for half of each input cycle. A *full-wave* rectifier utilizes both the positive and the negative halves of the alternating input voltage.

There are two common full-wave rectifier circuits. The first uses two diodes and requires a transformer with a centre-tap (see fig. 7.33(*a*)). During half of the a.c. cycle, v_1 and v_2 are positive. Then diode D_1 is forward biased and conducts while D_2 is reverse biased and is off. During the other half cycle, D_2 conducts and D_1 is off. Regardless of which diode is conducting, the voltage drop across R is always of the polarity shown in fig. 7.33(*a*). The total secondary transformer voltage and the current in the load are shown in fig. 7.33(*b*). In this circuit only one diode is conducting at any instant, so the resistance of only one diode must be considered in calculating the current. The peak inverse voltage is V_m.

The bridge rectifier shown in fig. 7.34 gets its name from the fact that the arrangement of circuit elements resembles the bridge used for measuring resistance or impedance. It requires four diodes but does not need a centre-tapped transformer. In fig. 7.34, D_1 and D_3 conduct during one half cycle while D_2 and D_4 conduct during the other half cycle.

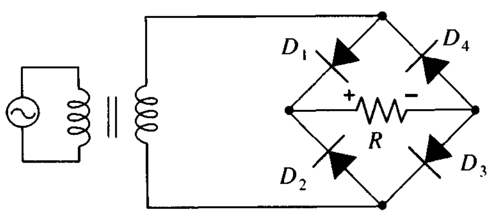
The output voltage waveform of the bridge circuit is identical with that of the full-wave rectifier. An advantage of the bridge circuit is that the inverse

Fig. 7.33. Full-wave rectifier: (*a*) circuit; (*b*) waveforms.



(*b*)

Fig. 7.34. Bridge rectifier.



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voltage is across two diodes in series so the PIV rating of each diode is half that required for the standard full-wave arrangement. The bridge circuit can be connected directly across the power line. Such a connection can be dangerous, however, since neither side of the load can then be grounded (because one side of the power line normally is grounded). Because the bridge circuit uses two diodes in series, an additional voltage drop appears in the circuit. This usually is not significant. The effective value of the load current is the same as has already been calculated for sinusoidal alternating current, that is, $I = I_m/\sqrt{2}$. The average value of the full-rectified current wave is

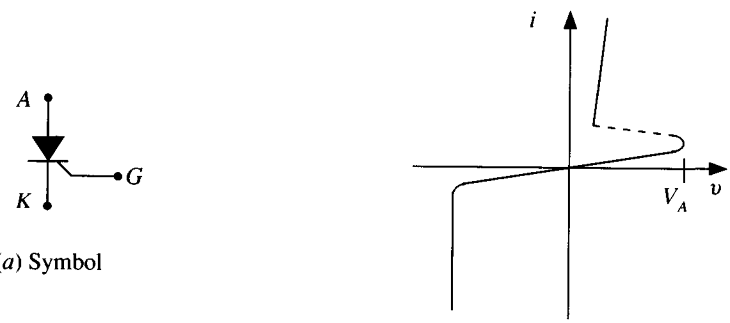
$$I_{av} = \frac{1}{2\pi}\int_{-\pi}^{\pi} I_m \cos\omega t \, d\omega t = \frac{2I_m}{\pi}$$

7.7 Thyristor circuits

With either a half-wave or a full-wave rectifier one may vary the average current supplied to a given load by (1) controlling the amplitude of the alternating voltage supply or (2) including a variable resistor in series with the load. The first method requires a variable transformer and the second is wasteful of energy. Another method of controlling the load current employs the controlled rectifier. The controlled rectifier is usually fabricated of silicon and so is often called, particularly in the U.S.A., a *silicon controlled rectifier*, abbreviated SCR. In the U.K. the term *thyristor* is more frequently used.

The two-terminal diode conducts whenever the anode is positive with respect to the cathode. The thyristor is a diode in which a voltage applied between a third terminal (the *gate*) and the cathode can affect the forward conducting characteristic (fig. 7.35(*a*)). The characteristic between anode

Fig. 7.35. Characteristic for a thyristor.



(*a*) Symbol

(*b*) Characteristic

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and cathode of the SCR when the gate–cathode voltage is negative has the form shown in fig. 7.35(*b*). Unless the anode–cathode voltage exceeds V_A the diode current is insignificant. The dashed line represents an unstable region. When the anode–cathode voltage exceeds V_A, the diode suddenly becomes fully conducting, the voltage drops sharply, and the current increases, being limited by whatever external resistance there is in series with the diode. The device now behaves like an ordinary two-terminal diode. Current continues until the anode–cathode voltage is reduced to zero.

For most practical applications, however, the thyristor is chosen so that the anode–cathode voltage will not exceed V_A. The diode is instead made to conduct by driving the gate positive with respect to the cathode by an amount sufficient to cause a *trigger current* of a few milliamps to flow in the gate circuit. This trigger current initiates forward conduction in the diode. In contrast with the turn-on process, the traditional thyristor can be turned off only if there is a *negative* gate current comparable in magnitude to the forward diode current. So it has been customary to say that, practically, once the diode conducts the gate loses control and conduction ceases only when the anode–cathode voltage drops to zero. Recently, however, there has appeared a controlled diode which can be turned off by a negative gate current of the order of 1% or less of the diode current. Such a device increases the possible applications of the thyristor.

Fig. 7.36(*a*) shows a thyristor circuit in which the gate is used to control the average current in the load of a half-wave rectifier. As shown there, the gate is connected through resistor R to the same side of the power line as the anode. During the negative half of the input voltage cycle the diode D_2 in the gate circuit is reverse biased and cannot conduct. As the input voltage enters its positive half cycle, D_2 conducts, the gate–cathode voltage becomes positive and the gate current increases. For a particular value of R the input voltage must reach a specific amplitude in order to furnish the required trigger current that initiates conduction in the diode. The larger R, the later in the positive half cycle conduction begins. It is apparent that if conduction has not occurred by the time the input voltage reaches its maximum value, conduction cannot occur at all. Thus, by varying R we may reduce the average load current smoothly from its maximum value to one-half the maximum value, but no lower. The circuit of fig. 7.36 is said to provide a 'retard angle' that can lie between zero and 90 degrees. Fig. 7.36(*b*) shows the currents and voltages when R is chosen for an angle of 45°.

Smooth control of the load current over almost the full range from zero to its maximum value may be achieved by *phase control* of the gate voltage.

Fig. 7.37(*a*) shows a half-wave rectifier using a thyristor. Now the gate voltage comes from the power line through an *RC* phase-shift circuit. When the thyristor is not conducting, the voltage $V_{G'K}$ depends upon *R* and *C*:

$$V_{G'K}=\frac{1/\mathrm{j}\omega C}{1/\mathrm{j}\omega C+R}V_{\mathrm{in}}=\frac{V_{\mathrm{in}}}{1+\mathrm{j}\omega CR}=\frac{V_{\mathrm{in}}}{\sqrt{[1+(\omega CR)^2]}}\underline{/\theta}$$

where $\theta=-\tan^{-1}(R\omega C)$. Thus, $V_{G'K}$ lags V_{in} and there can be no gate current until $V_{G'K}$ becomes positive. The maximum angle of lag (90°) is approached as (ωCR) becomes much greater than 1.0.

Now *R* both adjusts the phase of $V_{G'K}$ and limits the gate current. Fig. 7.37(*b*) shows the case where *R* is set for $\theta=-45°$. There is an additional delay in onset of thyristor conduction as the gate current rises to the 'trigger' level. Even after *R* has reached a value such that $\theta\approx-90°$, further delay in 'firing' is achieved by further increase in *R*. In a practical case, *R* may consist of a small fixed resistor in series with a variable resistor, so that the gate current is always held to a safe value.

Fig. 7.36. Gate current control of a thyristor.

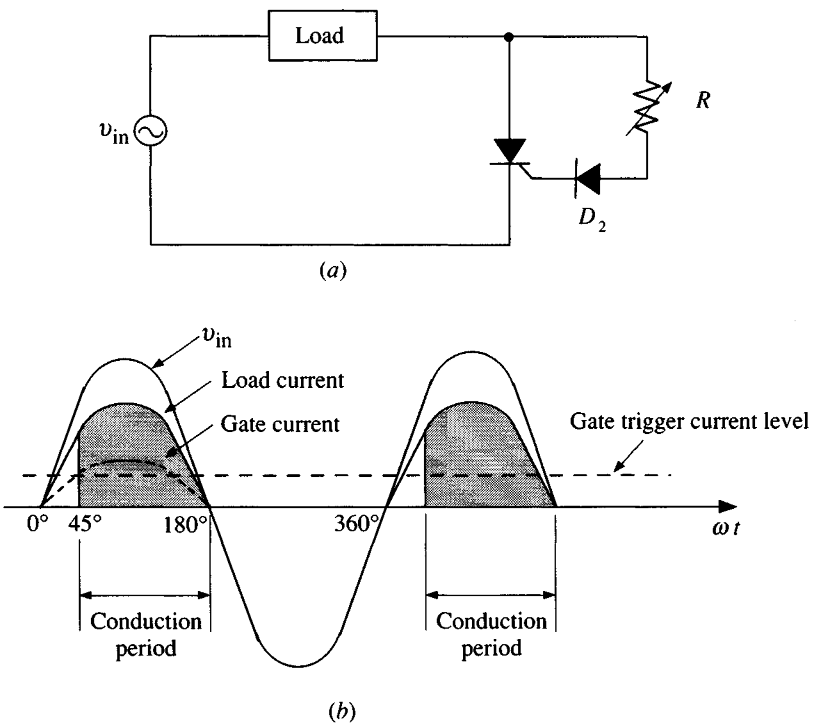

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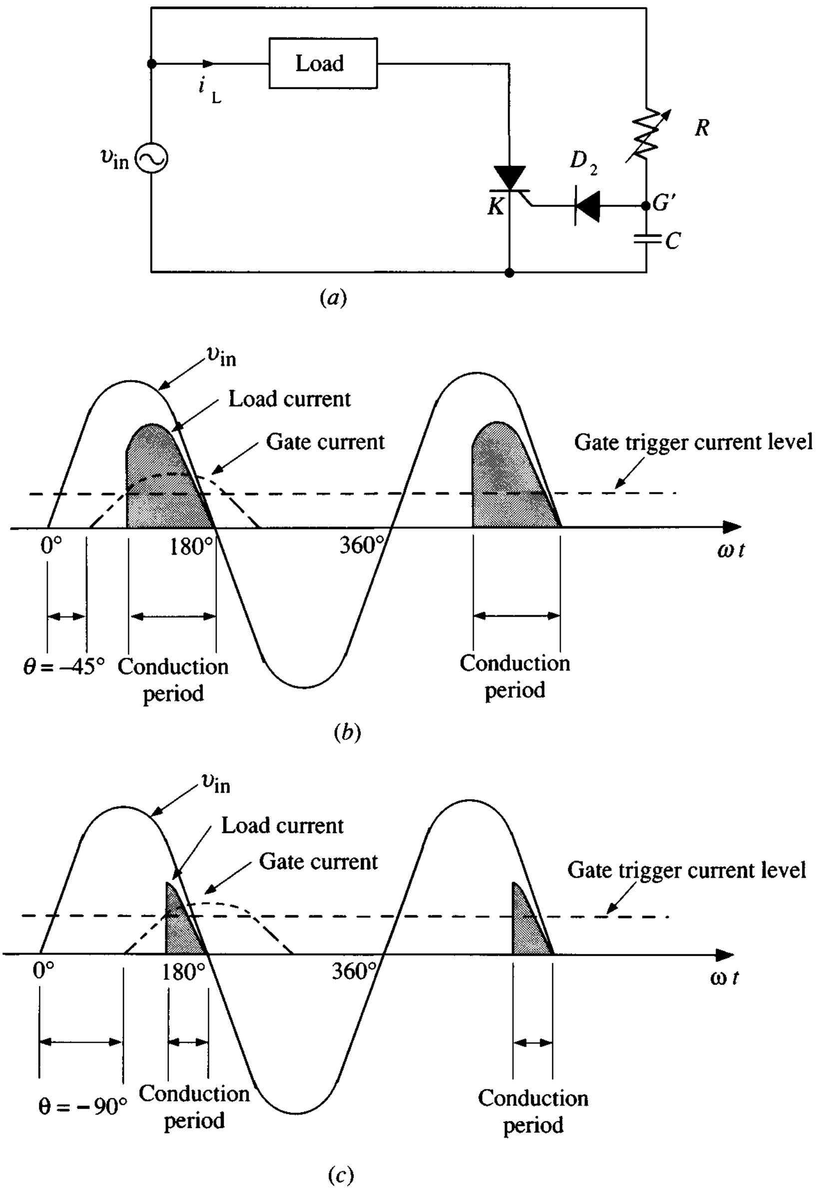
Fig. 7.37. Phase control of thyristor: (*a*) circuit; (*b*) waveform for 45° phase angle; (*c*) waveform for 90° phase angle.



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In addition to providing control of average rectified current in a load, the circuit of fig. 7.37(*a*) finds application in dimming circuits for incandescent lamps and in the control of the speed of fractional horsepower motors (those used in hand-held electric drills, for example). Wider range of control in these applications is achieved by use of the *triac*, a three terminal device that exhibits identical bidirectional conduction and control characteristics. Its characteristic in either direction is the same as that of the thyristor. Fig. 7.38 shows the triac connected in a full-wave control circuit. The drive for the common gate is obtained from a phase shift circuit as in fig. 7.37(*a*). Note that diode D_2 in fig. 7.37(*a*) has been replaced by resistor R_2 because now the gate must carry current in both directions.

Because it is a bidirectional device, the triac does *not* function as a full-wave rectifier. However, this may be achieved by using two thyristors in place of the conventional diodes of fig. 7.33(*a*). Such a circuit is shown in fig. 7.39 where each gate is supplied from a separate phase-shift circuit. The two phase-shift resistors may be identical and may be mounted on the same shaft so that a single knob controls both. Usually it is desirable to have the two diodes conduct for equal fractions of the positive and negative half cycles.

Fig. 7.38. Triac with phase control.

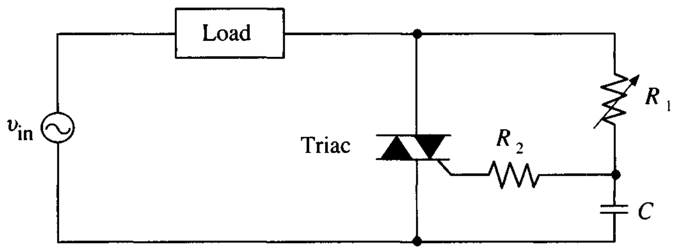
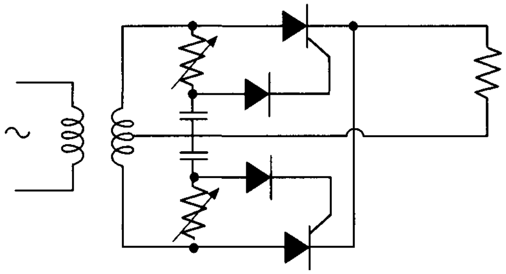


Fig. 7.39. Full-wave rectifier using phase-controlled SCRs.



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7.8 Fourier analysis of periodic waves

7.8.1 Fourier expansion

Periodic non-sinusoidal waveforms are encountered in many branches of science and engineering. The rectifier circuits that have been discussed in the preceding sections, for example, give output voltages that are periodic but non-sinusoidal. The techniques of a.c. circuit analysis, introduced in chapter 3, apply only to sinusoidal waveforms and we cannot employ them directly to find the response of a circuit to a non-sinusoidal waveform. We can, however, transform the problem so that such techniques are applicable.

By means of Fourier analysis a periodic, non-sinusoidal function may be expressed as a series containing a constant term together with a number of sinusoidal terms. The response to each component term of the series may be found using standard techniques. Provided the circuit is linear, the response to the original function may then be obtained from the superposition of the separate responses. This approach to the analysis of circuits is of great power and generality.

As an introduction to Fourier analysis consider fig. 7.40(*a*), which shows a constant and several sinusoids whose frequencies are integral multiples of the lowest frequency shown. In fig. 7.40(*b*) we have plotted the algebraic sum of the functions of time shown individually in fig. 7.40(*a*). The sum is a periodic but non-sinusoidal wave whose period is the period of the lowest frequency in fig. 7.40(*a*). The waveform of fig. 7.40(*b*) may be written as the sum of its components

$$f(\omega t)=f_1+f_2+f_3+f_4=1+3\sin\omega t+1\sin 2\omega t-\tfrac{1}{2}\sin 3\omega t \qquad (7.29)$$

This expression illustrates the following general statement: within certain limits of finiteness and continuity that always are met in practice, periodic functions may be expressed as infinite series of the form:

$$\begin{aligned} f(\omega t)=\frac{a_0}{2}&+(a_1\cos\omega t+b_1\sin\omega t)+(a_2\cos 2\omega t+b_2\sin 2\omega t)+\dots \\ &+(a_n\cos n\omega t+b_n\sin n\omega t)+\dots \end{aligned} \qquad (7.30)$$

The series in (7.30) is the *Fourier series*, or the *Fourier expansion*, for $f(\omega t)$, and the a_n and b_n are the *Fourier coefficients*.

Equation (7.30) may also be written more compactly as

$$f(\omega t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t) \qquad (7.31)$$

or in the alternative form

$$f(\omega t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} K_n\cos(n\omega t+\phi_n) \tag{7.32}$$

where

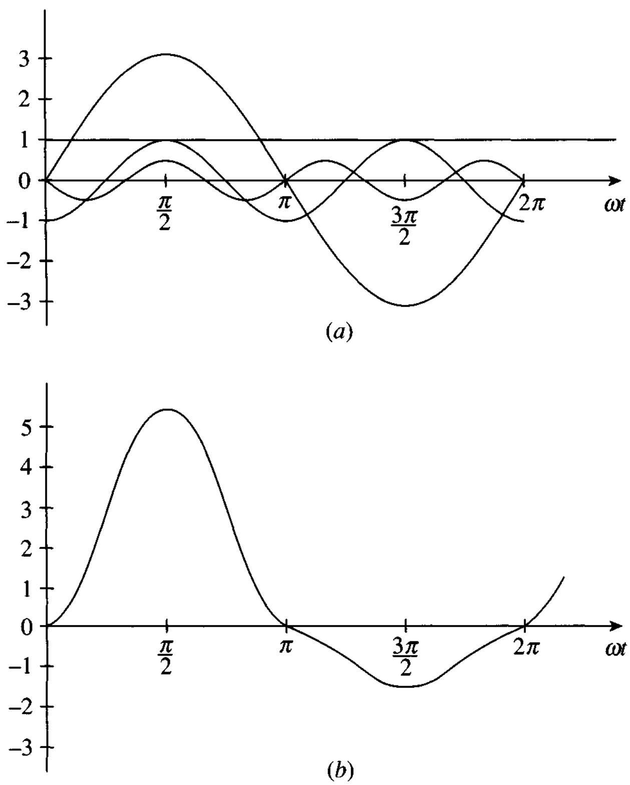
$$K_n=\sqrt{(a_n^{\,2}+b_n^{\,2})} \text{ and } \phi_n=\tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

Since the average value of each sine or cosine term in the above equations is zero, it follows that the term $a_0/2$ must be the average value of $f(\omega t)$. The term $K_1\cos(\omega t+\phi_1)$ in (7.32) is called the *fundamental* or *first harmonic* and has the same period as the original function. The nth term in the summation is the nth harmonic; its frequency is n times the fundamental frequency.

Interpreted as a periodic function of voltage, (7.32) may be written as

$$v(\omega t)=V_0+\hat{V}_1\cos(\omega t+\phi_1)+\hat{V}_2\cos(2\omega t+\phi_2)+\ldots$$
$$+\hat{V}_n\cos(n\omega t+\phi_n)+\ldots$$

Fig. 7.40. Addition of sinusoids.



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In this expression $V_0(=a_0/2)$ is the average value or *d.c. component* of the waveform; $\hat{V}_n(=K_n)$ is the *amplitude* of the nth harmonic ($\hat{V}_n = V_n\sqrt{2}$, where V_n is the r.m.s. magnitude), and ϕ_n is the phase of the nth harmonic.*

The Fourier expansions represented by (7.30) and (7.32) are infinite series. In many cases of practical interest in electrical engineering the amplitudes of the terms in the series decrease rapidly as the order of the harmonics increases, and often the first three or four terms will provide a satisfactory representation of the original function.

The process of Fourier analysis consists essentially in finding the coefficients a_n and b_n. This process is somewhat simplified if we express the variable of the given function in terms of angle rather than time. In the following theory we shall therefore, where appropriate, put $\omega t = \theta$. In this case (7.31) becomes

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{7.33}$$

To obtain an expression for a_n, let the function $f(\theta)$ be defined over one complete period, conveniently over the range $-\pi < \theta < \pi$. Multiply both sides of (7.33) by $\cos m\theta \, d\theta$ (where m is an integer not equal to zero) and integrate between the limits $-\pi$ to π. The resulting expression will contain, on the right-hand side, integrals of the form:

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \pi \qquad \text{for } m = n$$
$$= 0 \qquad \text{for } m \neq n$$

and

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta = 0 \qquad \text{for all } m \text{ and } n$$

(The values of these integrals for the given m and n may be confirmed by simple integration.) We see that this procedure eliminates all terms except the one containing a_n, hence,

$$\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta = a_n \pi$$

or

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \tag{7.34}$$

* The notation $\hat{V}$ (rather than V_m) to denote amplitude avoids the confusion of subscripts in Fourier analysis.

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We have previously noted that $a_0/2$ is the average value of $f(\theta)$, therefore,

$$\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta)$$

or

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

From this we see that a_0 also can be found from (7.34) for the special case $n=0$. (Note that this result is obtained because we have expressed the first term of the series as $a_0/2$, not simply as a_0.)

A similar procedure allows us to find b_n: multiply (7.33) by $\sin m\theta\,d\theta$ and integrate from $-\pi$ to π. Again all terms on the right-hand side of the resulting expression vanish except that containing b_n. The result is:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \tag{7.35}$$

For purposes of evaluating a_n and b_n, any point on the periodic function $f(\theta)$ may be taken as the origin of time or angle. The amplitudes of the harmonics (represented by $K_n=\sqrt{(b_n{}^2+a_n{}^2)}$) depend upon the form of the original function and are independent of the choice of origin. The phase angle ϕ_n, however, will depend upon the location of the origin. The above statements are equivalent to saying that it is sometimes possible to choose the origin so that either all the a_n or all the b_n in (7.31) are identically zero.

Equations (7.34) and (7.35) are useful only when: (1) we have an explicit expression for the original function $f(\theta)$, and (2) we can perform the integrations indicated by the equations. These conditions are met for a wide variety of waveforms encountered in electrical engineering practice.

7.8.2 Worked example

Derive the Fourier expansion for the square wave (fig. 7.41(*a*)) defined by

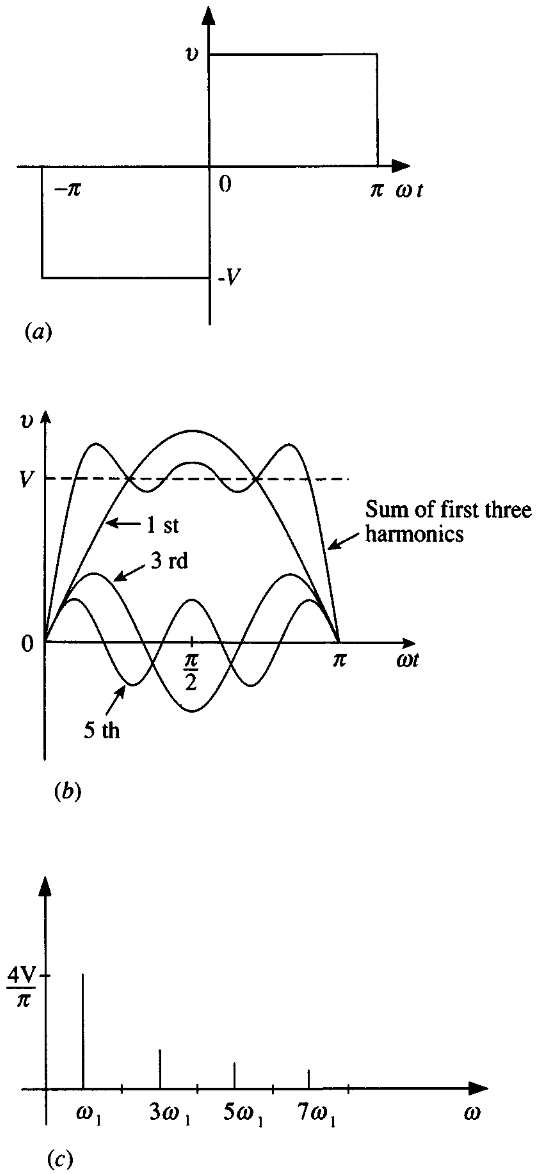
$$\begin{aligned} v(\omega t) &= -V \qquad -\pi<\omega t<0 \\ &= +V \qquad 0<\omega t<\pi \end{aligned}$$

Solution

Let $\omega t=\theta$, then, by (7.34),

$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}(-V)\cos n\theta\,d\theta+\frac{1}{\pi}\int_{0}^{\pi}(+V)\cos n\theta\,d\theta$$

Fig. 7.41. Diagrams for worked example: (*a*) square wave; (*b*) Fourier components; (*c*) line frequency spectrum.



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$$=\frac{V}{n\pi}[-\sin n\theta]_{-\pi}^{0}+\frac{V}{n\pi}[\sin\theta]_{0}^{\pi}$$

$$=0$$

and, by (7.35)

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}(-V)\sin n\theta\,d\theta+\frac{1}{\pi}\int_{0}^{\pi}(+V)\sin n\theta\,d\theta$$

$$=\frac{V}{n\pi}[\cos n\theta]_{-\pi}^{0}+\frac{V}{n\pi}[-\cos n\theta]_{0}^{\pi}$$

$$=\frac{2V}{n\pi}(1-\cos n\pi)$$

$$b_n=\begin{cases}0 & n \text{ even}\\ \dfrac{4V}{n\pi} & n \text{ odd}\end{cases}$$

Hence, using (7.33)

$$v(\theta)=\frac{4V}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\theta \qquad n \text{ odd}$$

In expanded form, the representation of the square wave is (with $\theta=\omega t$)

$$v(\omega t)=\frac{4V}{\pi}[\sin\omega t+\tfrac{1}{3}\sin 3\omega t+\tfrac{1}{5}\sin 5\omega t+\ldots] \tag{7.36}$$

We observe that there are no even harmonics. It is true in general that there are no even harmonics when the function has identical positive and negative parts, that is, when $f(\omega t)=-f(\omega t+\tfrac{1}{2}\omega T)$.

Fig. 7.41(*b*) shows a partial sum of the first three terms of the expansion (7.36); the resulting wave is already a good approximation to the original function. Increasing the number of terms improves the approximation; in particular, the slope of the transitions between positive and negative values is increased.* But it must be remembered that real pulse waveforms, for which the square wave is an idealized representation, possess finite transition times and it is, therefore, unnecessary to include more terms in

* In the case of the square wave an infinite number of terms will not produce a perfect representation. A detailed mathematical treatment shows that the infinite Fourier series (7.36) is not uniformly convergent and that there is appreciable overshoot (about 18%) at the discontinuities between positive and negative values. This result is known as the *Gibbs phenomenon* (see reference 14).

the Fourier representation than is warranted by the transition times occurring in the real waveforms under consideration.

An informative way of displaying graphically the relative amplitudes of the harmonics in a waveform is by means of the *line frequency spectrum.* This is shown for the first few terms of the square wave in fig. 7.41(c).

7.8.3 Odd and even functions

It is often possible to choose the origin so that the wave being analyzed is symmetrical about $\omega t = \theta = 0$. If the symmetry is such that $f(\theta)$ is an *even* function (that is, $f(\theta) = f(-\theta)$), all the b_n are zero. Furthermore, integration from $-\pi$ to 0 gives identical results to integration from 0 to π. To find a_n in this case, therefore, it is convenient to change the lower limit in (7.34) to zero and double the result. When this is done, we obtain *for an even function*:

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta \, d\theta \tag{7.37}$$

and

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \tag{7.38}$$

If the wave is an odd function ($f(\theta) = -f(-\theta)$), then all the a_n are zero and, again, similar remarks apply concerning the interval of integration. In this case we obtain *for an odd function*:

$$b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin n\theta \, d\theta \tag{7.39}$$

and

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta \tag{7.40}$$

The square wave of section 7.8.2 was an example of an odd function resulting in an expansion containing only sine terms. It should be noted that if $f(\theta)$ is even, the amplitude of the nth harmonic is $K_n = a_n$; if $f(\theta)$ is odd, then $K_n = b_n$.

7.8.4 Worked example

Find the Fourier cosine series and Fourier sine series for a triangular waveform of current having peak values $\pm I$ and period T.
Solution: The function of current $i(\omega t)$ is shown in figs. 7.42(a) and (b) where $\omega = 2\pi/T$. In fig. 7.42(a) the origin has been chosen so that the function is

even; thus, all b_n in the expansion are zero and we obtain the Fourier cosine series.

Let $\omega t = \theta$, then, in the range $0 < \theta < \pi$, the slope of the function $i(\theta)$ is $2I/\pi$ and the intercept on the vertical axis is $-I$. The function may therefore be described by

$$i(\theta) = \frac{2I}{\pi}\theta - I = I\left(\frac{2\theta}{\pi} - 1\right) \qquad 0 < \omega t < \pi$$

The coefficients a_n are given for an even function by (7.37), which in this case becomes:

$$a_n = \frac{2I}{\pi}\int_0^{\pi}\left(\frac{2\theta}{\pi} - 1\right)\cos n\theta \, d\theta$$

$$= \frac{2I}{\pi}\left[\frac{2}{\pi}\int_0^{\pi}\theta\cos n\theta \, d\theta - \int_0^{\pi}\cos n\theta \, d\theta\right]$$

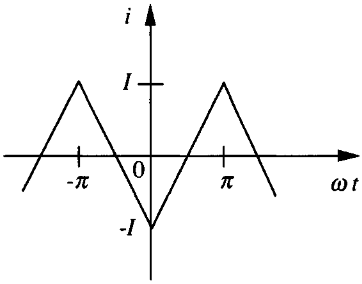
Because of symmetry about the horizontal axis the second integral in this expression is obviously zero. The first integral is readily evaluated using integration by parts (putting $u = \theta$; $dv = \cos n\theta$). This gives

$$a_n = \frac{4I}{\pi^2}\left[\frac{1}{n^2}(\cos n\pi - 1)\right]$$

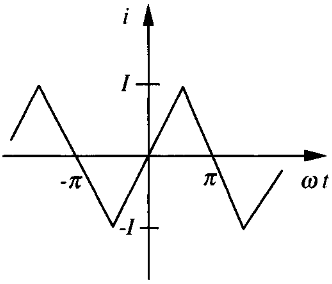
Therefore,

$$a_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8I}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

Fig. 7.42. Diagrams for worked example: triangular wave.



(*a*) Even function



(*b*) Odd function

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So the Fourier cosine series for the triangular wave is (with $\theta = \omega t$)

$$i(\omega t) = -\frac{8I}{\pi^2}\left[\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \ldots\right] \qquad (7.41)$$

To find the Fourier sine series we choose the origin as shown in fig. 7.42(*b*). In the range $0 < \theta < \frac{\pi}{2}$ the slope of the function is $2I/\pi$, while in the range $\frac{\pi}{2} < \theta < \pi$ it is $-2I/\pi$, hence, the current is described by:

$$i(\theta) = \frac{2I}{\pi}\theta \qquad 0 < \theta < \frac{\pi}{2}$$

$$i(\theta) = I - \frac{2I}{\pi}\left(\theta - \frac{\pi}{2}\right) = \frac{2I}{\pi}(\pi - \theta) \qquad \frac{\pi}{2} < \theta < \pi$$

The coefficients b_n are given for an odd function by (7.39):

$$b_n = \frac{2I}{\pi}\left[\frac{2}{\pi}\int_0^{\pi/2} \theta \sin n\theta \, d\theta + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - \theta)\sin n\theta \, d\theta\right]$$

Integration by parts (putting $u = (\pi - \theta)$; $dv = \sin n\theta$, in the second integral) gives

$$b_n = \frac{4I}{\pi^2}\left[\frac{2}{n^2}\sin n\frac{\pi}{2}\right]$$

$$b_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{+8I}{\pi^2 n^2} & n = 1, 5, 9 \ldots \\ \dfrac{-8I}{\pi^2 n^2} & n = 3, 7, 11 \ldots \end{cases}$$

Hence, the Fourier sine series for the triangular wave is

$$i(\omega t) = \frac{8I}{\pi^2}\left[\sin\omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\sin 5\omega t + \ldots\right] \qquad (7.42)$$

Comparing (7.41) and (7.42) we see that in the cosine series all terms are of the same sign whereas in the sine series terms alternate in sign. The reason for this will become apparent if the first three terms of the series for this wave are sketched out for a full period of the fundamental (in a fashion similar to that shown in fig. 7.41(*b*) for the square wave). It will be seen that a shift in the origin of $\pi/2$ (fig. 7.42) changes the terms in the summation from cosine to sine and at the same time changes the signs of the first and fifth harmonics leaving the third harmonic unchanged.

Mathematically (7.42) may be derived from (7.41) using the relationship $\cos n\theta = \sin\left(\frac{\pi}{2} - n\theta\right) = -\sin\left(n\theta - \frac{\pi}{2}\right)$. A phase shift of $\pi/2$ (corresponding to a shift in the origin of $\pi/2$) referred to the fundamental, corresponds to a phase shift of $n\pi/2$ referred to the nth harmonic. Hence the nth term of the cosine series transforms to

$$-\sin\left[n\theta - \frac{\pi}{2} + n\frac{\pi}{2}\right] = -\sin\left[n\theta + (n-1)\frac{\pi}{2}\right]$$

For

$$\begin{array}{ll} n=1 & \cos\theta \rightarrow -\sin\theta \\ n=3 & \cos 3\theta \rightarrow -\sin(3\theta+\pi) = \sin 3\theta \\ n=5 & \cos 5\theta \rightarrow -\sin(5\theta + 2\pi) = -\sin 5\theta \text{ etc.} \end{array}$$

The series for the triangular wave converges more rapidly (as $1/n^2$) than does the series for the square wave. The difference in the rates of convergence is related to the fact that the square wave is discontinuous whereas the triangular wave is continuous but has discontinuous derivatives. In general, the smoother the original wave the more accurately it can be represented by a few terms in the series. The square wave requires higher harmonics to fill in the corners.

Observe that in both examples the constant term is zero. This could have been predicted from the fact that in both cases the positive and negative half cycles are identical and so the average value of the wave is zero.

7.8.5 Fourier expansion for rectifier output

Of particular interest are the Fourier expansions for the output voltages of half-wave and full-wave rectifiers.

(a) *Half-wave rectifier*. For convenience the amplitude of the output waveform (fig. 7.43) is taken to be unity. We choose the origin where the voltage is maximum, which results in an even function so that all the b_n are zero. The a_n are derived from (7.37).

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta \cos n\theta \, d\theta \qquad (7.43)$$

Integration of (7.43) is accomplished by making use of the identity $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$. The result is:

$$a_n = \frac{2}{\pi(1-n^2)}\cos\frac{n\pi}{2} \qquad (7.44)$$

Now

$$\cos\frac{n\pi}{2}=\begin{cases} 0 & n \text{ odd } (n\neq 1) \\ -1 & n=2, 6, 10\ldots \\ 1 & n=4, 8, 12\ldots \end{cases}$$

For $n=1$, (7.44) is indeterminate; therefore, we return to (7.43) which, with $n=1$, becomes

$$a_1=\frac{2}{\pi}\int_{\theta}^{\pi/2}\cos^2\theta\,d\theta$$

This, after integration and substitution of limits, gives $a_1=\frac{1}{2}$. The other coefficients, including a_0, are evaluated using (7.44). The Fourier expansion for the half-wave rectifier output is then (with $\theta=\omega t$)

$$f(\omega t)=\frac{1}{\pi}\left[1+\frac{\pi}{2}\cos\omega t+\frac{2}{3}\cos 2\omega t-\frac{2}{15}\cos 4\omega t+\ldots\right] \tag{7.45}$$

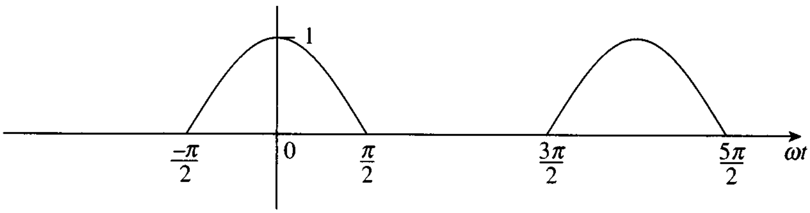
(b) *Full-wave rectifier.* We choose the origin as shown in fig. 7.44. Here again the wave is an even function and so there are no sine terms. From (7.37)

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}\sin\theta\cos(n\theta)\,d\theta \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\sin(1+n)\theta\,d\theta+\int_0^{\pi}\sin(1-n)\theta\,d\theta\right] \\ &= \frac{-1}{\pi(1+n)}[\cos(1+n)\pi-1]+\frac{-1}{\pi(1-n)}[\cos(1-n)\pi-1] \end{aligned}$$

When n is odd, both terms in the brackets are zero. When n is even, each bracket has the value -2. Therefore,

$$a_n=\frac{4}{\pi(1-n^2)}$$

Fig. 7.43. Half-wave rectifier output represented as an even function.



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The Fourier expansion for the full-wave output is then

$$f(\omega t)=\frac{2}{\pi}\left[1-\frac{2}{3}\cos 2\omega t-\frac{2}{15}\cos 4\omega t-\frac{2}{35}\cos 6\omega t\ldots\right] \tag{7.46}$$

7.8.6 Expansion of functions of time

Sometimes it is convenient to specify waveforms as functions of time rather than of angle. In this case the expressions for the Fourier expansion take a slightly different form.

For a periodic function $f(t)=f(t+T)$ we may put $\omega t=2\pi t/T$ in (7.31) to give

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi t}{T}+b_n\sin\frac{2n\pi t}{T}\right) \tag{7.47}$$

Since π, the half-period in angle, corresponds to $T/2$ in time, (7.34) and (7.35) become

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt \tag{7.48}$$

and

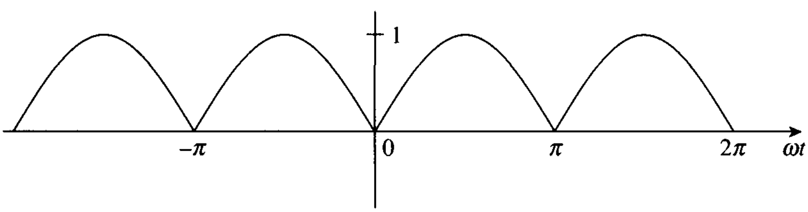
$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt \tag{7.49}$$

These equations allow one to evaluate the Fourier coefficients directly in terms of time; however, they are usually more cumbersome to use than the corresponding equations (7.34) and (7.35) in terms of angle. For this reason it is usually preferable in a practical problem to specify functions in terms of angle.

7.8.7 Complex exponential form of Fourier series

The Fourier series expansion (7.31) or (7.33) may also be written in terms of the complex exponential. This approach to Fourier expansion is of particular importance in the theory of communication networks.

Fig. 7.44. Full-wave rectifier output represented as an even function.



The pair of general terms in (7.33) may be written as

$$a_n \cos n\theta + b_n \sin n\theta = a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} + b_n \frac{e^{jn\theta} - e^{-jn\theta}}{j2}$$

$$= \frac{a_n - jb_n}{2} e^{jn\theta} + \frac{a_n + jb_n}{2} e^{-jn\theta}$$

Now if we define *complex* coefficients

$$c_n = \frac{a_n - jb_n}{2} \text{ and } c_{-n} = \frac{a_n + jb_n}{2} \tag{7.50}$$

then the expansion (7.33) may be written

$$f(\theta) = c_0 + \sum_{n=1}^{\infty} (c_n e^{jn\theta} + c_{-n} e^{-jn\theta})$$

where $c_0 = a_0/2$.

This expression may, by allowing n to have all integral values from $-\infty$ to $+\infty$, including zero, be written in the compact form:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta} \tag{7.51}$$

By combining (7.50) with (7.34) and (7.35) we obtain

$$c_n = \frac{a_n - jb_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)(\cos n\theta - j \sin n\theta) \, d\theta$$

or

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} \, d\theta \tag{7.52}$$

It may be shown that this equation holds for all n, positive, negative and zero.

Although (7.52) provides a direct method for deriving the coefficients c_n, in practice it is usually easier to evaluate a_n and b_n and then to derive c_n from (7.50).

The Fourier series expansion in the form (7.51) may be interpreted in a mathematical rather than a physical sense, as a series containing terms of both positive and negative frequency. The individual harmonic components of the trigonometric Fourier series are, however, composed of pairs of terms from the positive and negative sequences, as will be appreciated if the coefficients c_n are expressed in terms of the amplitudes and phases of the harmonics. In the form (7.32) (with $\omega t = \theta$) the Fourier expansion is

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} K_n \cos(n\theta + \phi_n) \tag{7.32}$$

where:

amplitude of nth harmonic $K_n = \sqrt{(a_n^2 + b_n^2)}$

and

phase of nth harmonic $\phi_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

From (7.50) and (7.32) we have

$$\begin{aligned} c_n &= \frac{a_n - \mathrm{j}b_n}{2} = \tfrac{1}{2}\sqrt{(a_n^2 + b_n^2)}\mathrm{e}^{\mathrm{j}\phi_n} = \frac{K_n}{2}\mathrm{e}^{\mathrm{j}\phi_n} \\ c_{-n} &= \frac{a_n + \mathrm{j}b_n}{2} = \tfrac{1}{2}\sqrt{(a_n^2 + b_n^2)}\mathrm{e}^{-\mathrm{j}\phi_n} = \frac{K_n}{2}\mathrm{e}^{-\mathrm{j}\phi_n} \end{aligned} \tag{7.53}$$

The sum of the two terms corresponding to the nth harmonic in the series is, therefore

$$\begin{aligned} c_n \mathrm{e}^{\mathrm{j}n\theta} + c_{-n}\mathrm{e}^{-\mathrm{j}n\theta} &= \frac{K_n}{2}\left(\mathrm{e}^{\mathrm{j}(n\theta + \phi_n)} + \mathrm{e}^{-\mathrm{j}(n\theta + \phi_n)}\right) \\ &= K_n \cos(n\theta + \phi_n) \end{aligned}$$

Thus, the sum of pairs of negative and positive frequency components contained in the complex exponential series corresponds to one harmonic component in the trigonometric series. We observe also, from (7.53) that the amplitude of each coefficient of the pair of terms is half that of the corresponding coefficient in the trigonometric series, that is,

$$|c_n| = |c_{-n}| = \frac{K_n}{2} \tag{7.54}$$

The magnitude of the phase angle is the same in both complex and trigonometric series.

As an example of the application of the complex form of Fourier series, let us consider again the square wave of amplitude $\pm V$ shown in fig. 7.41 and analyzed in section 7.8.2. The origin was chosen so that the wave was an odd function thus giving $a_n = 0$. The coefficients b_n were found to be

$$b_n = \begin{cases} \dfrac{4V}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Therefore, from (7.50),

$$\left.\begin{aligned} c_n &= \frac{-\mathrm{j}}{2}\frac{4V}{n\pi} = -\mathrm{j}\frac{2V}{n\pi} \\ c_{-n} &= \frac{\mathrm{j}}{2}\frac{4V}{n\pi} = +\mathrm{j}\frac{2V}{n\pi} \end{aligned}\right\} n \text{ odd}$$

The complex Fourier expansion is then, from (7.51),

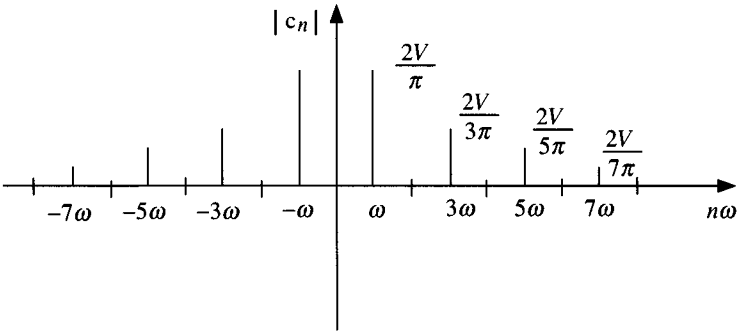
$$\begin{aligned} f(\omega t) &= \sum_{n=-\infty}^{\infty} -\mathrm{j}\frac{2V}{n\pi}\,\mathrm{e}^{\mathrm{j}n\omega t} \\ &= \ldots + \mathrm{j}\frac{2V}{5\pi}\mathrm{e}^{-\mathrm{j}5\omega t} + \mathrm{j}\frac{2V}{3\pi}e^{-\mathrm{j}3\omega t} + \mathrm{j}\frac{2V}{\pi}\mathrm{e}^{-\mathrm{j}\omega t} \\ &\quad -\mathrm{j}\frac{2V}{\pi}\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{j}\frac{2V}{3\pi}\mathrm{e}^{\mathrm{j}3\omega t} - \mathrm{j}\frac{2V}{5\pi}\mathrm{e}^{\mathrm{j}5\omega t} \ldots \end{aligned} \tag{7.55}$$

It is seen that the phases of the complex coefficients c_n and c_{-n} are $-90°$ and $+90°$ respectively. Amplitude and phase spectra are plotted in fig. 7.45. Plots of this type find application in the theory of communication networks. In fig. 7.45(*c*) amplitude and phase information has been combined in a single diagram, the phase change of $+90°$ to $-90°$ being indicated by oppositely directed amplitude components. This form of representation is possible only if the original function is either even or odd, otherwise the c_n coefficients possess both real and imaginary parts and the phase changes continuously throughout the spectrum.

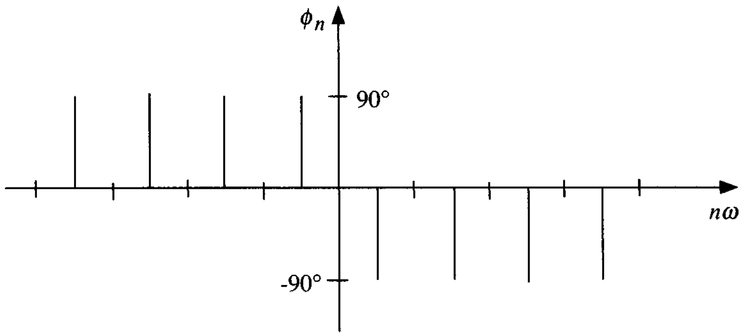
†7.8.8 Expansions for r.m.s. values and power

It is frequently of interest to know the r.m.s. or effective value of a periodic, non-sinusoidal waveform of current or voltage. This may be calculated (from the definition given in section 3.1) using the analytical function describing the waveform. Such calculation yields an r.m.s. value that includes the effect of all of the harmonics contained in the waveform. It may, however, be of interest to know the r.m.s. value of just a portion of the spectrum and for this a series expansion is required. We may also wish to know the power developed at a terminal pair in a circuit where voltage and current are periodic at the same fundamental frequency but which possess different waveforms. This may be found by taking the instantaneous

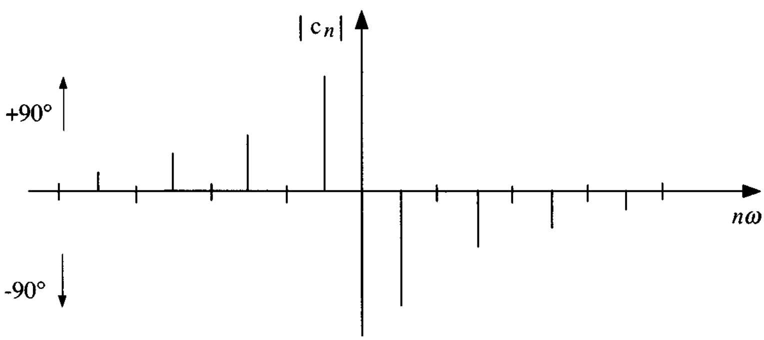
Fig. 7.45. Line spectra for the square wave of fig. 7.41.



(*a*) Amplitude



(*b*) Phase



(*c*) Combined amplitude and phase

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product of the two waveforms and integrating to find the average value of the product over one period of the waveform. But, again, it may be of interest to know the power contributed by one or more harmonics present in either of the waveforms.

We now proceed to establish expressions that will allow us to calculate the r.m.s. values and powers associated with one or more harmonics in non-sinusoidal waveforms.

Consider two functions, $f(\omega t)$ and $g(\omega t)$, represented by their Fourier expansions in the form (7.32):

$$f(\omega t)=\frac{a_{of}}{2}+\sum_{n=1}^{\infty} K_{nf}\cos(n\omega t+\phi_{nf}) \tag{7.56}$$

$$g(\omega t)=\frac{a_{0g}}{2}+\sum_{m=1}^{\infty} K_{mg}\cos(m\omega t+\phi_{mg}) \tag{7.57}$$

In these expressions $a_{0f}/2$ and $a_{0g}/2$ are the average values or d.c. components of the two functions; K_{nf} and K_{mg} are the amplitudes, and ϕ_{nf} and ϕ_{mg} are the phase angles of the nth and mth harmonic components of the two functions respectively.

The average value of the product of the two functions is given by

$$\overline{f(\omega t)g(\omega t)}=\frac{1}{2\pi}\int_0^{2\pi} f(\omega t)g(\omega t)\,\mathrm{d}\omega t \tag{7.58}$$

When we multiply (7.56) by (7.57), the right-hand side of (7.58) is seen to contain:

(a) self-product terms of the same frequency ($m=n$);
(b) cross-product terms of different frequency ($m\neq n$);
(c) terms consisting of products of the d.c. components with harmonics;
(d) the product of the two d.c. components.

Type (a) terms are of the form:

$$\frac{1}{2\pi}\int_0^{2\pi} K_{nf}\cos(n\omega t+\phi_{nf})K_{ng}\cos(n\omega t+\phi_{ng})\,\mathrm{d}\omega t$$

$$=\frac{K_{nf}K_{ng}}{2\pi}\int_0^{2\pi} \tfrac{1}{2}[\cos(2n\omega t+\phi_{nf}+\phi_{ng})+\cos(\phi_{nf}-\phi_{ng})]\,\mathrm{d}\omega t$$

(using the identity $\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$). The first term within square brackets is a cosine function of time whose average value is zero; the second term is a constant. Since the average value of a constant is the constant itself, the average value of the nth self-product term is

$$\frac{K_{nf}K_{ng}}{2}\cos(\phi_{nf}-\phi_{ng}) \tag{7.59}$$

Type (b) terms are of the form:

$$\frac{1}{2\pi}\int_0^{2\pi} K_{nf}\cos(n\omega t+\phi_{nf})K_{mg}\cos(m\omega t+\phi_{mg})\,\mathrm{d}\omega t$$

$$=\frac{K_{nf}K_{mg}}{2\pi}\int_0^{2\pi}\tfrac{1}{2}[\cos\{(n+m)\omega t+\phi_{nf}+\phi_{mg}\}$$

$$+\cos\{(n-m)\omega t+\phi_{nf}-\phi_{mg}\}]\,\mathrm{d}\omega t=0$$

since for $n\neq m\neq 0$, each of the terms within the square brackets is a cosine function of time whose average is zero.

Type (c) terms also give rise to cosine functions of time with zero average. With regard to type (d) terms, clearly, the average value of the product of the two d.c. components is the product itself, namely,

$$\frac{a_{0f}a_{0g}}{4} \tag{7.60}$$

Combining (7.59) and (7.60) we obtain a series representing the average value of the product of two functions:

$$\overline{f(\omega t)g(\omega t)}=\frac{a_{0f}a_{0g}}{4}+\sum_{n=1}^{\infty}\frac{K_{nf}K_{ng}}{2}\cos(\phi_{nf}-\phi_{ng}) \tag{7.61}$$

This expression may be used to find the r.m.s. value of a current $i(\omega t)$ in terms of its harmonic components.

Let $i(\omega t)=f(\omega t)$, then, from (7.56), with $a_{of}/2=I_0$ and $K_{nf}=\hat{I}_n$ (where $\hat{I}_n$ is the *amplitude* of the nth harmonic), the expansion of $i(\omega t)$ may be written as

$$i(\omega t)=I_0+\sum_{n=1}^{\infty}\hat{I}_n\cos(n\omega t+\phi_n)$$

Equation (7.61), with $i(\omega t)=f(\omega t)=g(\omega t)$, then gives:

$$\overline{i(\omega t)^2}=I_0{}^2+\sum_{n=1}^{\infty}\frac{\hat{I}_n{}^2}{2} \tag{7.62}$$

By definition, the r.m.s. value of a current $i(\omega t)$ is

$$i(\omega t)_{\text{r.m.s.}}=\sqrt{[\text{average value of } i(\omega t)^2]}$$

$$=\sqrt{\left[\mathrm{I}_0{}^2+\sum_{n=1}^{\infty}\frac{\hat{I}_n{}^2}{2}\right]} \tag{7.63}$$

Expressing the harmonic components in terms of their r.m.s. values $I_n=\hat{I}_n/\sqrt{2}$, (7.63) becomes

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$$i(\omega t)_{\text{r.m.s.}} = \sqrt{\left[\sum_{n=0}^{\infty} I_n{}^2\right]}$$

or

$$i(\omega t)_{\text{r.m.s.}} = \sqrt{[I_0{}^2 + I_1{}^2 + I_2{}^2 + \ldots]} \tag{7.64}^*$$

A similar expression holds for the r.m.s. value of a voltage function.

Equation (7.61) enables us to find also an expansion for the average power P at a terminal pair in a circuit, given expressions for the instantaneous voltage and current $v(\omega t)$ and $i(\omega t)$.

Using (7.56) and (7.57), voltage and current functions may be expressed as

$$v(\omega t) = V_0 + \sum_{n=1}^{\infty} \hat{V}_n \cos(n\omega t + \phi_{nv})$$

and

$$i(\omega t) = I_0 + \sum_{m=1}^{\infty} \hat{I}_m \cos(m\omega t + \phi_{mi})$$

Then from (7.61)

$$P = \overline{v(\omega t)i(\omega t)} = V_0 I_0 + \sum_{n=1}^{\infty} \frac{\hat{V}_n \hat{I}_n}{2} \cos(\phi_{nv} - \phi_{ni}) \tag{7.65}$$

If we let $\phi_n = \phi_{nv} - \phi_{ni}$ be the phase angle of the nth harmonic of voltage with respect to current, and if we let $V_n = \hat{V}_n/\sqrt{2}$ and $I_n = \hat{I}_n/\sqrt{2}$, then (7.65) becomes

$$P = V_0 I_0 + \sum_{n=1}^{\infty} V_n I_n \cos\phi_n$$

or

$$\begin{aligned} P &= V_0 I_0 = V_1 I_1 \cos\phi_1 + V_2 I_2 \cos\phi_2 + \ldots \\ &= \text{Sum of powers for each harmonic component alone} \end{aligned} \tag{7.66}$$

If voltage and current are pure sinusoids, then there are no harmonics and this expression reduces to $VI\cos\phi$, which is the expression for power obtained by other means in section 4.2.

An important conclusion which may be drawn from the foregoing theory

* Equations (7.64) is a special case of *Parseval's theorem* (see, for example, references 2, 8, 10)

is that only voltages and currents of the same frequency can interact to produce finite average power. In particular, d.c. components do not interact with a.c. components. Products of quantities of different frequency average over a period of time to zero.

7.8.9 Summary of formulae

Basic forms:

$$f(\omega t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t) \tag{7.31}$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}K_n\cos(n\omega t+\phi_n) \tag{7.32}$$

where

$$K_n=\sqrt{(a_n{}^2+b_n{}^2)} \text{ and } \phi_n=\tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

and

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\omega t)\cos n\omega t\,\mathrm{d}\omega t \tag{7.34}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\omega t)\sin n\omega t\,\mathrm{d}\omega t \tag{7.35}$$

For an even function

$$a_n=\frac{2}{\pi}\int_{0}^{\pi}f(\omega t)\cos n\omega t\,\mathrm{d}\omega t;\ b_n=0 \tag{7.37}$$

For an odd function

$$b_n=\frac{2}{\pi}\int_{0}^{\pi}f(\omega t)\sin n\omega t\,\mathrm{d}\omega t;\ a_n=0 \tag{7.39}$$

Complex form:

$$f(\omega t)=\sum_{n=-\infty}^{\infty}c_n\mathrm{e}^{\mathrm{j}n\omega t} \tag{7.51}$$

where

$$\begin{aligned} c_n&=\tfrac{1}{2}(a_n-\mathrm{j}b_n) && n>0\\ &=\tfrac{1}{2}(a_n+\mathrm{j}b_n) && n<0\\ &=a_0/2 && n=0 \end{aligned}$$

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or

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega t)e^{-jn\omega t}\,d\omega t \tag{7.52}$$

$$|c_n| = K_n/2 \tag{7.54}$$

Square wave (unit amplitude; odd function)

$$f(\omega t) = \frac{4}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right] \tag{7.36}$$

Triangular wave (unit amplitude; odd function)

$$f(\omega t) = \frac{8}{\pi^2}\left[\sin\omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\sin 5\omega t - \ldots\right] \tag{7.42}$$

Half-wave rectified sine wave (unit amplitude; even function)

$$f(\omega t) = \frac{1}{\pi}\left[1 + \frac{\pi}{2}\cos\omega t + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \ldots\right] \tag{7.45}$$

Full-wave rectified sine wave (unit amplitude; even function)

$$f(\omega t) = \frac{2}{\pi}\left[1 - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \ldots\right] \tag{7.46}$$

Average value of product of two functions. If

$$f(\omega t) = \frac{a_{0f}}{2} + \sum_{n=1}^{\infty} K_{nf}\cos(n\omega t + \phi_{nf}) \tag{7.56}$$

and

$$g(\omega t) = \frac{a_{0g}}{2} + \sum_{m=1}^{\infty} K_{mg}\cos(m\omega t + \phi_{mg}) \tag{7.57}$$

then

$$\overline{f(\omega t)g(\omega t)} = \frac{a_{0f}a_{0g}}{4} + \sum_{n=1}^{\infty} \frac{K_{nf}K_{ng}}{2}\cos(\phi_{nf} - \phi_{ng}) \tag{7.61}$$

R.M.S. value of a function in terms of harmonic components. If $i(\omega t)$ is a function of current with r.m.s. value I, then

$$I = [\overline{i(\omega t)^2}]^{\frac{1}{2}} = \sqrt{(I_0{}^2 + I_1{}^2 + I_2{}^2 + \ldots)} \tag{7.64}$$

where I_0, I_1, I_2, etc., are the r.m.s. values of harmonic components.

Power in terms of harmonic components.

If, at a terminal pair, $v(\omega t)$ and $i(\omega t)$ are functions of voltage and current with the same periodicity, then the average power is given by

$$P = \overline{v(\omega t)i(\omega t)} = V_0 I_0 + V_1 I_1 \cos\phi_1 + V_2 I_2 \cos\phi_2 \ldots \tag{7.66}$$

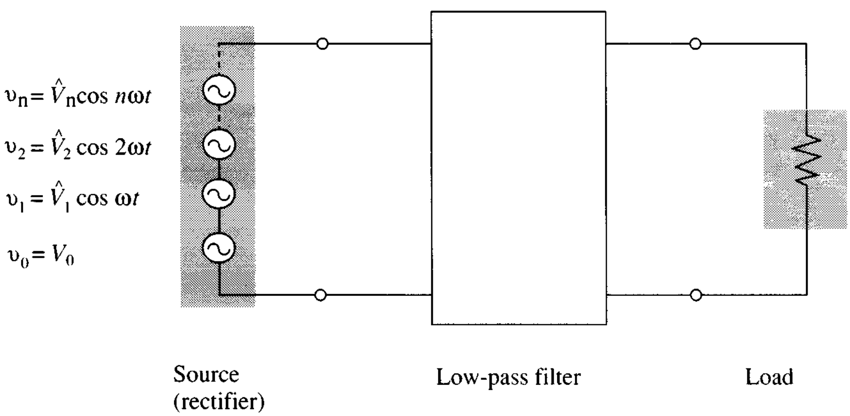
where (V_1, I_1); (V_2, I_2); etc. are the r.m.s. values of harmonic components of like frequency.

†7.9 Filter circuits for rectifiers

Half-wave or full-wave rectifier circuits of the type discussed in section 7.6 provide the basis for most d.c. power supplies used in electronic equipment, but for many applications the unidirectional but fluctuating voltages provided by such circuits is not sufficiently constant, and provision must be made for 'smoothing' their outputs. This is accomplished by means of a filter circuit designed to attenuate the fluctuating harmonic components of the output voltage waveform whilst passing the constant, d.c. component.

Referring to fig. 7.46, each harmonic component in the Fourier expansion of the rectifier output may be regarded as deriving from a separate ideal voltage source. The attenuation of the filter may then be calculated separately for each input and, assuming that filter and load are composed of linear circuit elements, the overall effect of all sources acting together may be obtained by superposition. In practice it is sufficient to know that the amplitude of a particular component is below a certain specified level. In the type of circuit considered here only the first two or three harmonic components are of interest since higher harmonics, firstly,

Fig. 7.46. Analysis of a rectifier/filter circuit using Fourier component representation of rectifier output waveform.



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have relatively small amplitudes (see equations (7.45) and (7.46)); and, secondly, they suffer greater attenuation by the filter.

Various combinations of inductive and capacitive elements are used in rectifier filter circuits; some of the more common arrangements are considered below.

7.9.1 Inductor

One of the simplest filter circuits consists of a single inductor in series with the load, assumed to be a pure resistance, as shown in fig. 7.47. We analyze first the smoothing effect of this circuit upon the output of the full-wave rectifier, making use of the series expansion (7.46) and considering each term separately.

The first term represents the d.c. component of the rectified wave with amplitude $2V_m/\pi$, where V_m is the peak value of the rectified wave. Since this is a direct voltage all of it appears across R (assuming that the inductor has negligible resistance).

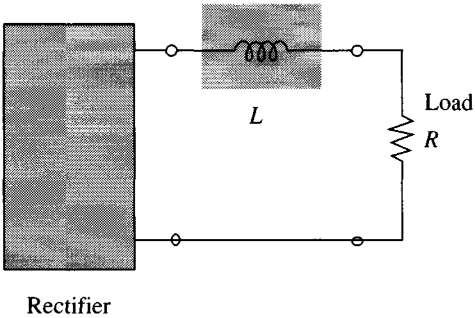
The next term in the expansion is the second harmonic component with amplitude $4V_m/3\pi$. If the supply frequency is ω, then the second harmonic is at frequency 2ω and the factor by which this component is reduced by the filter is $R/\sqrt{[R^2+(2\omega)^2L^2]}$. Thus, the amplitude V_{2m} of the ripple across R is given by

$$V_{2m}=\frac{4V_m}{3\pi}\cdot\frac{R}{\sqrt{(R^2+(2\omega)^2L^2)}}=\frac{4V_m}{3\pi}\cdot\frac{1}{\sqrt{[1+(2\omega L/R)^2]}} \qquad (7.67)$$

Usually, with this type of circuit, the reactance of the inductor ($2\omega L$) is arranged to be large in comparison with R, in which case V_{2m} is given to a sufficiently good approximation by

$$V_{2m}\approx\frac{4V_m}{3\pi}\cdot\frac{R}{2\omega L}=\frac{2V_mR}{3\pi\omega L} \quad (2\omega L\gg R)$$

Fig. 7.47. Inductor smoothing of rectifier output.



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The fourth harmonic component has amplitude $4V_m/15\pi$, and the amplitude of the voltage appearing across R due to this component is approximately

$$V_{4m} \approx \frac{4V_m}{15\pi} \cdot \frac{R}{4\omega L} = \frac{V_m R}{15\pi\omega L}$$

From the above expressions we see that the amplitude of the fourth harmonic component across the load is only about one tenth that of the second harmonic component and can usually be ignored.

It is customary to describe the effectiveness of the smoothing action of a filter circuit by stating the peak-to-peak value of the most significant ripple component as a percentage of the d.c. component.

$$\text{percentage ripple} = \frac{\text{peak-to-peak ripple amplitude}}{\text{magnitude of direct component}} \times 100 \quad (7.68)$$

For the full-wave rectifier with series inductor smoothing the percentage ripple is

$$2 \times \frac{2V_m R}{3\pi\omega L} \times \frac{\pi}{2V_m} \times 100 = \frac{2R}{3\omega L} \times 100\%$$

As an example, consider a full-wave rectifier designed to supply a 50 Ω load. If $L = 10$ H and the supply frequency is 50 Hz, then the percentage ripple will be

$$\frac{2 \times 50 \times 100}{3 \times 2\pi \times 50 \times 10} = 1\%$$

Note that the full-wave rectifier with inductor smoothing produces a lower ripple than the half-wave rectifier because, for the former, the fundamental component of ripple is at twice supply frequency, while for the latter it is at the same frequency as supply frequency (compare series expansions (7.46) and (7.45)).

Carrying out a similar analysis for the half-wave rectifier, using the expansion (7.45), gives the following results:

D.C. component	$V_0 = V_m/\pi$
Amplitude of first harmonic	$V_{1m} = V_m/2$
Reduction factor (for first harmonic)	$= \dfrac{R}{\omega L}$

$$\text{Percentage ripple} = 2 \times \frac{V_m}{2} \times \frac{R}{\omega L} \times \frac{\pi}{V_m} \times 100 = \frac{\pi R}{\omega L} \times 100\%$$

$$= 5\% \text{ (for } R = 50\,\Omega;\ L = 10\,\text{H)}$$

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7.9.2 L-section

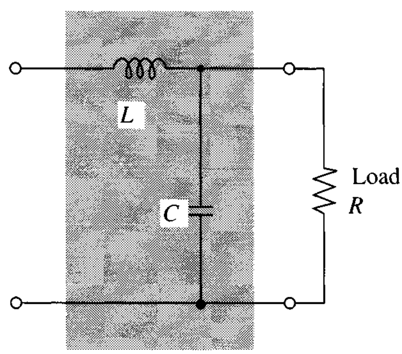
Considering again the full-wave rectifier, a further degree of smoothing is obtained by the addition of a capacitor to the circuit of fig. 7.47, as shown in fig. 7.48. It is usual to make the reactance of this capacitor $(1/2\omega C)$ small in comparison with R so that substantially the whole of the second harmonic component of current goes through the capacitor. The inductor and capacitor then constitute a voltage divider and the factor by which the second-harmonic voltage is reduced is, approximately,

$$\frac{1/2\omega C}{\sqrt{[(1/2\omega C)^2+(2\omega L)^2]}} \approx \frac{1}{4\omega^2 LC} \tag{7.69}$$

7.9.3 Capacitor

The output of a rectifier circuit may be smoothed by placing a capacitor across the load, as shown in fig. 7.49(*a*). With this type of circuit the diode conducts for only a fraction of each half-cycle of the supply voltage, and the output waveform can no longer be represented by the Fourier series expansions given in section 7.8.5. The action of the circuit is illustrated in fig. 7.49(*b*). On positive half cycles of the input voltage the capacitor charges to the peak value V_m of the supply. As the supply voltage falls, the diode ceases to conduct and the capacitor discharges through the load R with a time constant RC. If this time constant is large compared with the period T of the input voltage, then the capacitor will lose only a small fraction of its initial charge before the next half cycle of the voltage appears and raises the capacitor voltage back to V_m. During the period in which the diode is non-conducting the capacitor supplies the whole of the current to the load, and for this reason it is often referred to as a 'reservoir' capacitor.

Fig. 7.48. L-section filter.



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The capacitor voltage during the discharge period is

$$v_c = V_m e^{-t/RC}$$

If $RC \gg T$, then we may approximate the exponential by the first two terms of its series expansion to give

$$v_c \approx V_m\left(1 - \frac{t}{RC}\right)$$

Now, if we further assume that the charging time t_1 is short compared with T, the decrease in capacitor voltage ΔV is approximately

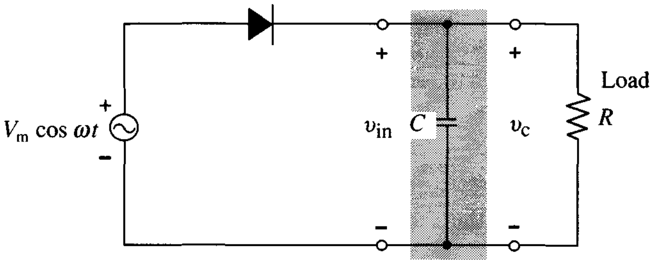
$$\Delta V \approx \frac{V_m T}{RC}$$

and the average voltage (d.c. component) across the load is

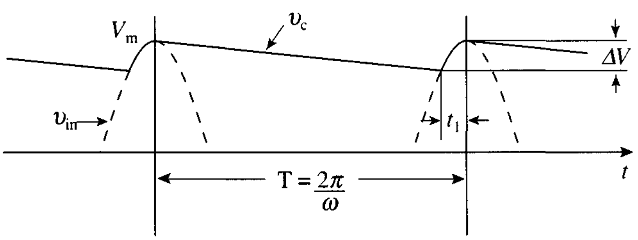
$$V_0 \approx V_m - \frac{\Delta V}{2} = V_m\left(1 - \frac{T}{2RC}\right) \tag{7.70}$$

Using the definition (7.68) the percentage ripple for the half-wave rectifier with capacitor smoothing is given by

Fig. 7.49. Half-wave rectifier with capacitor smoothing: (*a*) circuit (*b*) waveforms.



(*a*)



(*b*)

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$$\text{Percentage ripple} = \frac{\Delta V}{V_0} \times 100 \approx \frac{(V_m T/RC) \times 100}{V_m} \approx \frac{T}{RC} \times 100\%$$

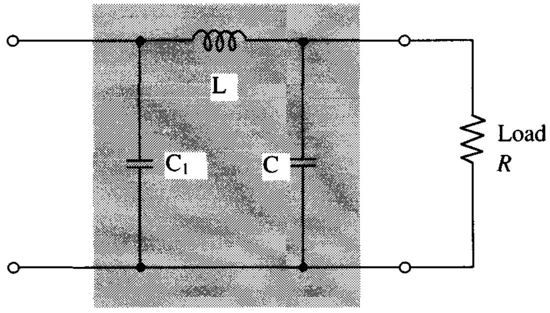
This approximate expression is usually sufficiently accurate to allow one to choose a suitable value for the filter capacitor. However, if a more detailed and accurate treatment is required, in particular if the power in the diode is to be calculated, then one may use methods similar to those employed in section 7.6.2 in relation to the battery charger problem. These methods allow one to calculate the conduction period t_1 and, given the diode resistance, to derive the instantaneous function of current during the charging period. The power dissipated in the diode may then be determined.

7.9.4 π-section

If a high degree of smoothing is required, the single capacitor of fig. 7.49(a) may be combined with the L-section filter of fig. 7.48 to produce the π-section filter circuit shown in fig. 7.50. In this circuit the voltage across C_1 will resemble the sawtooth voltage of fig. 7.49(b). In order to estimate the ripple across the load it is usual to make the simplifying assumption that the fundamental component of ripple across C_1, (for the half-wave rectifier, this is the first harmonic Fourier component of the sawtooth at supply frequency) has an amplitude equal to half the peak-to-peak voltage of the sawtooth. (Such an assumption gives, of course, an overestimate of the ripple.) Then, making also the assumptions discussed in section 7.9.2, we may deduce that the factor by which this ripple component is reduced is $1/(\omega^2 LC)$. (For a full-wave rectifier the fundamental component of ripple is at twice supply frequency and the corresponding factor is $1/(4\omega^2 LC)$.)

In order to reduce weight and cost of components, π-section filters are often designed with a resistor R_F in place of the inductor. In this case the ripple is reduced by the factor $1/\omega C R_F$ for the half-wave rectifier, or

Fig. 7.50. π-section filter.



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$1/2\omega CR_F$ for the full-wave rectifier. A disadvantage of this circuit is that R_F and R now form a voltage divider for the d.c. component of the rectifier output and so the d.c. voltage across the load may be reduced significantly if a low ripple, and therefore a high value of R_F, is required.

7.10 Summary

The analysis of circuits containing non-linear elements requires special techniques. These fall into four major categories: graphical analysis, small-signal models, piecewise-linear techniques, and analytical techniques.

By means of graphical analysis, voltages and currents can be found in circuits containing simple series and parallel combinations of linear and non-linear elements; this method depends upon a knowledge of the complete voltage–current characteristic of the non-linear devices in the circuit. For a two-terminal non-linear device, such as a diode, a single characteristic curve relating voltage and current is all that is required. For a three-terminal device, such as a bipolar transistor, two sets of characteristic curves are required. Graphical analysis is commonly used to determine the *bias* and *operating point* in circuits containing transistors, the procedure in this case being often referred to as the *load line* method.

If the incremental voltage and current swings in a non-linear device are small, then over a limited region, its voltage–current characteristic may be considered as being of straight-line form and a linear relationship may be assumed between voltage and current. This is the basis of the small-signal approach to the analysis of non-linear circuits in which the non-linear devices in the circuit are described by one or more linear parameters. This approach is of particular importance in the case of transistor circuits. The *hybrid-parameter* (small-signal) models of figs. 7.13 and 7.14 are commonly employed in this context.

The small-signal model of a device uses a linear approximation that is valid over a narrow region of the device characteristic. By this means circuits including such devices may be treated as linear and all the techniques of linear circuit analysis become applicable. An extension of this approach is to approximate the voltage–current characteristic of a device by a series of straight-line segments extending over an arbitrarily wide region of the characteristic. Within each segment the device is modelled by appropriate linear parameters. This *piecewise-linear* approximation then allows linear circuit analysis to be applied over any desired range of operating voltages and currents. The analysis of circuits containing diodes is one important area of application for this technique, in particular rectifier circuits. The piecewise-linear approach is also useful when we wish to synthesize circuits that will reproduce given non-linear characteristics by means of combinations of resistors and diodes.

The characteristics of many non-linear devices may be expressed by means of analytical functions or approximated by power series. In such cases the circuits of which they form part may be solved using either algebraic or numerical techniques. The characterization of a non-linear device by means of a power series provides a method for the analysis of modulators and frequency changer circuits of the type used in communication networks.

Rectifiers, employing diodes and thyristors for a.c.–d.c. power conversion, comprise a broad class of circuits requiring non-linear techniques of analysis. In such circuits it is often sufficiently accurate to approximate a diode or thyristor by a two-segment piecewise-linear approximation, which in its simplest form is equivalent to treating the device as a switch.

Many types of non-linear circuit produce waveforms that are periodic but non-sinusoidal; rectifier circuits being but one example. The response of linear circuits to such waveforms cannot be determined directly using standard techniques of d.c. or a.c. circuit analysis. However, by means of Fourier series analysis a periodic waveform may be resolved into a d.c. component plus a series of harmonic components of sinusoidal form. The response to such a waveform may then be conveniently found by determining the response to each Fourier component separately and then combining these individual responses by superposition. The utility of the technique lies in the fact that it is usually necessary to consider only the first few harmonic components in order to obtain a sufficiently accurate knowledge of circuit behaviour for design purposes. This approach is particularly useful in the design of filter circuits for rectifiers.

Finally, Fourier theory may be extended to allow the calculation of the r.m.s. value of an arbitrary number of harmonic components, and the power associated with the harmonic components of periodic non-sinusoidal voltage and current waveforms.

7.11 Problems

1. A source of e.m.f. 4.5 V is connected in series with a 225 Ω resistor and a diode operating in the forward conducting region. The diode has the voltage–current characteristic given in table 7.1. Determine by a graphical method the current in the circuit and the voltage across the diode.

Current (mA)	0	1	2	3	4	5	6	7	8	9	10	11	12
Voltage (V)	0	0.45	0.78	1.02	1.23	1.38	1.56	1.74	1.92	2.04	2.16	2.28	2.36

Table 7.1 for problems 1 and 2

2. Each of the diodes in the circuit of fig. 7.51 has the voltage–current characteristic given in table 7.1. Use a graphical method to determine the combined v–i characteristic for the two diodes and the 1.5 V battery. Hence determine the value of the current I and the voltage V. Determine also the dissipation in each diode.
(London University)

3.

Current (mA)	0	1	2	3	4	5	6	7	8	9	10
Voltage D_1	0	0.7	1.3	1.7	2.1	2.3	2.6	2.9	3.2	3.4	3.6
Voltage D_2	0	1.8	2.8	3.6	4.3	4.9	5.3	5.8	6.2	6.6	6.9

Table 7.2 for problem 3

The diodes in the circuit of fig. 7.52 have the characteristics given in table 7.2. Using a graphical method to combine the diode characteristic, determine the current I, the voltage V and the voltage across each diode. Find also for this bias condition the effective d.c. resistance and the incremental a.c. resistance of the diode combination.
(London University)

Fig. 7.51. Circuit for problem 2.

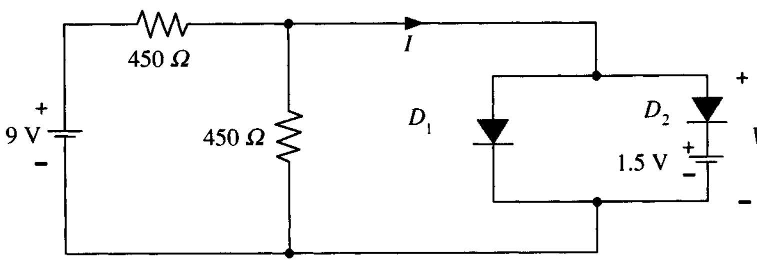
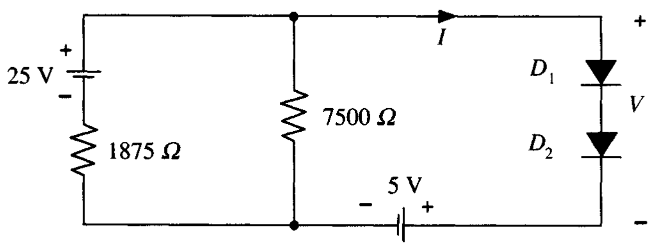


Fig. 7.52. Circuit for problem 3.



4. A tunnel diode having the characteristic shown in fig. 7.53 is connected in series with a voltage source V_0 and a resistor R.
(a) If $V_0 = 0.75$ V, what range of values of resistance R allows the circuit to have two stable states?
(b) If $R = 1$ kΩ, what change in V_0 is necessary to change the state of the circuit?
5. The Zener diode in the circuit of fig. 7.54(*a*) has the characteristic shown in fig. 7.54(*b*).
(a) What is the voltage across the diode if $R_L = \infty$?
(b) What value of R_L will reduce the diode voltage to 90% of the open circuit value?
6. Determine the slopes and break points of the v–i relationship for the piecewise-linear circuit shown in fig. 7.55. (Assume ideal components.)
7. Determine the slopes and break points of the v–i relationship for the piecewise-linear circuit shown in fig. 7.56. (Assume ideal components.)
8. Design a circuit consisting of ideal sources, diodes and resistors that will have the piecewise-linear characteristic shown in fig. 7.57.
9. Fig. 7.58(*a*) shows two ideal voltage generators V_1 and V_2 connected to a load resistance R_3 through R_1 and R_2 plus a perfect diode. The voltages are shown in fig. 7.58(*b*).

Draw the waveform of the voltage across R_2 and the currents in R_1 and R_2, when $R_1 = R_2 = R_3 = R$. What power is consumed in R_3?
(Oxford University)

Fig. 7.53. Graph for problem 4.

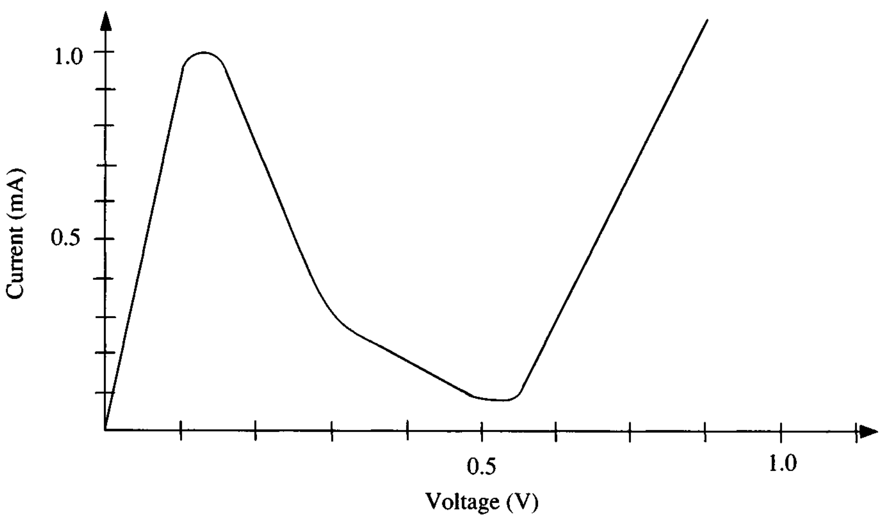


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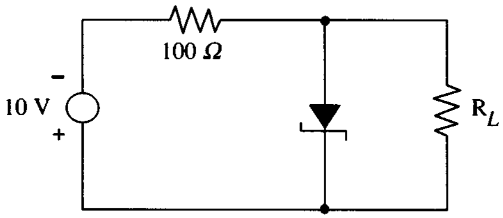
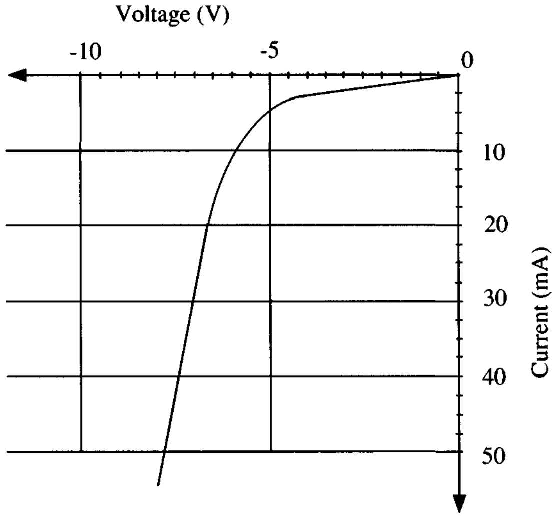
Fig. 7.54. Circuit and graph for problem 5.

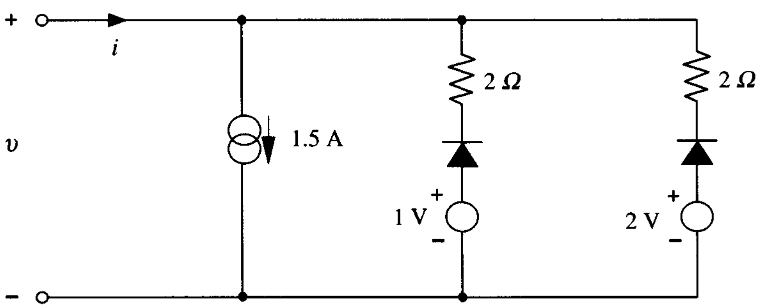


(*a*)



(*b*)

Fig. 7.55. Circuit for problem 6.



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10. A diode whose forward characteristic is given in fig. 7.59(*a*) is connected as shown in the circuit of fig. 7.59(*b*).
(a) What will be the diode voltage, current and power dissipation?
(b) Approximate the diode characteristic, over the full range shown, by three straight-line segments, keeping the errors in the diode current to within 0.4 mA. Hence draw a piecewise-linear equivalent circuit for the diode when conducting in the forward direction. If the diode is replaced by the equivalent circuit, what will be the power dissipated in the equivalent circuit?
(Cambridge University: Second year)

Fig. 7.56. Circuit for problem 7.

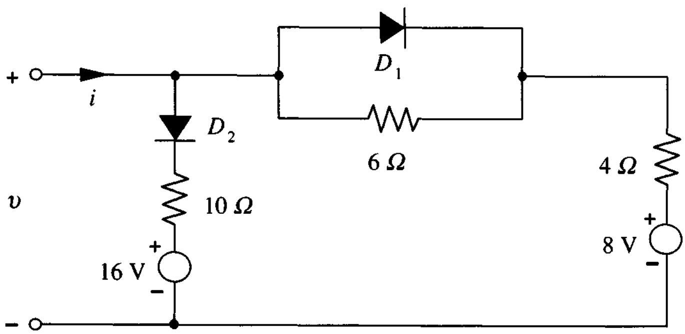
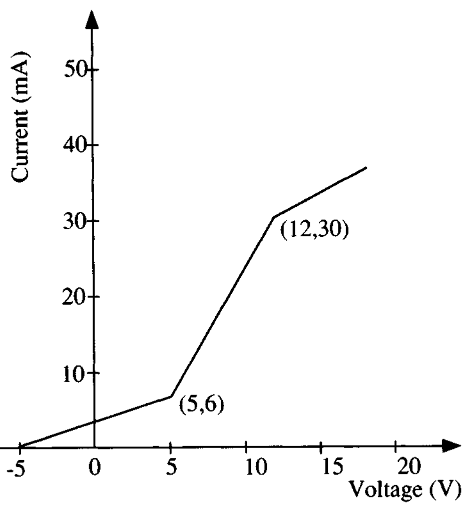


Fig. 7.57. Graph for problem 8.



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11. The circuit shown in fig. 7.60 is employed to approximate the characteristic $i = v^2/10^4$ amperes over the range $0 \leqslant v \leqslant 10$ V by a piecewise-linear approximation. Assuming that the diodes are ideal and that D_2 is to conduct at 10/3 V and D_3 at 20/3 V, calculate suitable values for the resistors.
(Cambridge University: Second year)

12. In the circuit of fig. 7.61 the non-linear resistor r has a characteristic described by $v = i(1+i^2)^{-\frac{1}{2}}$. Determine the current I.

Fig. 7.58. Circuit and graph for problem 9.

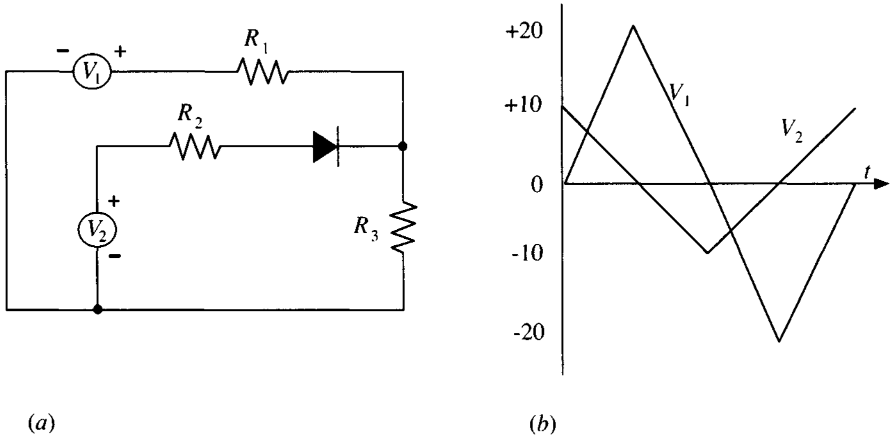
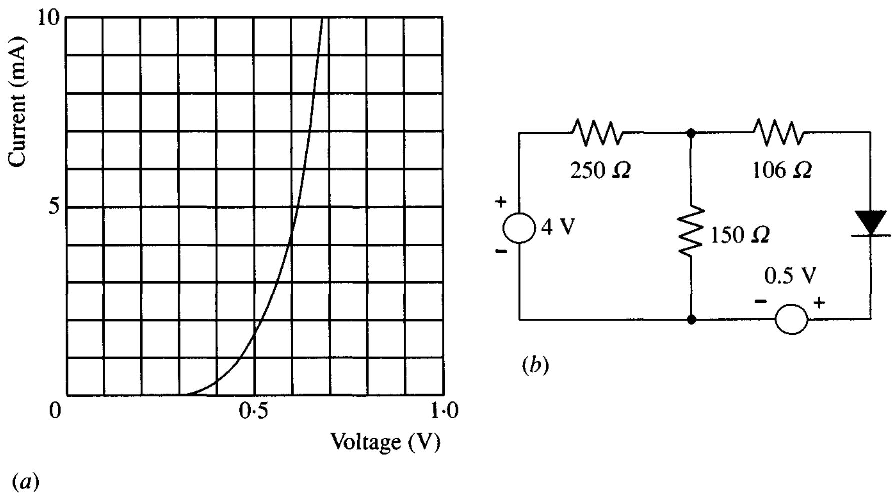


Fig. 7.59. Graph and circuit for problem 10.



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13. Two non-linear devices d_1, d_2 have the following v–i characteristics:

$$d_1: \quad i_1 = 0.3(e^{v_1} - 1)$$
$$d_2: \quad i_2 = v_2 + 0.4v_2^2$$

The devices are connected in series across a voltage source.
(a) If the voltage source is ideal with e.m.f. 3 V, find the current in the circuit and the voltage across each device.
(b) If the voltage source has an e.m.f. of 3 V and internal resistance 0.5 Ω, find the current in the circuit and the voltage across each device.
14. In the circuit shown in fig. 7.62, if the output load resistance R_L is constant, show that the incremental variation of output voltage V_0 with a change of input voltage V_{in} is equal to

$$x\left(x + R + \frac{Rx}{R_L}\right)^{-1}$$

Fig. 7.60. Circuit for problem 11.

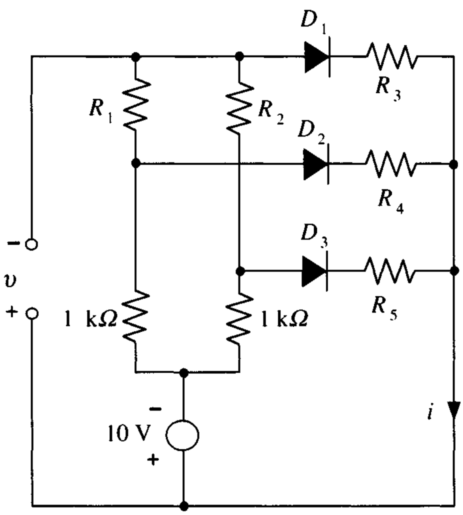
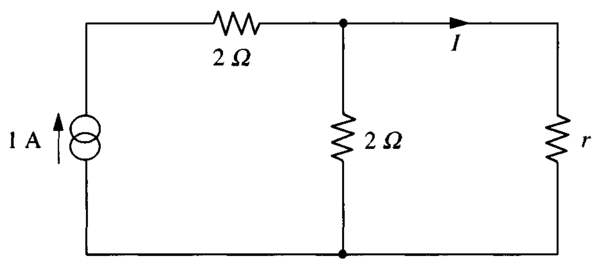


Fig. 7.61. Circuit for problem 12.



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where x is the incremental impedance of the Zener diode.
(Sheffield University: Second year)

15. (a) Derive a two-slope circuit model for the diode whose characteristic is given in fig. 7.63(*a*).

(b) The diode is used in the simple rectifying circuit of fig. 7.63(*b*). What is the average current in the circuit if the supply voltage $v(t)$ is:

(i) $1.0\cos\omega t$ volts;

(ii) $100\cos\omega t$ volts?

Fig. 7.62. Circuit for problem 14.

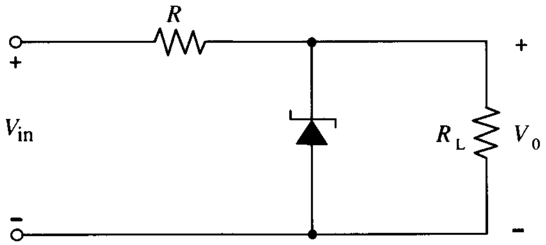
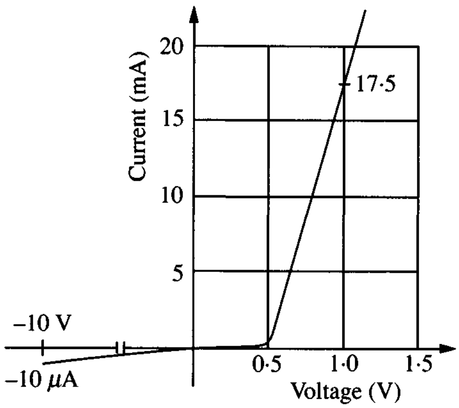
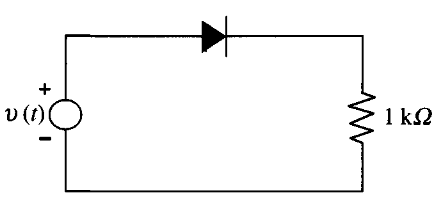


Fig. 7.63. Graph and circuit for problem 15.



(*a*)



(*b*)

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16. Switch contacts may be damaged by the opening of a switch that supplies current to a coil. Protection to the contacts is afforded by installation of a diode as shown in fig. 7.64.
(a) Without the diode, the switch contacts would 'arc' when the switch is opened. Why?
(b) Explain how the diode solves the 'arcing' problem but does not interfere with normal operation of the coil.
(c) What is the steady state current in the coil with the switch closed?
(d) How long after the switch is opened will be required for the coil current to reach 3 mA?
17. In the circuit shown in fig. 7.65 the two voltage sources are equal, the diode is perfect and the resistance is small compared to $\frac{1}{2}\sqrt{(LC)}$. Initially there is no charge on the capacitor. Calculate and sketch the variation of capacitor voltage with time after the switch is closed.
(Oxford University)
18. (a) A half-wave rectifier consists of a silicon diode in series with a load resistance of 2 Ω and is supplied from the secondary of a 50 Hz transformer which has an open-circuit voltage of 5.0 V r.m.s. and an effective resistance of 0.1 Ω. The diode characteristic may be represented by an ideal diode in series with a resistance of 0.04 Ω. Determine the mean current flowing in the load.

Fig. 7.64. Circuit for problem 16.

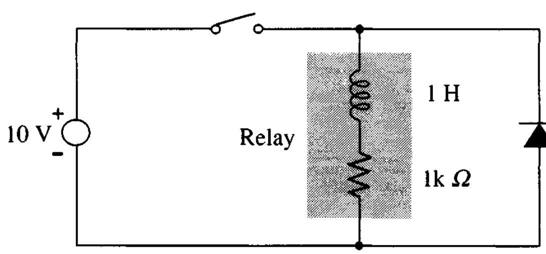
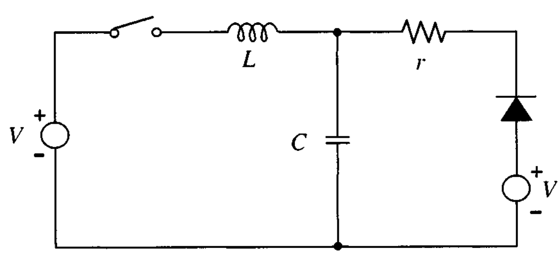


Fig. 7.65. Circuit for problem 17.



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(b) A smoothing capacitor, connected across the load, is needed to reduce the peak-to-peak ripple voltage to 10% of the mean voltage at the load. Determine the capacitance to be used and comment on the result. Show that the mean voltage at the load is now about 5 V.
(Cambridge University: Second year)

19. A half-wave rectifier supplies a load resistor $R = 5\,\text{k}\Omega$ in parallel with a filter capacitor $C = 32\,\mu\text{F}$. The a.c. supply is $300\sin(100\pi t)$ V and the combined forward resistance of the rectifier diode and the resistance of the supply is $30\,\Omega$. Estimate:
(a) the average current in the load resistor;
(b) the percent amplitude of the ripple voltage;
(c) the peak current in the rectifier;
(d) the r.m.s. current in the rectifier;
(e) the percent ripple if a second 32 microfarad capacitor is added in parallel;
(f) the percent ripple if a second 32 microfarad capacitor and a 20 henry choke are used in the circuit. (The two capacitors and the choke are arranged to make an L-section filter.)

20. (a) A transformer with centre-tapped secondary each half of which has an e.m.f. of 400 V r.m.s. at 50 Hz is used with a 20 henry inductor and two diodes of zero forward resistance to provide d.c. power to a $500\,\Omega$ load. Calculate the average voltage across the load and the amplitude of the lowest frequency component of the ripple voltage. It may be assumed that one or the other rectifier is conducting at all times.
(b) A filter capacitor is used instead of the inductor and the transformer e.m.f. is altered to give approximately the same average voltage and percent ripple across the $500\,\Omega$ load. What is the new e.m.f. and what capacitance is required?

21. (a) For the general bridge circuit of fig. 7.66(*a*) show that balance occurs when $Z_1 Z_4 = Z_2 Z_3$. The bridge is unbalanced by changing Z_4 by a small increment δZ_4. Show that the unbalance voltage $|V_{AB}|$ is, to a good approximation, proportional to $|\delta Z_4|$. If $Z_1 = Z_2 = Z_3 = (1000 + \text{j}0)\,\Omega$, $Z_4 = (990 + \text{j}0)\,\Omega$, and V is a sinusoidal voltage of r.m.s. magnitude 10 V, what is the unbalance voltage?
(b) In the phase-sensitive detector circuit of fig. 7.66(*b*) the signal and reference voltages are sinusoidal and of the same frequency but differ in phase by an angle ϕ. V_s and V_r are the r.m.s. magnitudes of the corresponding transformer secondary voltages ($V_r > V_s$). The time constant CR is long compared with the period of these waveforms. Derive an expression for the d.c. output voltage V_d in terms of V_s, V_r and ϕ. It may be assumed that the diodes are ideal and that no current is drawn from the output terminals.

(c) Explain the advantages of using the circuit of fig. 7.66(*b*) as a detector for the bridge circuit of fig. 7.66(*a*). Indicate how the circuits are connected, and suggest one possible application of such an arrangement.
(Cambridge University: Second year)

22. In the circuit of fig. 7.67 the resistance R controls the mean power to the load. The gate current required to trigger the thyristor is 20 mA. What approximately is the maximum angle of delay achievable with this circuit, and to what value must R be set to obtain this angle?

Fig. 7.66. Circuits for problem 21.

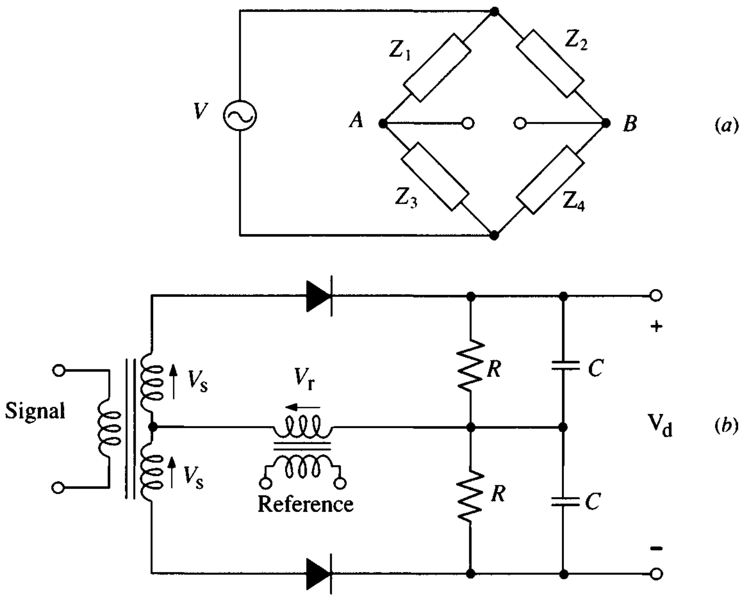
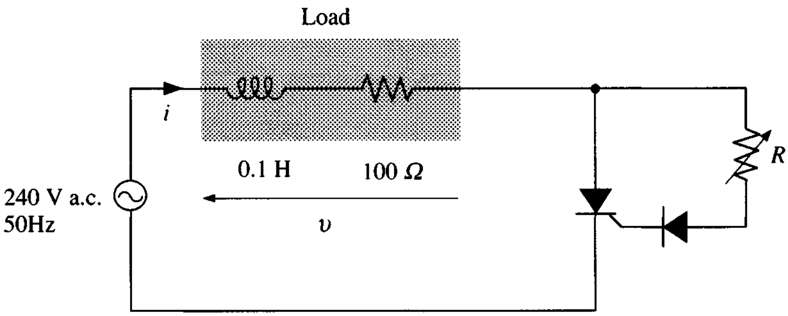


Fig. 7.67. Circuit for problem 22.



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With the angle of delay set to 45°, derive an expression for the load current i during the period of conduction. Sketch the waveforms of i and v for one complete period of the a.c. mains supply. Indicate on your sketch the approximate angle at which the current reaches a maximum and the angle at which conduction ceases.
(Cambridge University: Second year)

23. The trapezoidal wave shown in fig. 7.68 has period T seconds, amplitude A and rise and fall times between zero and A of p seconds. Derive a Fourier expansion for this waveform choosing a time axis that will give rise to sine terms only. Hence find the sine series for:
(a) a square wave;
(b) a triangular wave.
Find values of p for which:
(c) there is no third harmonic;
(d) there is no fifth harmonic.

Fig. 7.68. Waveform for problem 23.

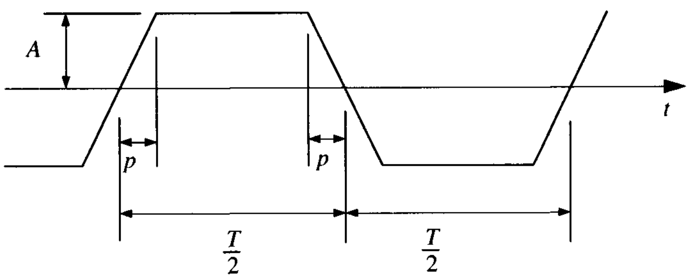
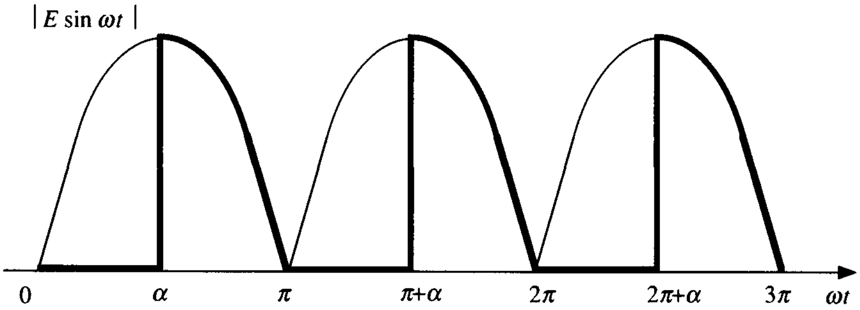


Fig. 7.69. Waveform for problem 24.



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24. The voltage waveform indicated by the heavy line in fig. 7.69 is produced by a thyristor circuit. Show that the amplitude of the fundamental ripple component is

$$c_2 = \frac{E}{3\pi}\sqrt{[(14 + 12\cos\alpha - 4\cos 3\alpha - 6\cos 2\alpha)]}$$

Hence calculate the angle α for which c_2 is a maximum and the ratio of c_2 to the mean value of the voltage at that angle.
(Cambridge University: Second year)

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