Appendix B

The general mesh equations and proofs of the network theorems

Mesh equations

The mesh equations for a general M-mesh network are:

$$
\begin{aligned}
&Z_{11}I_1+Z_{12}I_2 \ldots +Z_{1M}I_M=V_{11}\\
&Z_{21}I_1+Z_{22}I_2 \ldots +Z_{2M}I_M=V_{22}\\
&\quad\vdots\\
&Z_{m1}I_1+Z_{m2}I_2 \ldots +Z_{mM}I_M=V_{mm}\\
&\quad\vdots\\
&Z_{M1}I_1+Z_{M2}I_2 \ldots +Z_{MM}I_M=V_{MM}
\end{aligned}
\tag{B.1}
$$

where V_{mm} is the net e.m.f. in the mth mesh, $I_1 \ldots I_M$ are the M dependent mesh currents, and the coefficients Z are the network self and mutual impedances (all quantities complex).

The network determinant is then

$$
\Delta=\begin{vmatrix}
Z_{11}Z_{12} & \ldots & Z_{1M}\\
Z_{21}Z_{22} & \ldots & Z_{2M}\\
\vdots & & \vdots\\
Z_{M1}Z_{M2} & \ldots & Z_{MM}
\end{vmatrix}
\tag{B.2}
$$

and the solution for the current I_m in the mth mesh is

$$
I_m=\frac{1}{\Delta}\begin{vmatrix}
Z_{11}\,Z_{12} & \ldots & Z_{1m-1}\,V_{11}\,Z_{1m+1} & \ldots & Z_{1M}\\
Z_{21}\,Z_{22} & \ldots & Z_{2m-1}\,V_{22}\,Z_{2m+1} & \ldots & Z_{2M}\\
\vdots & & \vdots & & \vdots\\
Z_{M1}Z_{M2} & \ldots & Z_{Mm-1}V_{MM}Z_{Mm+1} & \ldots & Z_{MM}
\end{vmatrix}
\tag{B.3}
$$

The network theorems may be deduced directly from this solution.

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The Superposition theorem

For a discussion of this theorem see section 2.6.1. Expanding the numerator of (B.3) about the mth column we obtain

$$(-1)^{m-1}I_m = V_{11}\frac{\Delta_{1m}}{\Delta} - V_{22}\frac{\Delta_{2m}}{\Delta} + V_{33}\frac{\Delta_{3m}}{\Delta} \ldots + (-1)^{M-1}V_{MM}\frac{\Delta_{Mm}}{\Delta} \tag{B.4}$$

where Δ_{1m} denotes the determinant remaining when the first row and mth column are deleted from (B.2). All the determinants in this expression are functions only of the complex impedances in the network and, for a given linear network, are therefore constants. Furthermore, each term contains only a single net e.m.f. (there are, for example, no squared or cross-product terms). The superposition theorem is therefore proved.

The Reciprocity theorem

For a discussion of this theorem see section 2.6.2. Consider two branches in the general network and let us choose our meshes such that one branch occurs only in one mesh which we may label mesh (1) and the other only in some other mesh which we may label mesh (2). (A little thought will show that this is always possible.) Under these conditions the current in branch (1) will be I_1 only, and that in branch (2) will be I_2 only. Furthermore, an e.m.f. V_1 in branch (1) will form part of the V_{11} only, and an e.m.f. V_2 in branch (2) will form part of V_{22} only. The superposition theorem tells us that currents caused by one e.m.f. are independent of all other e.m.f.s, so that without loss of generality we may set all e.m.f.s except V_1 and V_2 to zero. Using (B.4) the solutions for the mesh current I_1 and I_2 are then:

$$I_1 = V_1\frac{\Delta_{11}}{\Delta} - V_2\frac{\Delta_{21}}{\Delta}$$

$$I_2 = -V_1\frac{\Delta_{12}}{\Delta} + V_2\frac{\Delta_{22}}{\Delta}$$

Now consider identical e.m.f.s, V say, acting in each of the branches. The current in branch (1) due to V acting *alone* in branch (2) will be

$$I_1' = -V\frac{\Delta_{21}}{\Delta}$$

and the current in branch (2) due to V acting alone in branch (1) will be

$$I_2' = -V\frac{\Delta_{12}}{\Delta}$$

Now, in the general mesh equations, for any pair of subscript values p and q, we have $Z_{pq} = Z_{qp}$, and recalling (from the theory of determinants) that rows and columns of a determinant may be interchanged without affecting its value, it may be readily seen that $\Delta_{pq} = \Delta_{qp}$, hence $I_1' = I_2'$. Since this same result may be obtained for any two meshes chosen arbitrarily, the theorem is proved.

Thévenin's theorem

For a statement and discussion of this theorem see section 2.7. Let the open circuit e.m.f. between the terminals of the network be V_T, and let the impedance measured between these terminals with all internal voltage sources short circuited be Z_T. (We assume that any current sources will have been transformed to voltage sources.) To prove the theorem we have to find expressions for V_T, Z_T and the current that will flow in an external load impedance Z connected between the terminals.

We first connect an impedance Z in series with a source of e.m.f. E between the terminals. This operation will create an additional mesh in the network, which we take as mesh (1), the total number of meshes in the network then being M. The current in Z is then I_1 which, by (B.3), is

$$I_1 = \frac{1}{\Delta} \begin{vmatrix} V_{11} & Z_{12} & Z_{13} & \dots & Z_{1M} \\ V_{22} & Z_{22} & Z_{23} & \dots & Z_{2M} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ V_{MM} & Z_{M2} & Z_{M3} & \dots & Z_{MM} \end{vmatrix} = \frac{\Delta'}{\Delta} \tag{B.5}$$

In the present case, E will form part of V_{11} only, and Z a part of Z_{11} only. Let

Δ_0' be the value of Δ' when E is zero;
Δ_0 be the value of Δ when Z is zero;
Δ_{11}' be the minor of Δ'

then,

$$\Delta' = \Delta_0' + E\Delta_{11}' = \Delta_0' + E\Delta_{11} \tag{B.6}$$

and

$$\Delta = \Delta_0 + Z\Delta_{11} \tag{B.7}$$

The voltage between the terminals of the network is now:

$$I_1 Z + E = \frac{\Delta'}{\Delta} Z + E = \left[\frac{\Delta_0' + E\Delta_{11}}{\Delta_0 + Z\Delta_{11}}\right] Z + E$$

$$=\frac{\Delta_0'+E\Delta_{11}}{\Delta_0/Z+\Delta_{11}}+E$$

The open circuit e.m.f. between the terminals is the value of this expression when Z becomes infinite and E is zero, so that

$$V_{\mathrm{T}}=\frac{\Delta_0'}{\Delta_{11}} \tag{B.8}$$

The impedance between the terminals with all internal voltage sources short-circuited is the value of E/I_1 with Z equal to zero. Under these conditions Δ_0' is zero and from (B.6) we obtain

$$E=\frac{\Delta'}{\Delta_{11}}$$

Also, from (B.5),

$$I_1=\frac{\Delta'}{\Delta_0}$$

hence,

$$Z_{\mathrm{T}}=\frac{E}{I_1}=\frac{\Delta_0}{\Delta_{11}} \tag{B.9}$$

Considering again the solution (B.5) for the current, we have

$$I_1=\frac{\Delta'}{\Delta}$$

which, from (B.6) and (B.7), becomes

$$\mathrm{I}_1=\frac{\Delta_0'+E\Delta_{11}}{\Delta_0+Z\Delta_{11}}=\frac{\Delta_0'/\Delta_{11}+E}{\Delta_0/\Delta_{11}+Z}$$

Substituting (B.8) and (B.9) in this expression yields

$$I_1=\frac{V_{\mathrm{T}}+E}{Z_{\mathrm{T}}+Z} \tag{B.10}$$

But (B.10) is precisely the equation which applies to a single-mesh network containing total impedance $Z_{\mathrm{T}}+Z$ and total e.m.f. $V_{\mathrm{T}}+E$. The theorem is therefore proved.