2

Theorems and techniques of linear circuit analysis

2.1 Introduction

By making the assumption that all of the elements in a circuit are linear, the analysis is greatly simplified. Although all real circuits are non-linear to some degree, in most cases a linear treatment gives sufficiently accurate results, and even for circuits containing highly non-linear elements, methods can often be devised for dealing with them on a linear basis. It is for these reasons that the study of linear circuit theory is of paramount importance in electrical engineering science.

The theorems and techniques of linear circuit analysis presented in this chapter, while being of general usefulness and validity, are developed in the context of d.c. circuits. The advantages of this approach are twofold: firstly, the theory can be developed on the simplest possible basis and in terms which will be familiar to most students. Secondly, the study of d.c. circuit theory is of great practical importance in its own right since it arises in many branches of power and electronic systems analysis.

D.C. linear circuits comprise assemblies of linear lumped resistances together with ideal direct voltage and current sources. The theory appertaining to such idealized circuits is concerned with real situations since many types of source found in practice, a battery for example, can be represented to a good approximation by an ideal source in combination with a lumped resistance.

A typical voltage–current characteristic, or load characteristic, for a *practical voltage source* is shown in fig. 2.1(*b*). The terminal voltage V falls with increasing load current I, but over a certain part of the working range, between points AB, the characteristic can be represented by a straight line. Over this region the voltage–current relationship is of the form

$$V = V_0 - IR_0 \tag{2.1}$$

where R_0 is the negative of the slope of the straight line.

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The circuit model which gives precisely this relationship is shown in fig. 2.1(*c*). On open circuit ($I=0$) there is no voltage drop across R_0, and the terminal voltage is V_0. When a load current is drawn, the internal voltage drop is IR_0, and the terminal voltage falls by this amount. R_0 is called the *internal resistance*, or *output resistance*, of the practical voltage source. It must be stressed that the model of fig. 2.1(*c*), sometimes referred to as a linear voltage source, is applicable only to practical sources that exhibit a straight-line load characteristic.

Before proceeding it is necessary to define some of the terms used in connection with the analysis described in this and subsequent chapters. A number of these have already been introduced in chapter 1 but are included here for the sake of completeness. The definitions given below are illustrated with reference to fig. 2.2.

Node: An equipotential junction, formed by perfect conductors, between two or more elements. A junction between three or more elements, for example node *B*, is termed a principal node.

Branch: A path containing one or more series-connected elements

Fig. 2.1. Practical voltage source (battery). The linear lumped model is valid for region *AB* of the load characteristic.

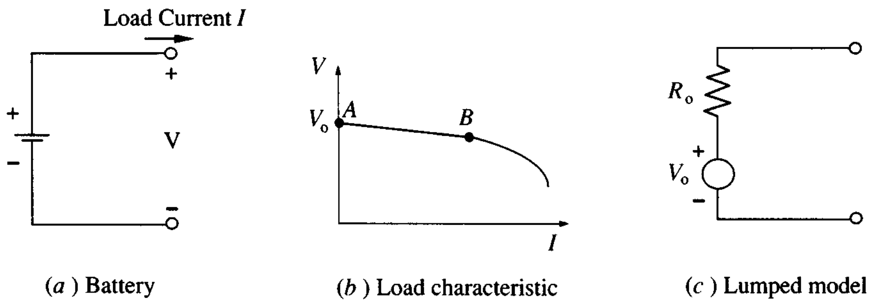
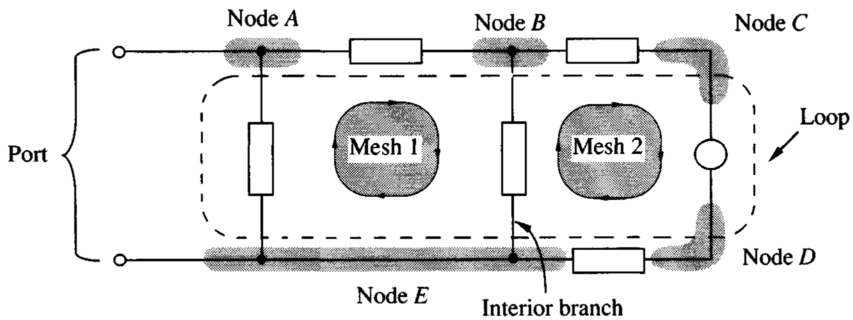


Fig. 2.2. Illustrating network terminology.



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joining two principal nodes; for example, path *BCDE*. Path *BE* is an *interior* branch.

Loop: Any connected, closed path in a circuit. The closed path *ABCDEA* forms a loop.

Mesh: A loop which cannot be subdivided into smaller loops by interior branches. The distinction between a mesh and a loop is a rather fine one. Simply by redrawing a circuit it is possible for a mesh to become a loop and vice versa. The loop shown in the figure is divided into two meshes by interior branch *BE*. This branch and branch *BCDE* may be interchanged, in which case what was formerly mesh 1 becomes a loop.

Port: A pair of terminals, or nodes, in a network, through which connections are made to external sources or other networks.

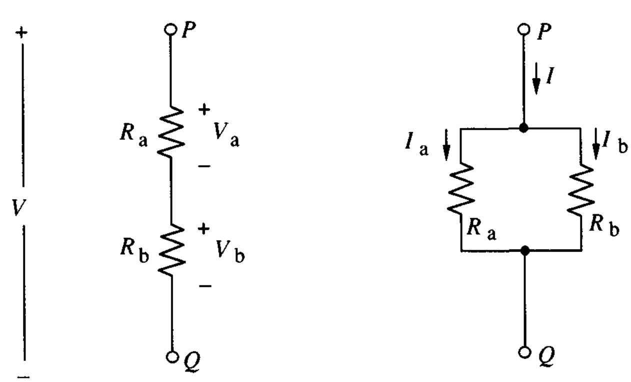
Network or *circuit*: Used in this text interchangeably. In more advanced analysis, a distinction is sometimes made (see for example reference 2).

2.2 Voltage and current dividers

Voltage and current dividers form two of the most common building blocks of electrical circuits. The basic voltage divider (also called a potential divider) is shown in fig. 2.3(*a*). With voltage V across terminals PQ, voltages V_a and V_b are established across resistances R_a and R_b. These voltages may be related to V by the methods discussed in section 1.7.

The combined resistance of R_a and R_b in series is R_a+R_b, therefore, the current is $V/(R_a+R_b)$ and the voltage across R_a is given by

Fig. 2.3. Divider circuits.



(*a*) Voltage divider (*b*) Current divider

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$$V_a = \frac{R_a}{R_a + R_b} V \tag{2.2}$$

Likewise,

$$V_b = \frac{R_b}{R_a + R_b} V \tag{2.3}$$

We see that in each of these expressions the voltage V is divided in the ratio of the particular resistance concerned to the total resistance.

The current divider is shown in fig. 2.3(*b*). To establish the relationships between the main current I and the branch currents I_a and I_b, we observe that the voltage across PQ is the product $I(R_a//R_b) = IR_aR_b/(R_a + R_b)$ (see section 1.7.3 for 'product-over-sum' rule). Therefore, the current through R_a is given by

$$I_a = \frac{IR_aR_b}{R_a + R_b} \cdot \frac{1}{R_a} = \frac{R_b}{R_a + R_b} I \tag{2.4}$$

Likewise the current through R_b is

$$I_b = \frac{IR_aR_b}{R_a + R_b} \cdot \frac{1}{R_b} = \frac{R_a}{R_a + R_b} I$$

In this case each branch current is found by taking the fraction of the total current equal to the resistance in the *opposite* branch divided by the sum of the branch resistances.

The divider circuits shown in fig. 2.3, one a series circuit the other a parallel circuit, are duals (see section 1.12). This will be readily apparent if (2.4) is expressed in terms of the conductances $G_a = 1/R_a$ and $G_b = 1/R_b$:

$$I_a = \frac{1/G_b}{1/G_a + 1/G_b} I = \frac{G_a}{G_a + G_b} I \tag{2.5}$$

Comparing (2.2) and (2.5) we see that these expressions are of similar form with voltage and current interchanged and resistance and conductance interchanged.

Voltage dividers are used extensively in electronic and power circuits. One common application, illustrated in fig. 2.4, is to provide a fixed or variable degree of voltage control (or attenuation). An input voltage V_1 is applied at the terminals AB (the input port) and a proportion of this voltage V_2 is extracted at terminals CD (the output port). We might use the circuit of fig. 2.4(a) for example, to measure a very high voltage utilizing a voltmeter capable of measuring only a relatively low voltage.

According to (2.3) the voltages at the input and output ports (fig. 2.4(*a*)) are related by

$$V_2 = \frac{R_2}{R_1 + R_2} V_1 \tag{2.6}$$

R_1 and R_2 may be adjusted to provide the requisite division or attenuation.

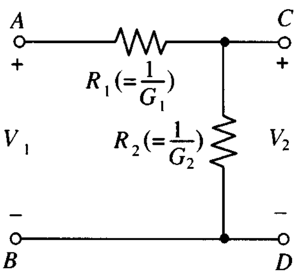
In practical circuits the resistances R_1 and R_2 forming the two 'arms' of the divider may each consist of combinations of separate series or parallel elements, in which case, before applying (2.6) the appropriate reduction formulae (section 1.7.3) must be used to find the two equivalent resistances. In cases where the two arms of the divider contain elements connected simply in parallel (see problem 3 at the end of this chapter) it is convenient to express the divider relationship (2.6) in terms of conductances rather than resistances. Let $R_1 = 1/G_1$ and $R_2 = 1/G_2$, where G_1 and G_2 are the total conductances in the two arms of the divider, then substituting in (2.6) gives

$$V_2 = \frac{1/G_2}{1/G_1 + 1/G_2} V_1 = \frac{G_1}{G_2 + G_1} V_1 \tag{2.7}$$

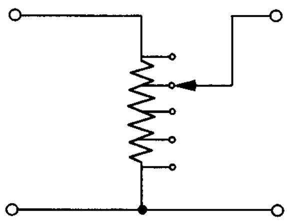
Note that G_1 now replaces R_2 in the numerator of the divider ratio.

It must be emphasized that (2.6) and (2.7) are true only if the terminals

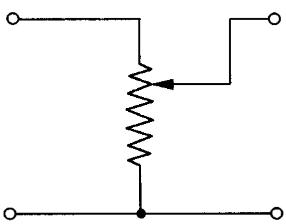
Fig. 2.4. Two-port voltage divider circuits.



(*a*) Fixed



(*b*) Stepped variable



(*c*) Continuously variable (potentiometer)

CD are open circuit. Generally, the output port will be connected to some other circuit which will draw current, and this must be taken into account when calculating the attenuation. If *R* is the effective resistance presented by the external circuit to the output port, then (2.7) becomes

$$V_2 = \frac{(R_2//R)}{R_1 + (R_2//R)} V_1 \tag{2.8}$$

The circuits shown in fig. 2.4 fall into the general category known as two-port networks; these are discussed in chapter 8.

2.3 Mesh analysis

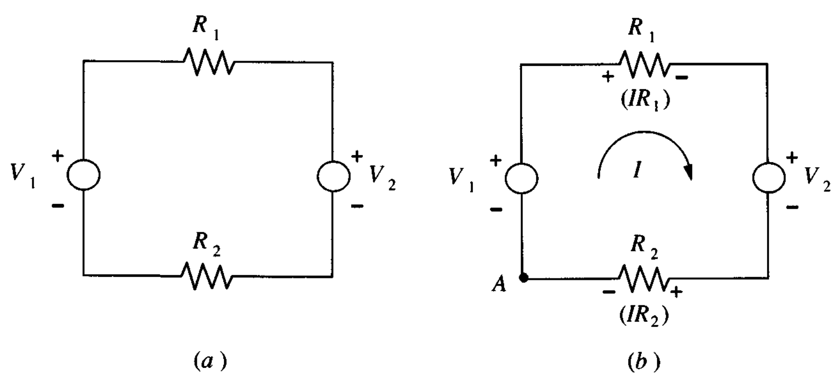
The general objective in circuit analysis is the establishment of a set of equations relating the circuit variables, voltages and currents, in terms of the circuit elements. This is achieved using Kirchhoff's two laws. The solution of these equations yields specific expressions for each of the circuit variables. In mesh analysis source voltages are specified and are treated in the equations as the independent variables; solutions are found for the currents in every branch, these being treated as the dependent variables.

The principles involved in mesh analysis may be illustrated with reference to the single-mesh circuit shown in fig. 2.5 in which source voltages V_1 and V_2 and elements R_1 and R_2 are specified.

First, the current *I* is assigned (fig. 2.5(*b*)) and then Kirchhoff's voltage law (KVL) is used in either of the forms (1.14) or (1.15) to write down the circuit equation. Choosing the latter, and traversing the circuit in a clockwise direction starting at point *A*, we have:

$$\Sigma v = 0 \tag{1.15}$$

Fig. 2.5. (*a*) Single-mesh circuit. (*b*) Circuit with assigned current and resulting voltage drops.



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that is

$$+V_1-IR_1-V_2-IR_2=0$$

In this equation a rise in potential is indicated by a positive sign, a fall by a negative sign. (Strict adherence to the conventions discussed in section 1.3 and 1.4 is necessary when setting up circuit equations of the above form.)

On rearrangement of this equation we obtain:

$$I=\frac{V_1-V_2}{R_1+R_2} \tag{2.9}$$

If, in (2.9) V_2 is numerically greater than V_1, the current is negative; this simply means that the direction of the conventional positive current in the circuit will be in the opposite sense to that assigned.

The above procedure for the analysis of a single-mesh circuit may be readily extended to circuits containing two or more meshes. Fig. 2.6 shows an example of a two-mesh circuit in which R_3 is common, or *mutual*, to meshes (1) and (2).

Two possible ways of assigning currents are shown in figs. 2.6(*b*) and 2.6(*c*). In the first and perhaps most obvious, a current is assigned to every branch in the circuit; in the second currents are assigned to meshes.

Considering first fig. 2.6(*b*) and applying KVL in the form (1.15), we have for mesh (1), traversing the path *ABCD*,

$$V_1-I_pR_1-I_rR_3=0 \tag{2.10}$$

and for mesh (2), traversing *DCEF*, we have

$$I_rR_3-I_qR_2-V_2=0 \tag{2.11}$$

In the above equations there are three current variables but these are not independent since, by application of Kirchhoff's current law (KCL) at node *C*, we can see that $I_p=I_q+I_r$ or

$$I_r=I_p-I_q \tag{2.12}$$

Substituting (2.12) in (2.10) and (2.11) we obtain

$$I_pR_1+(I_p-I_q)R_3=V_1 \tag{2.13}$$
$$I_qR_2-(I_p-I_q)R_3=-V_2 \tag{2.14}$$

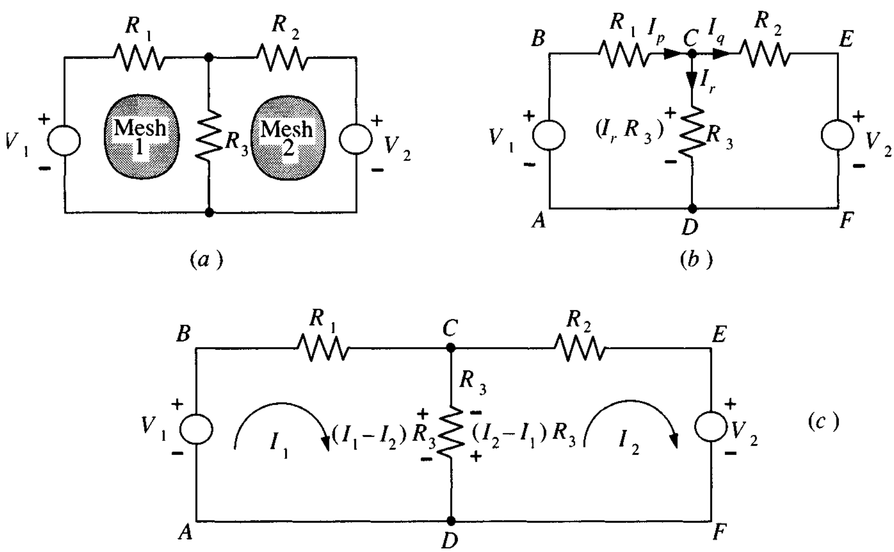
Thus, in reality, there are only two independent variables and these may be evaluated from (2.13) and (2.14). Having found I_p and I_q, we may determine the current in the mutual resistance R_3 using (2.12).

The two paths we have chosen here, to set up the two independent equations necessary to achieve a solution, are not the only possible ones; for example, we might have chosen instead of the mesh *DCEF*, the loop *ABEF*. (It is important to note that the paths chosen must be such that every circuit element is traversed at least once.) The method we are discussing here is, therefore, more generally termed loop analysis, but for circuits that can be drawn on a flat surface to form a series of 'windows' (as in this example) a more convenient and systematic solution is achieved by choosing meshes rather than loops.

It will be evident from the foregoing argument that it is unnecessary to specify every branch current when setting up the mesh equations; it is sufficient to assign a current to each mesh as in fig. 2.6(*c*). The mesh currents I_1 and I_2 are identical with the branch currents I_p and I_q in fig. 2.6(*b*), but in the mutual element R_3 the actual current flowing from *C* to *D*, corresponding to I_r, is $(I_1 - I_2)$. The relationship (2.12) is automatically satisfied and the mesh equations may be written down directly in terms of the mesh currents. This approach to mesh analysis was originally due to James Clerk Maxwell and is sometimes referred to as *Maxwell's cyclic current method.*

We now repeat the analysis using this method. In applying KVL in the form (1.15) to any particular mesh we traverse the mesh in the direction of the assigned current; where mesh currents meet in a mutual resistance such as R_3, the appropriate current difference is taken to calculate the voltage drop.

Fig. 2.6. (*a*) Two-mesh circuit; (*b*) with branch currents assigned; (*c*) with mesh current assigned.



Traversing mesh (1), path *ABCD*,

$$V_1 - I_1R_1 - (I_1 - I_2)R_3 = 0 \tag{2.15}$$

and traversing mesh (2), path *DCEF*,

$$-(I_2 - I_1)R_3 - I_2R_2 - V_2 = 0 \tag{2.16}$$

Note carefully the polarity of the voltage drops in fig. 2.6(*c*), and the order in which the mesh currents appear in the difference terms. In (2.15), relating to mesh (1), the first member of the term representing the difference current in R_3 is I_1; in (2.16), relating to mesh (2), the first member is I_2. It may be remarked also that, according to (2.16), there is no rise of potential in mesh (2); this, however, is simply a consequence of the particular choices made concerning direction of assigned current and direction of traverse.

The above procedures demonstrate the principles underlying mesh analysis using either branch currents or mesh currents. In practical circuit analysis it is useful to be able to write down the mesh equations in a consistent fashion, and in a form requiring a minimum of algebraic manipulation to reach a solution. This can be accomplished, using Maxwell's cyclic current method, by adopting two working rules:

Rule 1. Place all resistive voltage drops (products of current and resistance) on one side of the mesh equation. These terms are always positive.

Rule 2. Place all source voltages on the other side of the equation attaching the appropriate ($\pm$) sign as follows: source voltages acting in the same sense as the direction of the assigned mesh current take a ($+$) sign otherwise they take a ($-$) sign.

Using these working rules (2.15) and (2.16) are written down directly as:

$$I_1R_1 + (I_1 - I_2)R_3 = V_1$$
$$(I_2 - I_1)R_3 + I_2R_2 = -V_2$$

Equations of this form, in which voltage drops appear on one side and source voltages on the other, are sometimes referred to as *balance* equations.

2.4 Worked example

Two batteries of nominal voltage 12 V are connected to a charger as shown in fig. 2.7(*a*). The charger has an open-circuit voltage of 14 V and an internal resistance of 1.2 Ω. Before being placed on charge one battery (*A*) has an e.m.f. (open circuit voltage) of 12.1 V and an internal resistance of 0.1 Ω; the other battery (*B*) has an e.m.f. of 11.9 V and an internal resistance of 0.15 Ω. At the end of the charging period the e.m.f. of battery (*A*) rises to

12.6 V, and the e.m.f. of battery (B) rises to 12.5 V. Assuming that the internal resistances of the batteries do not change during charging, and that batteries and charger can be represented by linear voltage sources, determine: (a) the initial charging currents; (b) the final charging currents; (c) the circulating current through the batteries when the charger is disconnected.

Solution: The linear circuit model of the practical circuit is shown in fig. 2.7(b). Mesh currents I_1 and I_2 are assigned in a clockwise direction as shown. Using the working rules presented above the mesh equations may be written immediately as:

$$1.2I_1 + 0.1(I_1 - I_2) = 14 - V_A$$

and

$$0.1(I_2 - I_1) + 0.15I_2 = V_A - V_B$$

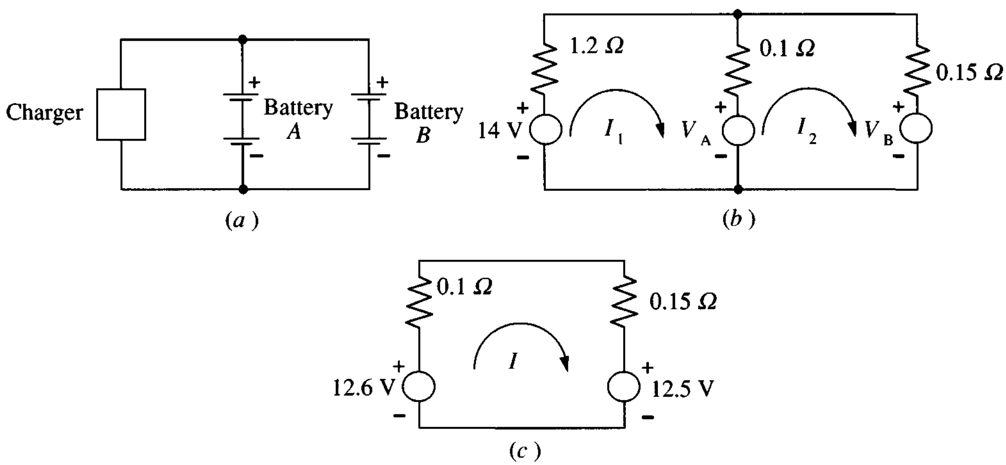
Rearranging we obtain

$$(1.20 + 0.1)I_1 - 0.1I_2 = 14 - V_A$$
$$-0.1I_1 + (0.1 + 0.15)I_2 = V_A - V_B$$

(a) *Initial charging currents.* Substituting the initial values of V_A and V_B gives

$$1.3I_1 - 0.1I_2 = 1.9$$
$$-0.1I_1 + 0.25I_2 = 0.2$$

Fig. 2.7. Circuits for worked example. For battery (A): V_A (initial) = 12.1 V; V_A (final) = 12.6 V. For battery (B): V_B (initial) = 11.9 V; V_B (final) = 12.5 V.



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Solving these for I_1 and I_2 (for example by Gaussian elimination and back substitution) we obtain: $I_1 = 1.57$ A and $I_2 = 1.43$ A.

Therefore, the initial charging current in battery (A) is $(I_1 - I_2) = 0.14$ A, and initial charging current in battery (B) is $I_2 = 1.43$ A.

(b) *Final charging currents.* Substituting the final values of V_A and V_B we obtain

$$1.3I_1 - 0.1I_2 = 1.4$$
$$-0.1I_1 + 0.25I_2 = 0.1$$

giving: $I_1 = 1.14$ A, and $I_2 = 0.86$ A.
Therefore, the final charging current in (A) is $(I_1 - I_2) = 0.28$ A, and final charging current in (B) is $I_2 = 0.86$ A.

(c) *Circulating current.* With the charger disconnected, the circuit reduces to that shown in fig. 2.7(c) for which the mesh equation is

$$(0.1 + 0.15)I = 12.6 - 12.5 = 0.1$$

giving: $I = 0.4$ A.
Note that this last result implies that energy is being transferred from battery (A) to battery (B).

2.5 The general mesh equations

When analysing a network containing a large number of meshes, it is advantageous to adopt a systematic approach to the formulation and solution of the mesh equations. These formal procedures will now be considered with reference to the three-mesh network shown in fig. 2.8.

With mesh currents assigned as shown and using the working rules enunciated previously we obtain:

for mesh (1)

$$I_1R_1 + (I_1 - I_3)R_3 + (I_1 - I_2)R_2 = V_1 - V_2$$

for mesh (2)

$$I_2R_4 + (I_2 - I_1)R_2 + (I_2 - I_3)R_5 = -V_3 + V_2$$

for mesh (3)

$$(I_3 - I_2)R_5 + (I_3 - I_1)R_3 = -V_4$$

Upon rearrangement these equations become:

$$(R_1 + R_2 + R_3)I_1 - R_2I_2 - R_3I_3 = V_1 - V_2$$
$$-R_2I_1 + (R_2 + R_4 + R_5)I_2 - R_5I_3 = V_2 - V_3$$
$$-R_3I_1 - R_5I_2 + (R_3 + R_5)I_3 = -V_4$$

These mesh equations conform to a standard pattern and it is often convenient to use a formal notation for their description and manipulation.

Let

$$R_{11} = R_1 + R_2 + R_3$$
$$R_{12} = -R_2$$
$$R_{13} = -R_3 \text{ and so forth.}$$

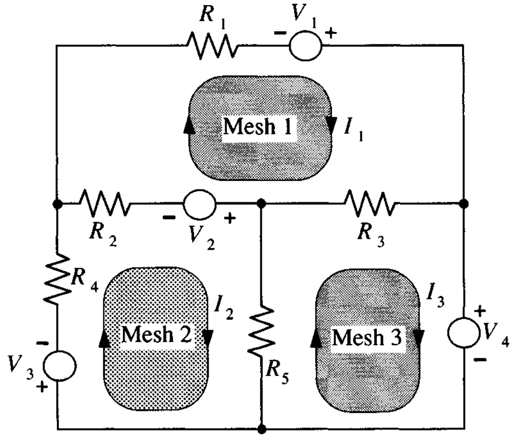
Then we obtain:

$$\begin{aligned} R_{11}I_1 + R_{12}I_2 + R_{13}I_3 &= V_1 - V_2 = V_{11} \\ R_{21}I_1 + R_{22}I_2 + R_{23}I_3 &= V_2 - V_3 = V_{22} \\ R_{31}I_1 + R_{32}I_2 + R_{33}I_3 &= -V_4 = V_{33} \end{aligned} \tag{2.17}$$

The coefficients R_{11}, R_{22}, R_{33}, lying along the leading diagonal of the array formed by the terms on the left-hand side of the equations, are called the *self resistances*; each is the sum of the separate resistances contained in the mesh indicated by the relevant subscripts. The coefficients $R_{12}(=R_{21})$, $R_{13}(=R_{31})$ etc., are symmetrically disposed about the leading diagonal and are called the *mutual resistances* since each is the resistance in the branch shared by the meshes indicated by the relevant subscripts. The mutual resistance terms are all negative if all cyclic currents are assigned in the same direction. Finally, notice that V_{11}, V_{22}, and V_{33}, are the net e.m.f.s acting round each of the meshes indicated by the attached subscripts. The extension of this formal notation to networks containing meshes of higher number than three will be obvious.

Various techniques are available for solving linear simultaneous equations of the form represented by (2.17). For only a small number of

Fig. 2.8. Example of a three-mesh network.



equations, a numerical solution would generally be obtained by means of Gaussian elimination and back substitution. A large number of numerical equations would be solved by standard routines available on most digital computers.* For the present purposes the method of determinants will be used. This will enable us to deduce the solution of (2.17) in symbolic form, and at the same time will allow us to introduce the notation and methods required to develop several important circuit theorems.

The solution for I_1 may be written as the ratio of two determinants, thus,

$$I_1 = \frac{\begin{vmatrix} V_{11} & R_{12} & R_{13} \\ V_{22} & R_{22} & R_{23} \\ V_{33} & R_{32} & R_{33} \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix}} \tag{2.18}$$

Putting the determinant in the denominator equal to Δ and expanding using Cramer's rule we obtain

$$I_1 = \frac{V_{11}}{\Delta}\begin{vmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{vmatrix} - \frac{V_{22}}{\Delta}\begin{vmatrix} R_{12} & R_{13} \\ R_{32} & R_{33} \end{vmatrix} + \frac{V_{33}}{\Delta}\begin{vmatrix} R_{12} & R_{13} \\ R_{22} & R_{23} \end{vmatrix}$$

$$= V_{11}\frac{\Delta_{11}}{\Delta} - V_{22}\frac{\Delta_{21}}{\Delta} + V_{33}\frac{\Delta_{31}}{\Delta} \tag{2.19}$$

where $\Delta_{11} = \begin{vmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{vmatrix}$ is the minor of Δ, that is, the determinant remaining when the first row and first column are deleted from Δ. Similar meanings may be attached to Δ_{21} and Δ_{31}.

Now if the determinants in (2.19) are expanded, it will be seen that Δ has the dimensions of (resistance)^3 whereas Δ_{11}, etc., have dimensions of (resistance)^2. Thus, we may write (2.19) as

$$I_1 = \frac{V_{11}}{r_{11}} - \frac{V_{22}}{r_{21}} + \frac{V_{33}}{r_{31}} \tag{2.20}$$

where r_{11}, r_{21}, r_{31} are coefficients having dimensions of resistance.

Solutions for currents I_2 and I_3 are found in a similar fashion. The general equations and solutions for a network containing any number of meshes are presented in Appendix B.

* A program, written in BASIC, for solving simultaneous equations is listed in Appendix C.

2.6 The superposition and reciprocity theorems

2.6.1 Superposition

Returning again to the single mesh circuit of fig. 2.5 and its solution (2.9) we see that the current may be written

$$I=\frac{V_1}{R_1+R_2}-\frac{V_2}{R_1+R_2}$$

The term $V_1/(R_1+R_2)$ represents a current due to V_1 acting alone and which flows in a clockwise direction, while the term $V_2/(R_1+R_2)$ represents a current due to V_2 acting alone and flowing in a counter-clockwise direction (indicated by the negative sign). The actual current I is formed by the *superposition* of these individual currents.

The same superposition principle is evident in the solution (2.20) for the current I_1 in the three-mesh network. This may be written in full as

$$\begin{aligned}I_1&=\frac{V_1}{r_{11}}-\frac{V_2}{r_{11}}-\frac{V_2}{r_{21}}+\frac{V_3}{r_{21}}-\frac{V_4}{r_{31}}\\&=\frac{V_1}{r_{11}}-V_2\left(\frac{1}{r_{11}}+\frac{1}{r_{21}}\right)+\frac{V_3}{r_{21}}-\frac{V_4}{r_{31}}\end{aligned}$$

We see that I_1 is composed of four individual currents, each due to one of the voltage sources acting in the circuit alone. Each of the individual currents depends only on the value of the relevant voltage source, and is independent of the values of the other voltage sources acting in the circuit. The general proposition demonstrated by these two examples is embodied in the superposition theorem which may be stated as follows:

The total current flowing in any branch of a network containing ideal voltage sources is equal to the algebraic sum of the currents which would flow in that branch if each of the ideal voltage sources in turn acted alone, the other sources being reduced to zero. (A formal proof of this theorem is contained in Appendix B.)

It follows from this theorem and Ohm's Law that the voltage between any two nodes in a network is equal to the algebraic sum of the voltages arising between those nodes due to each of the voltage sources in the network acting alone.

The superposition theorem is of considerable importance in the theory of linear network analysis since it provides a starting point for the development of several other useful theorems and techniques. It is also sometimes used as a practical alternative to the method of mesh analysis for finding the current in a specified branch of a network. To illustrate this we consider again the circuit of fig. 2.6 (repeated in fig. 2.9(*a*)). To find the current in, say, the branch CD containing R_3 we determine the current in this branch due to each of the sources V_1 and V_2 acting alone. Let I_1 be the current due to V_1 with V_2 reduced to zero as in fig. 2.9(*b*). (Note that V_2 is reduced to zero by replacing it with a short circuit *not* by open circuiting the branch EF.) In this modified circuit we see that R_2 and R_3 form a parallel combination whose resistance is given by $R_2R_3/(R_2+R_3)$. The total resistance across V_1 is therefore $R_1+R_2R_3/(R_2+R_3)$ and the current delivered by the source V_1 is given by

$$\frac{V_1}{R_1+R_2R_3/(R_2+R_3)}$$

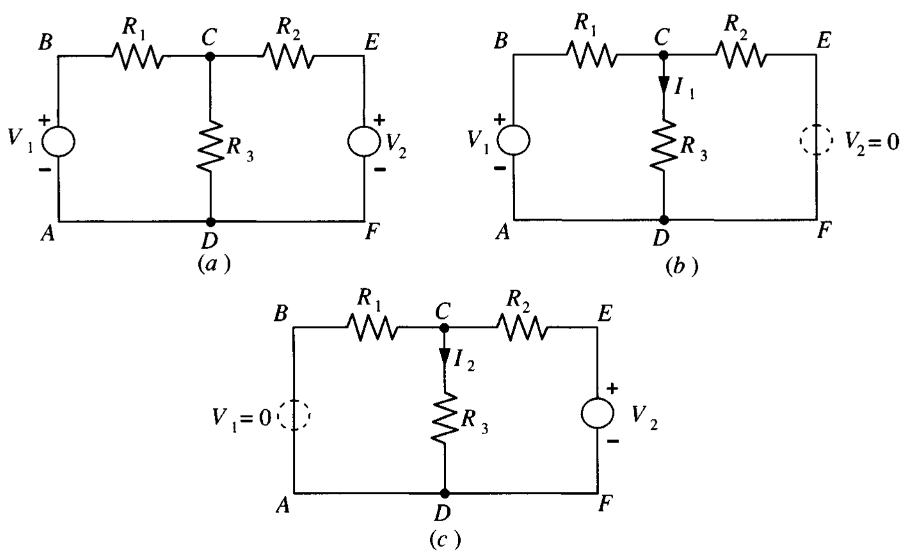
Now R_2 and R_3 together form a current divider hence, by (2.4), the current I_1 is given by the fraction of the total current equal to $R_2/(R_2+R_3)$, that is

$$I_1=\frac{V_1}{R_1+R_2R_3/(R_2+R_3)}\cdot\frac{R_2}{R_2+R_3}$$

or

$$I_1=\frac{R_2V_1}{R_1(R_2+R_3)+R_2R_3}$$

Fig. 2.9. Illustrating the superposition theorem.



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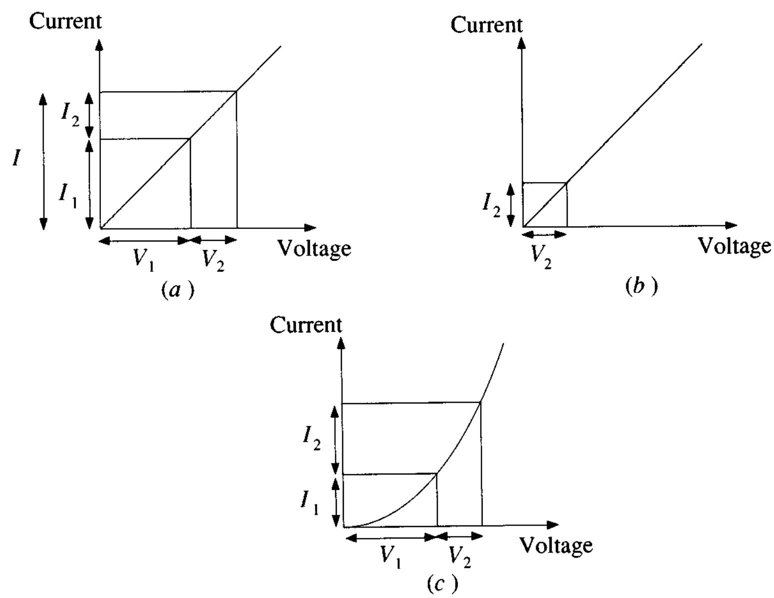
Similarly, with V_2 acting alone as shown in fig. 2.9(*c*), the current I_2 is given by

$$I_2 = \frac{R_1 V_2}{R_2(R_1 + R_3) + R_1 R_3}$$

The total current when both sources act together is, according to the superposition theorem, $(I_1 + I_2)$ directed from C to D. The same result may, of course, be obtained by solving (2.15) and (2.16) for the two mesh currents and taking the appropriate difference. Generally speaking, the use of the superposition principle in practical problems is of advantage only when it is desired to find one particular branch current in a network involving not more than two or three meshes.

The superposition principle is a direct consequence of the properties of a linear network. Consider the situation depicted in fig. 2.10 in which I_1 and I_2 are the contributions to the total current I which flow in one branch of a network as a result of sources V_1 and V_2 acting together in that network. In the linear case (fig. 2.10(*a*)) we see that each individual contribution is unaffected by the value of the source voltage producing the other. In other

Fig. 2.10. Voltage–current characteristics of linear and non-linear circuits. (*a*) Linear: voltages V_1 and V_2 produce currents I_1 and I_2. (*b*) V_1 reduced to zero, V_2 still produces I_2. (*c*) Non-linear: response I_2 to V_2 depends on value of V_1.



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words, V_1 could take any value including zero (fig. 2.10(*b*)) without affecting the contribution I_2 due to V_2. In the non-linear case, however, (fig. 2.10(*c*)) it is clear that the contribution I_2 to the total current is dependent not only on V_2 but also on the particular value of V_1. The current due to each source acting alone cannot, therefore, be superposed to find the total current when both act together.

2.6.2 Reciprocity

The reciprocity theorem states that:

The current produced in any one branch of a network by an e.m.f. acting in a second branch, is equal to the current which would be produced in the second branch if the e.m.f. were transferred to the first branch.

Alternatively, we may state: *the voltage produced at any one node of a network by a current source acting at another node, is equal to the voltage at the first node if the current source were transferred to the second node.* A proof of this theorem is presented in Appendix B.

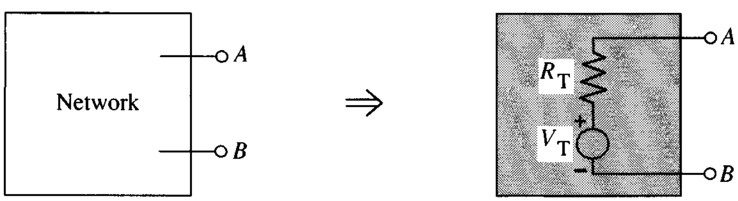
Like the superposition theorem the reciprocity theorem can occasionally save work in practical problems but it is mainly of value for the theoretical insights which it can provide. An example of this will be encountered in the theory of bridge circuits contained in section 3.10.

2.7 Thévenin's theorem

According to Thévenin's theorem any network consisting of linear resistances and ideal sources, having two terminals AB (fig. 2.11) may be replaced by an equivalent network consisting of a single resistance R_T in series with a single ideal voltage source V_T; in other words the network may be replaced by a practical voltage source of the form shown in fig. 2.11(*b*).

The theorem asserts that: *so far as any external network connected across AB is concerned, the given network and its equivalent are indistinguishable if V_T is made equal to the e.m.f. that would appear across AB on open circuit, and*

Fig. 2.11. Thévenin's theorem.



(*a*) *General linear network* (*b*) *Thévenin equivalent network*

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if R_T is made equal to the resistance that would exist between AB when all sources internal to the given network are rendered inoperative. By 'inoperative' we mean that voltage sources must be replaced by short circuits and current sources must be replaced by open circuits.

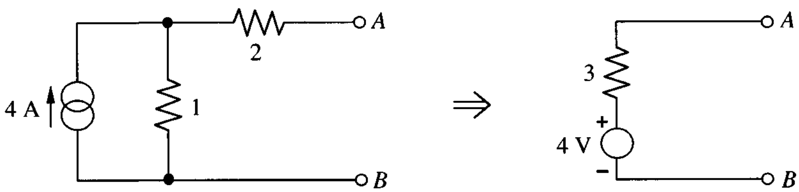
If the detailed configuration of a circuit is known, the Thévenin equivalent may be found theoretically. For example, the circuit shown in fig. 2.12(*a*) contains one current source and two resistances, the values of which are given. We may deduce by inspection that the voltage across *AB* on open circuit is 4 V, the terminal *A* being positive with respect to *B*. (No current flows through the 2 Ω resistance when *AB* are open circuit so that under these conditions the open circuit voltage must be identical to that across the current source.) The resistance across *AB* when the current source is made inoperative is, by inspection, 3 Ω. The Thévenin equivalent network is, therefore, as shown in fig. 2.12(*b*).

Frequently the internal details of a practical two-terminal circuit may not be known with exactitude because of limitations of the lumped modelling technique, variations associated with manufacturing tolerances of components, etc. In such cases the circuit in question can be completely characterized by its Thévenin equivalent, the elements of which can be determined by measurements made external to the circuit.

One method of finding the Thévenin equivalent of a network is illustrated in fig. 2.13(*a*). A variable resistor R_L is connected to the terminals *AB*, and the current drawn by this resistor is measured by means of an ammeter. The terminal voltage is measured by a voltmeter the resistance of which must be sufficiently high that the current flowing through it does not affect the measurement of the current through R_L.

A series of measurements of voltage and current are made for various settings of the resistor, and the results plotted in the form of a graph of voltage versus current. Provided the network under test is linear, the graph will be of straight-line form similar to that shown in fig. 2.13(*b*). The slope of the line gives R_T and the intercept on the voltage axis gives V_T.

Fig. 2.12. Application of Thévenin's theorem.



(*a*) Actual circuit

(*b*) Thévenin equivalent

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To prove Thévenin's theorem we consider the arrangement shown in fig. 2.14. P represents the given linear network with terminals AB, and Q is an external linear network to be connected to P. Let V_T be the open circuit e.m.f. across AB, and let R_T be the resistance between AB when all sources inside P are made inoperative. We may assume, without loss of generality, that Q does not contain any sources since, by the superposition theorem, the effects of these would be independent of any currents caused by the sources in P. Let the resistance across CD be R_Q. We wish to show that when Q is connected to P, the resulting current that flows between the two networks is precisely the same as that which would flow if P were replaced by the series combination of V_T and R_T.

We now consider the situation when an ideal voltage source V is connected between A and C, as indicated by the dashed lines in fig. 2.14, the circuit being completed by joining B to D. The resulting current that flows

Fig. 2.13. Determination of the Thévenin equivalent circuit by measurement.

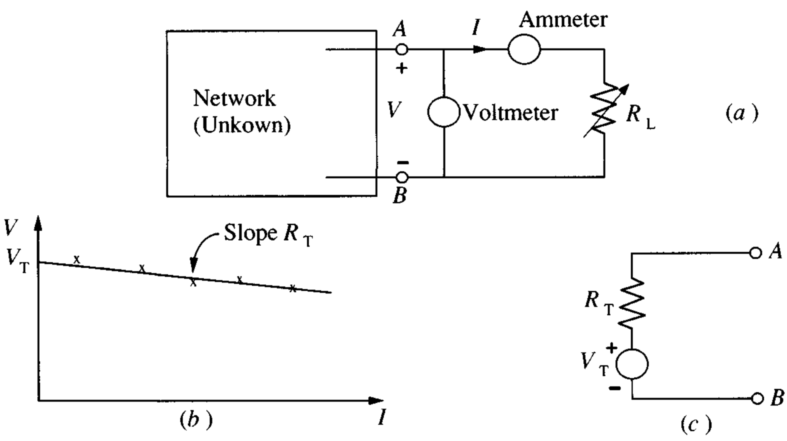
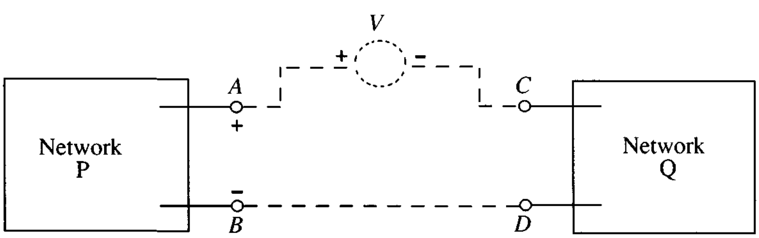


Fig. 2.14. Circuit for proof of Thévenin's theorem.



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round the path $ACDB$ can, according to the superposition theorem, be regarded as being made up of two components: (1) a current due to the combined action of all the sources in P; (2) a current due to the additional source V. Now let V be adjusted so that it is equal to V_T; with the polarities shown the current must fall to zero since there is then no net e.m.f. acting round the path $ACDB$. Thus, since the total resistance round this path is $(R_T + R_Q)$, the current (2) must be $V_T/(R_T + R_Q)$ flowing counterclockwise, and this must be equal to the current (1) flowing in the opposite direction. But $V_T/(R_T + R_Q)$ is precisely the current that would flow between P and Q if P were replaced by an ideal voltage source V_T in series with a resistance R_T. This proves the theorem. A formal, mathematical proof of this theorem is presented in Appendix B.

2.8 Worked example

A temperature sensitive resistor (thermistor) is used in a Wheatstone Bridge circuit for the measurement of the temperature of a water bath as shown in fig. 2.15(*a*). The temperature is indicated on the meter M, which has a sensitivity of 50 μA at full-scale deflection and a resistance of 300 Ω. A 4 V battery, of negligible internal resistance, is used to energize the bridge. The thermistor has a resistance of 1000 Ω at a water temperature of 50 °C and its resistance decreases by 5% for each degree increase in temperature.
(a) Find the value of the resistance to which R_4 must be set to give zero reading on the meter when the temperature of the bath is 50 °C.
(b) With R_4 set as in (a) above, find the temperature of the bath corresponding to full-scale deflection on the meter.

Solution. (a) We first label the nodes of the circuit $ABCD$ as shown. The required condition of zero current through the meter is obtained when the potential of node A is the same as that of B, that is, (referring potentials to the node D) when $V_{AD} = V_{BD}$. These two voltages are most easily found by recognizing that the bridge, under the given zero current condition, constitutes two separate voltage divider circuits as shown in fig. 2.15(*b*). To emphasize this the circuit has been split into two parts with a separate source for each part; it will be appreciated that this makes no essential difference to the operation of the circuit.

The required expressions for V_{AD} and V_{BD} may be found using (2.3). Thus, for zero meter current:

$$V_{AD} = V_{BD}$$

$$\frac{R_2}{R_1 + R_2} V_s = \frac{R_4}{R_3 + R_4} V_s$$

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which upon rearrangement becomes

$$R_1R_4 = R_2R_3 \tag{2.21}$$

This relationship expresses the so-called *balance condition* for the Wheatstone bridge. Note that (2.21) is independent of the source voltage V_s.

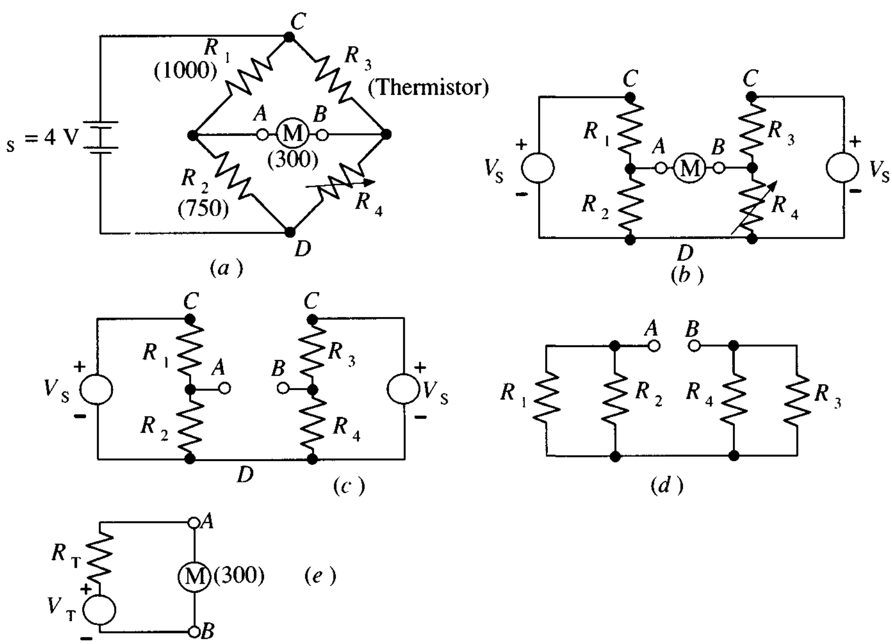
At a temperature of 50 °C the thermistor has a resistance of $R_3 = 1000\,\Omega$ hence, substituting into (2.21) the given resistance values for R_1 and R_2 we obtain

$$R_4 = \frac{R_2R_3}{R_1} = \frac{750 \times 1000}{1000} = 750\Omega$$

(b) We are required to find a relationship between the current flowing through the meter and the resistance R_3 of the thermistor from which the temperature for full-scale deflection may be deduced. Thévenin's theorem may be used to find such a relationship.

The general approach in applying Thévenin's theorem to a problem of this kind consists in removing from the circuit in question the branch through which it is desired to find the current. The Thévenin equivalent is then found of the remaining network that exists across the two terminals

Fig. 2.15. Circuits for worked example.



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exposed as a result of removing the branch. Finally, the branch is reconnected to the equivalent circuit, thus forming a single mesh circuit from which the current is easily found.

In this example the branch of interest is that containing the meter; on its removal, terminals AB are exposed and we see that the circuit remaining is essentially that of fig. 2.15(c).

Since the Thévenin equivalent voltage V_T is, by definition, that voltage which exists across AB under open circuit conditions, we may again (as in part (a) of this example) use the voltage divider principle to determine the voltage across AB. Thus, V_T is given by

$$V_T = V_{AD} - V_{BD} = \frac{R_2}{R_1 + R_2} V_s - \frac{R_4}{R_3 + R_4} V_s$$

Substituting actual values gives:

$$V_T = \frac{750 \times 4}{1000 \times 750} - \frac{750 \times 4}{R_3 + 750}$$

To find the Thévenin equivalent resistance R_T across AB we render the internal source V_s inoperative by replacing it by a short circuit. The circuit of fig. 2.15(c) then reduces to that shown in fig. 2.15(d). It is seen that R_1 and R_2 now form a parallel combination; likewise R_3 and R_4 form a parallel combination. R_T is therefore given by (using the 'product-over-sum' rule):

$$R_T = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}$$

substituting actual values:

$$R_T = \frac{1000 \times 750}{1000 + 750} + \frac{R_3 \times 750}{R_3 + 750}$$

The simplified circuit with the meter branch reconnected is shown in fig. 2.15(e). Since the current for full-scale deflection is $50\,\mu$A we obtain the following relationship for the circuit:

$$50 \times 10^{-6} = \frac{V_T}{R_T + 300}$$

Substituting the expressions for V_T and R_T into this equation and performing some algebraic manipulation we find that $R_3 = 1062\,\Omega$.

Now the relationship between the temperature T and the resistance of the thermistor is $R_3 = 0.95^{(T-50)} \times 10^3$, hence the temperature corresponding to a resistance of $R_3 = 1062\,\Omega$ is given by

$$T=\frac{\log\dfrac{1062}{1000}}{\log 0.95}+50=48.8\,^{\circ}\mathrm{C}$$

An alternative approach to this problem would be to use mesh analysis, but this involves setting up and solving three mesh equations – a somewhat tedious procedure. The power of the Thévenin approach lies in the fact that removal of one branch from a circuit often renders the remaining part of the circuit amenable to a simple form of analysis from which the equivalent circuit can be found. In this example, removing the branch containing the meter reduces the number of meshes from three to two; furthermore, by employing the artifice of the voltage divider, mesh analysis is avoided altogether.

2.9 Network transformations

If two networks have the same Thévenin equivalent circuit at corresponding pairs of terminals or ports, then, so far as any external connections are concerned, the two networks are indistinguishable. This corollary of Thévenin's theorem allows us to establish the conditions for which two or more networks are electrically equivalent. We are thus able to replace a network or part of a network with a different but electrically equivalent network, and this is of considerable practical significance in the analysis of circuits. Such a procedure is known as network transformation. Two important examples of network transformation will now be considered.

2.9.1 The Thévenin–Norton transformation

The circuit shown in fig. 2.16(*a*) is a practical voltage source of the type introduced in section 2.1. Another type of source, called a *practical current source*, is shown in fig. 2.16(*b*). This consists of an ideal current source in *parallel* with a linear resistance. According to Thévenin's theorem the two circuits are electrically equivalent if they both present the same open circuit voltage at their terminals, and if they both present the same terminal resistance when their sources are made inoperative; in other words, if they both possess the same Thévenin equivalent circuit. The circuit of fig. 2.16(*a*) is, of course, its own Thévenin equivalent.

The open circuit voltage of the practical current source is IR_2, and its resistance when the ideal current source is replaced by an open circuit is R_2. Therefore, the two circuits are equivalent if

$$IR_2=V \text{ and } R_2=R_1 \tag{2.22}$$

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They are, of course, equivalent only so far as external connections are concerned; internally the two circuits are fundamentally dissimilar since the practical current source dissipates power continually in its own resistance even when its terminals are open circuit. Note that the equivalence holds only if the direction of I in fig. 2.16(b) is such that the same voltage polarity is produced at the terminals of the two circuits.

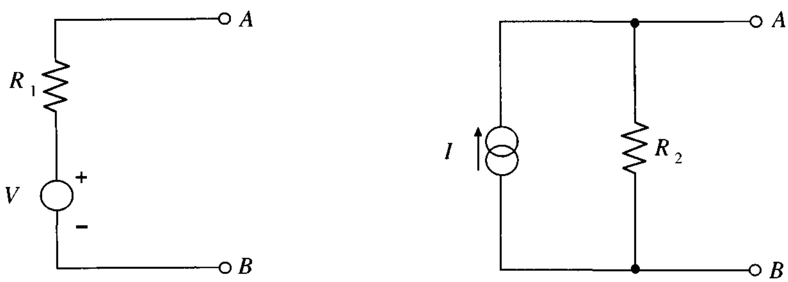
We conclude, therefore, that an ideal voltage source of magnitude V in series with a resistance R is equivalent to an ideal current source of magnitude V/R in parallel with a resistance R (or a conductance $G = 1/R$). The two circuits are duals of one another. Thévenin's theorem may therefore be restated in the following form:

Any network with two accessible terminals AB may, so far as external circuits are concerned, be replaced by an ideal current source I in parallel with a conductance G, where I is the current that would flow if the terminals AB were short-circuited, and G is the conductance across AB if the current source were open-circuited.

The theorem in this form was first stated by E.L. Norton, and it is consequently known as Norton's theorem, although it should be realized that it is not fundamentally different from Thévenin's theorem. The procedure of replacing the circuits of fig. 2.16, one by the other, using the relationships (2.22), is known as the Thévenin–Norton transformation.

Many practical voltage sources exhibit near ideal characteristics over part of their working range, and it is permissible to represent them by an ideal voltage source without series resistance. In this case the Thévenin–Norton transformation cannot be applied since the parameters of the Norton circuit are indeterminate.

Fig. 2.16. The Thévenin–Norton transformation. By Thévenin's theorem the circuits are equivalent at AB if $I = V/R_2$ and $R_2 = R_1$.



(a) Practical voltage source (Thévenin circuit)

(b) Practical current source (Norton circuit)

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2.9.2 The star-delta transformation

The use of Thévenin's theorem for establishing the relationships expressing equivalence between networks is not confined to those possessing a single accessible pair of terminals. The same principle may be applied to multi-terminal, or multi-port networks, by considering each corresponding pair of ports in turn, all other ports being open-circuited. This procedure may be illustrated with reference to the two circuits shown in fig. 2.17; one a star-connected arrangement of three resistances, the other a delta-connected arrangement. Since there are no current or voltage sources included in these circuits, we need consider only the resistances presented at each of the corresponding ports.

Considering first port(1) (terminals AC), with terminal B open-circuit, the resistance at this port is (R_a+R_c). At the corresponding port(1′) (terminals $A'C'$), with the terminal B' open-circuit, the resistance is $R_1//(R_2+R_3)$. The condition for equivalence is, therefore,

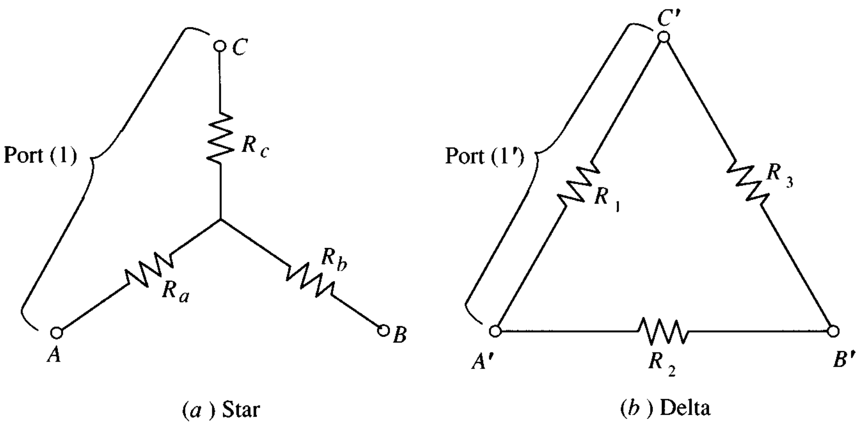
$$R_a+R_c=\frac{R_1(R_2+R_3)}{R_1+R_2+R_3}$$

Similarly, by comparing the resistances at the two remaining ports, we obtain

$$R_c+R_b=\frac{R_3(R_1+R_2)}{R_3+R_1+R_2}$$

and

Fig. 2.17. The star–delta transformation.



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$$R_b+R_a=\frac{R_2(R_3+R_1)}{R_2+R_3+R_1}$$

After algebraic manipulation the following sets of relations are established:

$$\begin{aligned} R_1&=R_a+R_c+\frac{R_aR_c}{R_b}; & R_a&=\frac{R_1R_2}{R_1+R_2+R_3} \\ R_2&=R_b+R_a+\frac{R_bR_a}{R_c}; & R_b&=\frac{R_2R_3}{R_1+R_2+R_3} \\ R_3&=R_c+R_b+\frac{R_cR_b}{R_a}; & R_c&=\frac{R_3R_1}{R_1+R_2+R_3} \end{aligned} \tag{2.23}$$

The procedure of conversion between the two circuits shown in fig. 2.17, using the relations (2.23), has come to be variously known as the star–delta, Y–Δ, Y–mesh transformation, and it finds application particularly in the analysis of power systems. The same transformation occurs in the theory of two-port networks and is there known as the Tee–Pi(T–π) transformation, so-called because in this context the circuits of fig. 2.17 are drawn rather differently and they resemble the shapes from which the name derives (see section 8.8).

The star–delta transformation is a particular case of a more general theory relating to multi-port networks, which has become established as *Rosen's theorem* (see ref. 1).

2.10 Nodal analysis

It will be recalled that in mesh analysis, currents are assigned to each of the meshes in the circuit under consideration, and the mesh equations are formulated by applying Kirchhoff's voltage law to each mesh in turn. In nodal analysis the 'dual' of this procedure is adopted: voltages are assigned to each node and the nodal equations are formulated by applying Kirchhoff's current law.

Node voltages are specified by choosing one node in the circuit as the reference with respect to which voltages at all other nodes are defined. Thus, in fig. 2.18, node O is chosen as the reference and V_A signifies the voltage of node A with respect to that of node O; similarly V_B is the voltage of node B with respect to node O. (It is not usual to employ a double-subscript notation (such as V_{AO}, V_{BO}) for this purpose since the second subscript would merely be repeated.) Any node may, in principle, be chosen as the reference but the nodal equations take their simplest form if the node to which the greatest number of elements is attached is selected. In many practical circuits this node will be the common (ground) terminal and will constitute an obvious choice.

In the circuit of fig. 2.18, the voltage at node C is specified and has the value V; V_A and V_B are the two unknown voltages that have to be determined by setting up and solving two independent nodal equations. We say that such a circuit contains two *independent nodes.*

In applying Kirchhoff's current law at a node it is convenient to use the form (1.13) (section 1.6.1)

$$\sum i = 0 \tag{1.13}$$

where $\sum i$ is interpreted as the algebraic sum of the currents flowing *away* from the node. Thus, at the node A in fig. 2.18, we have

$$I_{AO} + I_{AB} + I_{AC} = 0$$

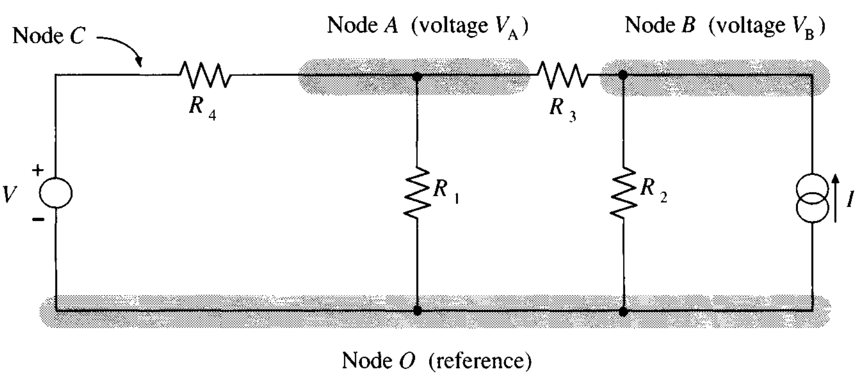
Now the current I_{AO} flowing through R_1 is clearly V_A/R_1; the current I_{AB} is $V_{AB}/R_3 = (V_A - V_B)/R_3$, and the current $I_{AC} = V_{AC}/R_4 = (V_A - V)/R_4$. Hence, the nodal equation at A is

$$\frac{V_A}{R_1} + \frac{V_A - V_B}{R_3} + \frac{V_A - V}{R_4} = 0 \tag{2.24}$$

Notice that the first member of each of the terms in this equation is the assigned voltage of node A itself. This is a consequence of choosing the positive direction of current as that flowing away from a node.

At node B we have a current source feeding current into the node; this may be treated as a current $(-I)$ flowing away from the node. As before, the current through each resistance attached to node B is found by taking the

Fig. 2.18. Nodal analysis: assignment of node voltages.



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difference voltage divided by the value of the resistance. Thus, at B the nodal equation is

$$-I+\frac{V_B}{R_2}+\frac{V_B-V_A}{R_3}=0 \tag{2.25}$$

Rearranging the above equations we obtain

$$\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\right)V_A-\frac{1}{R_3}V_B=\frac{V}{R_4} \tag{2.26}$$

$$-\frac{1}{R_3}V_A+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)V_B=I \tag{2.27}$$

Solving these two equations will yield specific expressions for V_A and V_B. If the currents in the various branches are required, these may be found by substitution in the appropriate difference terms in the nodal equations; for example, the current I_{BA} through R_3 is given by $(V_B-V_A)/R_3$.

In the above nodal equations we observe that the coefficient of V_A in (2.26) is simply the sum of the conductances attached to node A; this is termed the *self-conductance* at A. Likewise the self-conductance at node B appears as the coefficient of V_B in (2.27). The coefficient of V_B in (2.26) and of V_A in (2.27), namely $(1/R_3)$, is the *mutual-conductance* between nodes A and B. The mutual-conductance terms in the nodal equations are always negative.

On the right-hand side of each nodal equation we have a term representing the current injected into the node concerned from the source attached to that particular node. In the case of node A (equation 2.26) the ideal voltage source V together with R_4 constitute the effective current source, as will be readily apparent if a Thévenin–Norton transformation is carried out according to the principles discussed in section 2.9.1. When this is done, we may redraw the circuit of fig. 2.18 as shown in fig. 2.19, replacing the practical voltage source by an equivalent practical current source. It is now immediately obvious that the current injected into node A has a magnitude V/R_4.

The equations obtained in nodal analysis possess a formal similarity to those obtained in mesh analysis (equations 2.17), and a similar subscript notation is employed when it is required to express them in a general form. Thus, for a circuit such as that in fig. 2.18 in which there are two independent nodes we may write

$$\left.\begin{aligned} G_{11}V_1+G_{12}V_2&=I_{11}\\ G_{21}V_1+G_{22}V_2&=I_{22}\end{aligned}\right\} \tag{2.28}$$

where G_{11}, G_{22} are the self-conductances at the first and second independent nodes; $G_{12}=G_{21}$ is the mutual-conductance between them; and I_{11} and I_{22} are the net currents injected into the first and second nodes from the ideal current sources attached to them.

It will be appreciated that by carrying out the transformation shown in fig. 2.19, (which may be done mentally), and by using the concept of self- and mutual-conductances, the nodal equations for the circuit of fig. 2.18 could have been written directly in the form (2.26)/(2.27). The reader unfamiliar with network analysis is, however, advised to set up the equations initially in the form (2.24)/(2.25) as in this way there is less likelihood of error.

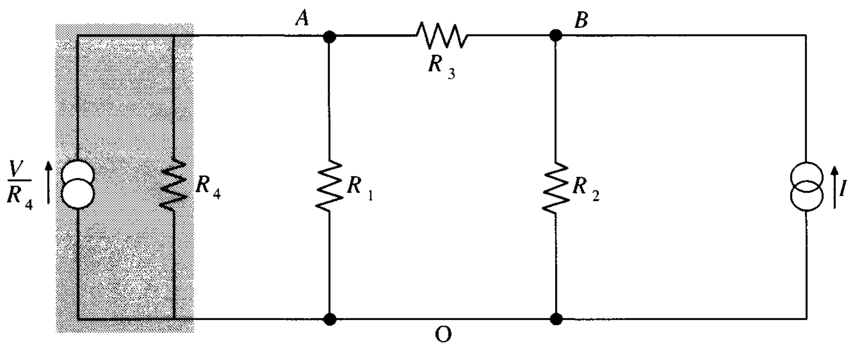
In general a circuit containing N independent nodes will give rise to N independent nodal equations, the equation at the kth node being of the form

$$\sum_{n=1}^{N} G_{kn}V_n = I_{kk} \tag{2.29}$$

This is a balance equation which expresses the continuity of current at a node and which corresponds to Kirchhoff's current law in the form (1.12).

As a final point of general interest in connection with the nodal equations, we may observe that each nodal equation is an expression of the superposition principle. Consider, for example, the first equation in (2.28). The first term represents the current flowing away from node (1) through all the conductances attached to that node, and with the second node voltage set to zero. The second term represents the current flowing from node (2) to node (1) with the first node voltage set to zero. The superposition of these two currents gives the net current flowing away from node (1), and this equals the current injected into this node from the attached current sources.

Fig. 2.19. The Thévenin–Norton transformation applied to the circuit of fig. 2.18.



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2.11 Comparison of mesh and nodal analysis

For circuits containing current sources, mesh analysis using the cyclic current method is generally less straightforward than nodal analysis. The voltage across a current source is constrained by the circuit in which it resides; it is necessary, therefore, to assign voltage drops to all current sources within the circuit before KVL can be applied to the loops in which they are contained. These unknown voltage drops must then be eliminated by combining the appropriate number of equations. Such complication can sometimes be avoided by assigning branch currents, rather than mesh currents, and by making a judicious choice of loops so as to avoid branches containing current sources. An alternative approach is to first transform practical current sources to practical voltage sources using the inverse Thévenin–Norton transformation, but rarely does this result in a more concise and labour saving solution than can be attained by other means.

Nodal analysis, on the other hand, suffers from no such constraints. It may be used freely for circuits containing both voltage and current sources, as we have seen in the case of the circuit of fig. 2.18, and it more often than not affords a method of solution involving fewer simultaneous equations than mesh analysis. Exceptions to this general rule include the symmetrical ladder type of circuit discussed in section 2.15.3.

We have seen that a network containing M independent meshes, that is, one in which there are essentially M unknown mesh currents to be found, requires the solution of M simultaneous equations. A network possessing N independent nodes leads to N nodal equations. By determining M and N for a particular circuit we are often able to make a rational choice as to which of the two methods of analysis to use. Unless we are dealing with a very large and complex circuit, it is an easy matter to determine N: count the total number of nodes N_T and the number of voltage sources N_V, then N is given by

$$N = N_T - N_V - 1 \tag{2.30}$$

The reason why one must subtract N_V nodes from the total N_T in this expression stems from the fact that each voltage source is connected to the circuit at two nodes; the voltage of one of these is, therefore, defined with respect to the other and only one can be counted as an independent node. The total N_T also contains the reference node and this must be subtracted as well.

The determination of M often presents considerably greater difficulties, particularly if the circuit is drawn with branches crossing one another. Two ways of drawing the Wheatstone bridge circuit are illustrated in fig. 2.20. We have no difficulty in distinguishing three independent meshes in fig.

2.20(*a*) but these are not nearly as apparent in fig. 2.20(*b*). Some circuit configurations are not mappable onto a plane surface without the necessity for crossing branches, so that it is not possible to get round this difficulty (as in this case) simply be redrawing the circuit.

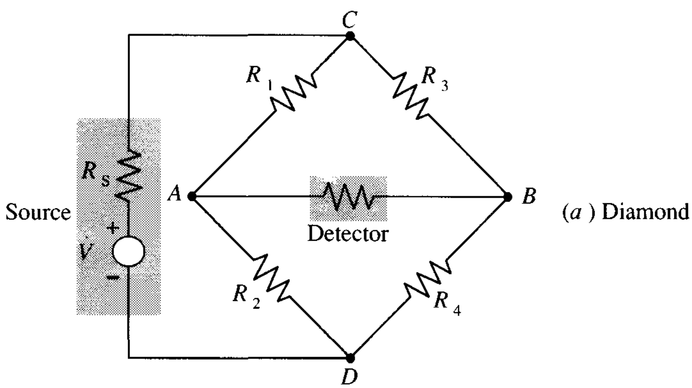
For such circuits it is often easier to determine M indirectly by first counting the total number of nodes N_T, and then applying the following relation (derived from a theorem of mathematical topology):

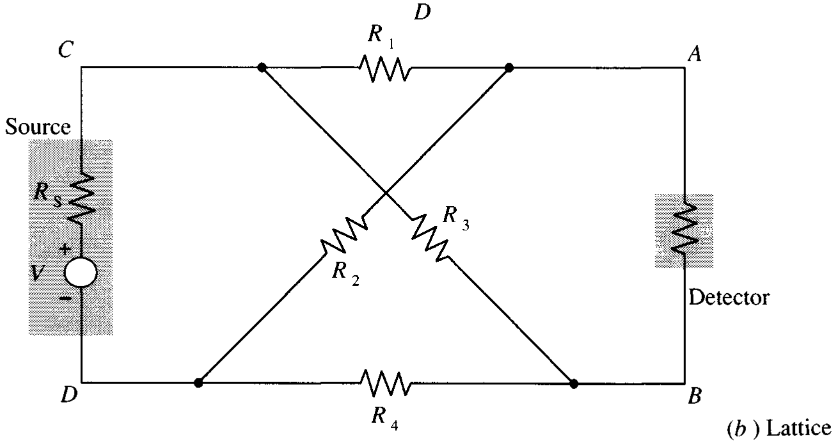
$$M = E - N_T + 1 \tag{2.31}$$

where E is the number of elements in the circuit including sources.

Thus, considering fig. 2.20(*b*), we see that $N_T = 5$, $E = 7$ whence from (2.31), $M = 3$. Also applying (2.30) with $N_V = 1$, we deduce that $N = 3$. For this circuit, the number of simultaneous equations required is the same; however, the reader may care to check that if the source internal resistance R_S is negligibly small, then nodal analysis confers an immediate advantage.

Fig. 2.20. The Wheatstone Bridge circuit drawn in alternative ways.





The worked example of section 2.4 is also slightly easier using nodal rather than mesh analysis because the circuit possesses essentially a single independent node.

Judicious choice of the reference node can sometimes lead to a simpler and more direct solution using nodal analysis even when N and M are the same. For example in fig. 2.20, if we were interested only in finding the voltage across the detector, we could choose B as the reference node and solve for the voltage V_A to give the detector voltage directly. Using mesh analysis on the other hand two cyclic currents would have to be found, their differences calculated and finally Ohm's law applied.

2.12 Worked example

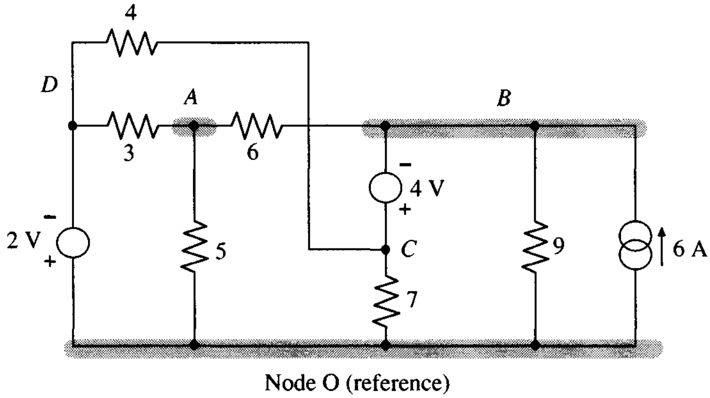
In the circuit of Fig. 2.21 find the magnitude and direction of the current through the 2 V source, and the magnitude and polarity of the voltage across the 6 A current source.

Solution:

The circuit contains a total of five nodes; these are identified and labelled in the figure. Since there are nine elements the number of independent meshes is, from (2.31), $M = 9 - 5 + 1 = 5$. Note, however, that one mesh contains a current source so that a solution by cyclic current mesh analysis would have to be preceded by an inverse Thévenin–Norton transformation. Alternatively, branch currents could be assigned. In either case the minimum number of simultaneous equations would be four.

There are two voltage sources in the circuit so that the number of

Fig. 2.21. Circuit for worked example on nodal analysis; A and B are the two independent nodes.



independent nodes is, from (2.30), $N=5-2-1=2$. Thus nodal analysis involves only two simultaneous equations.

Node O is chosen as the reference because, (a) it has the greatest number of elements attached, (b) by solving for the node voltage V_B, the voltage across the current source can be found directly. Of the remaining nodes, D is specified (-2 V), and C is specified with respect to B ($+4$ V). Hence, nodes A and B are identified as the two independent nodes.

At A the nodal equation is

$$I_{AB}+I_{AO}+I_{AD}=0$$

$$\frac{V_A-V_B}{6}+\frac{V_A}{5}+\frac{V_A-(-2)}{3}=0$$

At B the nodal equation is

$$I_{BA}+I_{BC}+I_{BO}-6=0$$

To find the current I_{BC} we note that this is equal to the current flowing away from node C through the 4 Ω and 7 Ω resistances. The voltage of C with respect to O is (V_B+4), hence,

$$I_{BC}=I_{CO}+I_{CD}=\frac{V_B+4}{7}+\frac{(V_B+4)-(-2)}{4}$$

The complete nodal equation at B is then

$$\frac{V_B-V_A}{6}+(V_B+4)(\tfrac{1}{7}+\tfrac{1}{4})+\tfrac{1}{2}+\frac{V_B}{9}-6=0$$

After algebraic manipulation the two nodal equations become:

$$(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{3})V_A-\tfrac{1}{6}V_B=-\tfrac{2}{3}$$
$$-\tfrac{1}{6}V_A+(\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{4}+\tfrac{1}{9})V_B=6-\tfrac{1}{2}-1-\tfrac{4}{7}$$

Solving we obtain $V_A=0.47$ V and $V_B=5.97$ V.

The voltage across the current source is therefore 5.97 V, B positive with respect to O.

The current through the 2 V source I_{DO} is given by

$$I_{DO}=I_{CD}+I_{AD}=\frac{(V_B+4)-(-2)}{4}+\frac{V_A-(-2)}{3}$$

Substituting the values for V_A and V_B found above we obtain $I_{DO}=3.82$ A, this is, the current flows through the source from D to O and has a magnitude of 3.82 A.

2.13 Analysis of networks containing dependent sources

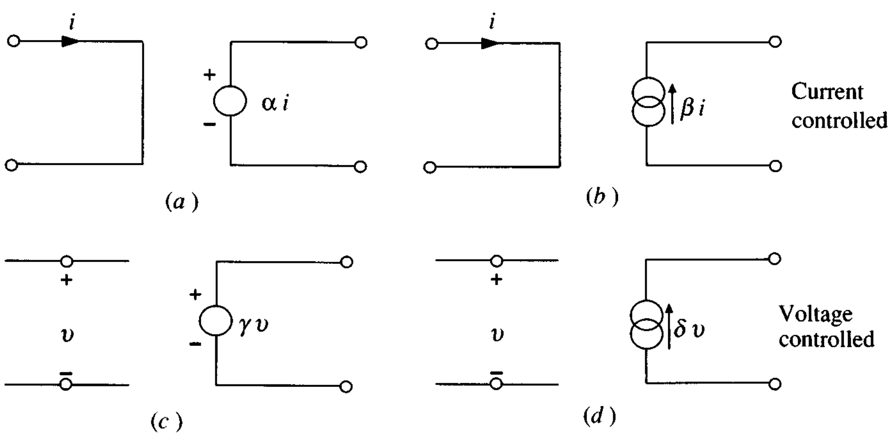
The theory presented in this chapter so far has been concerned with networks containing only independent ideal current and voltage sources. If dependent sources are present (see section 1.5) then the general mesh and nodal equations take a slightly different form, but the techniques of analysis remain essentially the same. The four possible types of dependent or controlled source are depicted in fig. 2.22. In each case the value of the source, voltage or current, is proportional to the value of a current or voltage in some other part of the network.

As an example of the analysis of such a circuit we consider the configuration shown in fig. 2.23. This type of circuit arises in the theory of bipolar transistors and is there termed the hybrid-π model (ref. 5). In this circuit the current source is controlled by the voltage established across the resistance R_1. V_1 is the voltage applied to the input port, and V_2 the voltage at the output port. Apart from the presence of the dependent current source, this circuit is similar in all respects to that shown in fig. 2.18, and the nodal analysis proceeds in a similar fashion. Voltages V and V_2, measured with respect to the reference node O, are assigned to the two independent nodes A and B.

The nodal equations are:

$$\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\right)V-\frac{1}{R_3}V_2=\frac{V_1}{R_4} \tag{2.32}$$

Fig. 2.22. Dependent (controlled) sources; α, β, γ, δ are control constants.



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$$-\frac{1}{R_3}V+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)V_2=-gV \tag{2.33}$$

These equations will be seen to be of similar form to (2.26) and (2.27), appertaining to the circuit of fig. 2.18. However, in this case we may transpose (2.33) to obtain

$$\left(g-\frac{1}{R_3}\right)V+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)V_2=0 \tag{2.34}$$

We now observe that the coefficients in (2.32) and (2.34) taken together no longer possess the symmetry about the leading diagonal, which is a characteristic of the equations for networks containing only independent sources. Solutions for V and V_2 are easily found since, from (2.34)

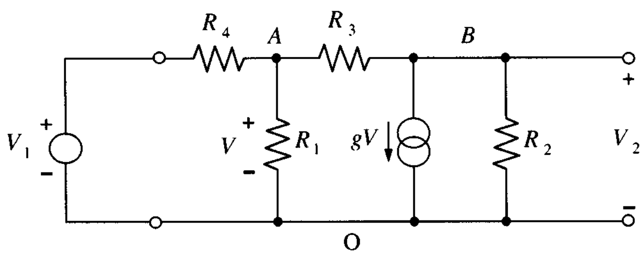
$$V=-V_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\Big/\left(g-\frac{1}{R_3}\right)$$

Hence, substitution in (2.32) gives an explicit expression for V_2.

It is frequently of interest to determine the Thévenin equivalent of a circuit containing a dependent source; in such a case however, the techniques that have been described so far in relation to circuits containing only independent sources are inadequate. In particular, the Thévenin equivalent resistance cannot be found simply by rendering all sources within the circuit inoperative and then determining the resistance of the network remaining. This is because for a circuit containing a dependent source the value of the Thévenin resistance, as well as the e.m.f. depends critically on the dependent source and its controlling parameters.

Two methods are commonly used for finding, analytically, the Thévenin equivalents of circuits containing dependent sources. These methods are applicable also to circuits containing only independent sources, but would not normally be used in such cases since they are rather more cumbersome than the techniques already described. However, the underlying principles

Fig. 2.23. Circuit containing a voltage-controlled current source with control constant g.



of the methods may be understood by reference to the circuits and techniques previously considered.

The first method to be described is conceptually similar to the practical method of finding the Thévenin equivalent of a circuit illustrated in fig. 2.13. If in this test circuit we make the load R_L infinite, the current will be zero and the voltmeter then registers the open-circuit voltage, which is identical to the Thévenin e.m.f. Let this voltage be $V_{o.c.}$ If next the load is made zero, the ammeter will register the short-circuit current; let this be $I_{s.c.}$. Now since the short-circuit current is that which results from the application of the full Thévenin e.m.f. across the Thévenin resistance, we have the following relations:

$$V_T = V_{o.c.} \text{ (open-circuit voltage)} \tag{2.35}$$

$$R_T = \frac{V_{o.c.} \text{ (open-circuit voltage)}}{I_{s.c.} \text{ (short-circuit current)}} \tag{2.36}$$

Note that placing a short circuit across the terminals of a circuit may, from a practical standpoint, result in damage. It must be stressed that the relation (2.36) provides the basis for an *analytical* approach to the determination of the Thévenin resistance, it does not represent a practical means of measuring this parameter.

The principles underlying the second method of finding the Thévenin equivalent of a circuit containing a dependent source may also be understood with reference to fig. 2.13. Suppose the load R_L is replaced by a voltage source of magnitude V_a, and suppose all internal sources in the circuit *with the exception of dependent sources* are made inoperative, then the ammeter will register a current I_a the magnitude of which will be determined by the effective Thévenin resistance according to

$$I_a = \frac{V_a}{R_T}$$

that is

$$R_T = \frac{V_a \text{ (applied voltage)}}{I_a \text{ (resulting current)}} \tag{2.37}$$

The Thévenin e.m.f. is found by determining the open-circuit voltage, but this is not, as was the case for the first method, a prerequisite for finding R_T. The relation (2.37) is sometimes used as a basis for the practical determination of R_T; here we are concerned only with its utility as a method of analysis.

Both of the methods described are illustrated in the following example.

2.14 Worked example

The characteristics of an operational amplifier may be modelled by the circuit shown in fig. 2.24(*a*). Resistances R_i and R_o are, respectively, the input and output resistances of the amplifier, and AV is a dependent source with control constant A.*

Figure 2.24(*b*) shows the circuit of a common type of electronic d.c. amplifier incorporating an operational amplifier modelled in accordance with fig. 2.24(*a*). The input signal to the amplifier is provided by the source V_1 at terminals AB, the output signal appears at terminals CD.

Find expressions for the Thévenin equivalent e.m.f. and the Thévenin equivalent resistance at the output terminals of the amplifier. Explain how these are related to the overall gain V_2/V_1 and the output resistance of the amplifier.

Solution: method 1

First we establish by means of a nodal analysis an expression for the open-circuit voltage V_2 which, according to (2.35), represents the Thévenin e.m.f. Note that for the purposes of this analysis the signal source V_1 must be regarded as an *internal independent* source.

Let V be the voltage at the node X, then, at node X

$$\frac{V-V_1}{R_1}+\frac{V-V_2}{R_2}+\frac{V}{R_i}=0 \tag{2.38}$$

and at node C

$$\frac{V_2-V}{R_2}+\frac{V_2-(-AV)}{R_o}=0 \tag{2.39}$$

Eliminating V from these equations we find

$$V_2=\left[\frac{R_o-AR_2}{AR_1+R_oR_1/R_i+R_1R_2/R_i+R_o+R_1+R_2}\right]V_1$$

The resistance R_i is, in practice, large in relation to the other resistances in the circuit in which case the above expression reduces to the simpler form:

$$V_2=\left[\frac{R_o-AR_2}{R_o+(1+A)R_1+R_2}\right]V_1 \tag{2.40}$$

* In the context of the theory of operational amplifiers the constant A is normally referred to as the *gain*. A full treatment of operational amplifiers and their application will be found in reference 5. The derivation of the model of fig. 2.24(*a*) is considered in section 8.3.

This expression gives the Thévenin equivalent circuit e.m.f. V_T. The quantity in square brackets gives the overall gain V_2/V_1 of the amplifier. If A is very large, the gain becomes, to a good approximation, $-R_2/R_1$.

To determine the Thévenin resistance we first find the current at the output port when terminals CD are short circuited. This current will be the sum of the currents flowing in R_2 and R_o, that is,

$$I_{s.c.}=\frac{V}{R_2}+\frac{(-AV)}{R_o}=\left[\frac{R_o-AR_2}{R_2R_o}\right]V \tag{2.41}$$

But from (2.38) with $V_2=0$, and again assuming that R_i is very large, V is given by

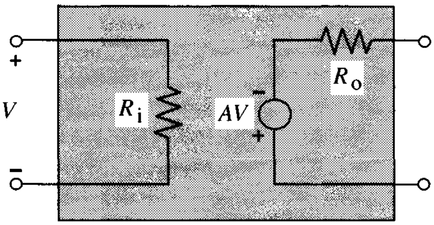
$$V=\frac{V_1}{R_1}\frac{1}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}=\frac{V_1R_2}{R_1+R_2}$$

Substituting for V in (2.41) we then obtain

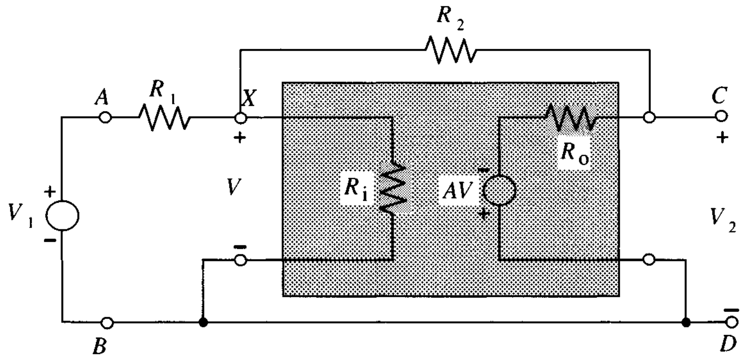
$$I_{s.c.}=\frac{V_1R_2}{(R_1+R_2)}\cdot\frac{(R_o-AR_2)}{R_2R_o}=\left[\frac{R_o-AR_2}{R_o(R_1+R_2)}\right]V_1 \tag{2.42}$$

Now by (2.36) the Thévenin resistance is given by the ratio of open-circuit voltage to short-circuit current, therefore, combining (2.40) and (2.42) gives

Fig. 2.24. Operational amplifier circuits for worked example.



(*a*) Model



(*b*) Amplifier

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$$R_T = \frac{V_{o.c.}}{I_{s.c.}} = \frac{(R_o - AR_2)V_1}{R_o + (1+A)R_1 + R_2} \cdot \frac{R_o(R_1+R_2)}{(R_o - AR_2)V_1}$$

or

$$R_T = \frac{R_o(R_1+R_2)}{R_o + (1+A)R_1 + R_2} \tag{2.43}$$

The Thévenin resistance, in the context of amplifier theory, is termed the *output* resistance of the overall amplifier.

Method 2

The Thévenin e.m.f. is found as in method 1. In order to determine R_T by means of the relation (2.37), terminals AB are short-circuited, thus removing the signal source, and a voltage source V_a is connected to the terminals CD. Note, however, that the dependent source is still active since the controlling voltage V is now derived from the applied voltage V_a via resistances R_1 and R_2. Let the polarity of V_a be such that the terminal C is positive with respect to D, and let the reference direction of the resulting current I_a be into the terminal C. For the reasons given above the effect of R_i will be ignored. The nodal equation at C is then

$$\frac{V_a}{R_1+R_2} + \frac{V_a - (-AV)}{R_o} - I_a = 0 \tag{2.44}$$

An expression for V is most easily obtained by observing that R_1 and R_2 form a voltage divider, hence we may write:

$$V = \left(\frac{R_1}{R_1+R_2}\right)V_a \tag{2.45}$$

Substituting (2.45) in (2.44) gives

$$I_a = \frac{V_a}{R_1+R_2} + \frac{V_a}{R_o} + \frac{A}{R_o} \cdot \frac{R_1}{(R_1+R_2)} \cdot V_a$$

Therefore, by (2.37)

$$R_T = \frac{V_a}{I_a} = \left[\frac{1}{R_1+R_2} + \frac{1}{R_o} + \frac{AR_1}{R_o(R_1+R_2)}\right]^{-1}$$

which upon rearrangement becomes

$$R_T = \frac{R_o(R_1+R_2)}{R_o + (1+A)R_1 + R_2}$$

This method for determining R_T is more direct and involves less work than method 1 because it is not necessary to first find the Thévenin e.m.f.

The amplifier circuit shown in fig. 2.24(*b*) contains only linear, bilateral elements and one may, as we have demonstrated, apply standard techniques of linear circuit analysis. However, the circuit contains also an active element, here represented by the dependent source AV, which gives it the properties of voltage and power gain. The consequence of this is that the reciprocity theorem does not apply to this circuit. We may easily check that this is so by considering the short-circuit currents that arise at the input and output ports as a result of the same voltage applied, in turn, at opposite ports. Consider first terminals AB short-circuited, and a voltage V_a applied at terminals CD. Making the assumption that R_i is infinite, the current in the short circuit at AB is $V_a/(R_1+R_2)$. Now consider CD short-circuited and V_a applied at AB; the current in this case is from (2.42)

$$\frac{V_a}{(R_1+R_2)}\cdot\frac{(R_o-AR_2)}{R_o}$$

Clearly, the two short-circuit currents are not the same, as would be the case if the reciprocity theorem were true. Circuits of this description, for which the reciprocity theorem does not hold, are said to be *non-reciprocal*. This is the subject of further discussion in chapter 8. It should be noted that the superposition theorem *is* applicable to such circuits.

2.15 Miscellaneous theorems and techniques

†2.15.1 The substitution and compensation theorems

We have seen that the analysis of a circuit can often be facilitated by judicious use of the appropriate linear network theorem; two useful additions to those theorems already discussed are presented below.

The *substitution theorem* is useful if it is required to change the values of the elements in one branch of a circuit, or substitute alternative kinds of elements, without changing voltages and currents elsewhere in the circuit.

Fig. 2.25(*a*) shows one branch of a circuit containing a resistance R and ideal voltage source V. The branch voltage is V_{AB} and the branch current I_{AB}, these being fixed values. The substitution theorem states that: *this branch may be replaced by another branch without anywhere changing voltages and currents provided the substitute branch also has voltage V_{AB} when carrying current I_{AB}.* This condition can be satisfied by various combinations of R and V which satisfy the branch equation $V_{AB}=I_{AB}R+V$. The maximum possible values are given by:

$$V_{AB}=I_{AB}R_{\max} \qquad (V=0); \qquad V_{AB}=V_{\max} \qquad (R=0)$$

Thus, either a resistance $R_{\max}$ alone or a source $V_{\max}$ alone may be substituted for the original elements (figs. 2.25(*b*) and (*c*)). Two other combinations of elements that may be substituted are shown in figs. 2.25(*d*) and (*e*). With these combinations the voltage across the current source adjusts automatically to the value required to satisfy the branch equation.

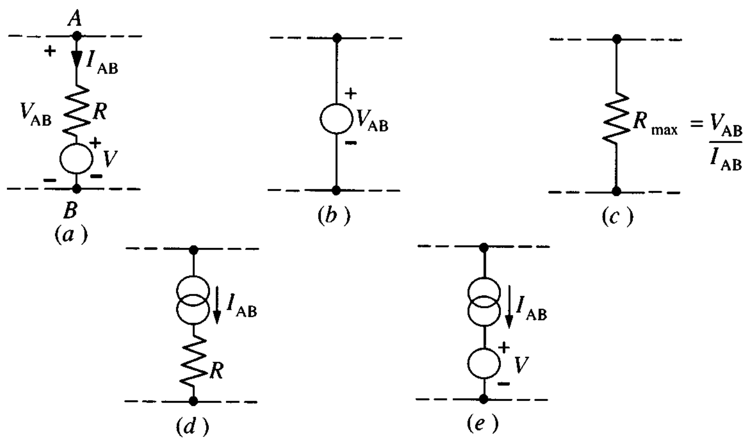
The *compensation theorem* may be employed when it is required to evaluate the effect which a modification in the resistance of one branch of a network has on the currents and voltages at any part of that network. For example we may wish to know how the insertion of an ammeter, possessing some small but finite resistance, will affect the operation of a circuit.

Let AB in fig. 2.26(*a*) be the branch that is to undergo modification. The sources in the remainder of the network will drive a current I through this branch. When the resistance of the branch is changed by an amount ΔR the current will change by some increment ΔI as shown in fig. 2.26(*b*). Note that if the resistance of the branch is increased then the current will be reduced, that is, ΔI will be negative.

The compensation theorem states that: *if the current in a branch of a network before modification is* I, *and the resistance in that branch is changed by an amount* ΔR, *the incremental change of current and voltage in any part of the network is that produced by an ideal voltage source of value* $I(\Delta R)$ *acting in the modified branch and directed in the opposite sense to* I.

The theorem may be proved by considering the change of e.m.f. necessary in AB to reduce ΔI to zero, that is, to restore the current to its original value I. This is accomplished by introducing an additional voltage source of magnitude $I(\Delta R)$, acting in the same sense as I (fig. 2.26(*c*)); this source voltage exactly compensates for the voltage drop across ΔR thereby effectively restoring the network to its original condition. Now, if the effect

Fig. 2.25. Illustrating the substitution theorem.



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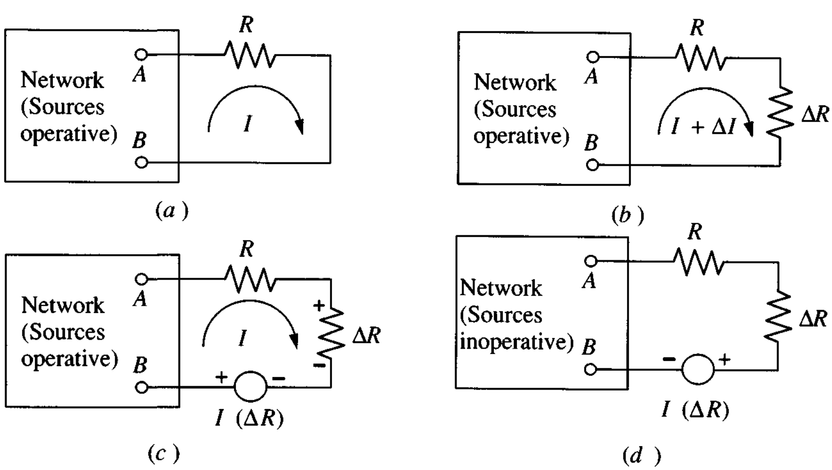
of introducing this source is to reduce ΔI to zero, we can say, according to the superposition theorem, that this source acting alone in the network must produce a current ΔI in AB flowing in the opposite direction to $(I+\Delta I)$. Correspondingly, an increase in current from I to $(I+\Delta I)$ must be that effected by a source of magnitude I (ΔR) acting in the opposite sense to I. The incremental current in AB, and all other incremental currents and voltages in the network, may therefore be found from the circuit of fig. 2.26(d) in which all sources other than the added source are made inoperative.

The compensation theorem cannot be applied to the situation where a branch is open-circuited since under such circumstances voltages and currents are indeterminate.

An illustration of the compensation theorem is provided by the circuit shown in fig. 2.27. The elements R_Z and V_Z represent a piecewise-linear circuit model of a Zener diode. (See chapter 7 for piecewise-linear circuit theory.) The circuit is designed to supply (ideally) a constant voltage to the load R_2. What is the change in load voltage if R_2 is decreased by 10%? This problem could, of course, be solved using the standard methods of analysis already presented in this chapter, however, the nature of the circuit renders it amenable to an approximate solution that is sufficiently accurate for most design purposes. We note that R_Z is small compared with both R_1 and R_2; the voltage V will therefore be very nearly equal to the Zener voltage V_Z. Thus, the current in R_2 (before branch modification) will be, to a good approximation, $5.6/10^3$ A. Since R_2 is reduced, ΔR_2 will be negative and the compensation voltage is $-100 \times 5.6/10^3 = -0.56$ V.

Fig. 2.26. Illustrating the compensation theorem.



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The incremental change of voltage ΔV is then found from the circuit shown in fig. 2.27(*b*). The modified branch resistance is now 900Ω and, because we are concerned only with the incremental change in load voltage, the other voltage sources in the circuit have been replaced by short circuits. The resulting combination of R_1 and R_Z in parallel is approximately equal to R_Z since $R_1 \gg R_Z$. We now recognize that the circuit is reduced to a simple potential divider from which

$$\Delta V = -0.56 \frac{22}{22+900} = -13.4\,\text{mV}$$

2.15.2 Circuit reduction

Circuits can often be rendered more tractable for purposes of analysis by first reducing them to a simpler form. A typical situation is depicted in fig. 2.28, in which the effect of varying parameters in one part of a network configuration is to be investigated whilst keeping the remainder fixed. In such a situation it is often convenient to reduce the fixed part of the network to its simplest possible form, usually its Thévenin equivalent, before proceeding with the analysis proper, since the subsequent analytical

Fig. 2.27. Application of the compensation theorem to a Zener diode voltage-stabiliser circuit.

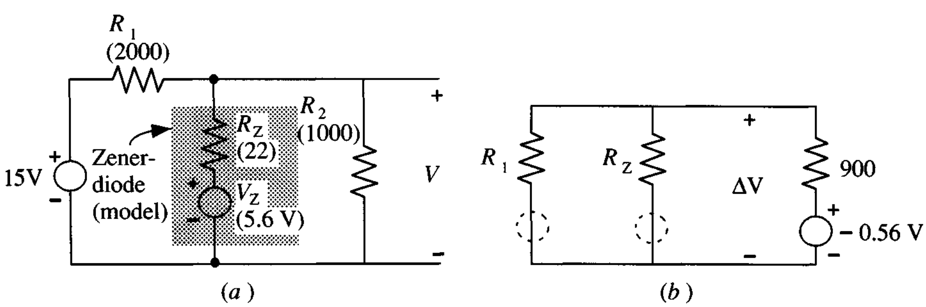
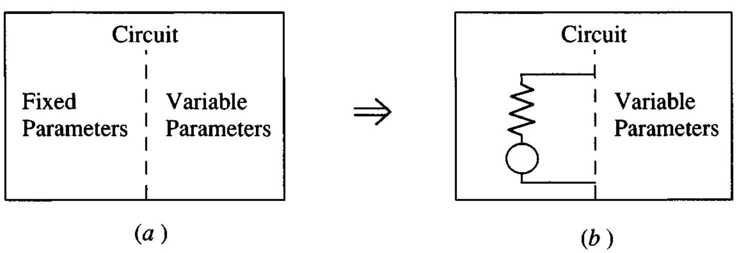


Fig. 2.28. Circuit reduction used to simplify part of a network.



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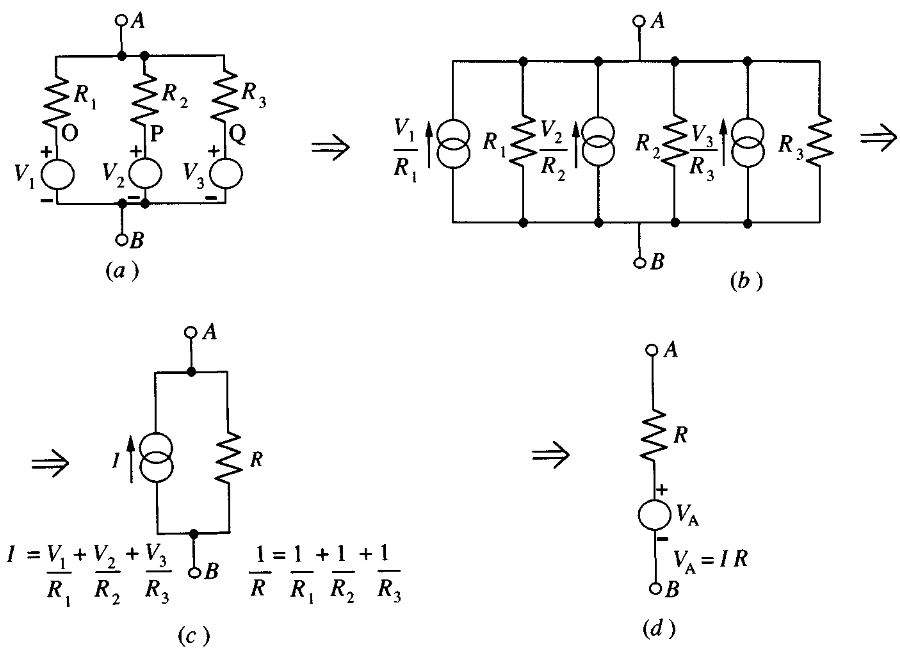
relations are thereby simplified and it becomes easier to investigate the effect of parameter changes.

Circuit reduction may be effected by the direct application of Thévenin's theorem to a complete part of a circuit (as illustrated in section 2.8) or by the use of mesh or nodal analysis. Alternatively, individual nodes and meshes may be eliminated as desired by repeated application of the Thévenin–Norton transformation or the star–delta transformation. The basis of the Thévenin–Norton transformation method is indicated in fig. 2.29. We suppose that the nodes *AB* are connected to some other part of a network and it is desired to simplify the portion between A and B.

First, the three practical voltage sources are transformed to their equivalent current sources (fig. 2.24(*b*)), which results in the elimination of nodes *O*, *P*, and *Q*. Current sources are then added together and resistances combined in parallel to produce a single practical current source (fig. 2.29(*c*)). Finally, if convenient, the current source may be transformed to a voltage source (fig. 2.29(*d*)).

Although the above step-by-step transformation procedure, making use of diagrams, can be useful and informative, the same process can be

Fig. 2.29. Circuit reduction by application of the Thévenin–Norton transformation.



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performed analytically by means of nodal analysis. Taking B as the reference node and solving for the node voltage V_A we obtain:

$$\frac{V_A-V_1}{R_1}+\frac{V_A-V_2}{R_2}+\frac{V_A-V_3}{R_3}=0$$

or

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)V_A=\frac{V_1}{R_1}+\frac{V_2}{R_2}+\frac{V_3}{R_3} \tag{2.46}$$

Putting

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} \quad \text{and} \quad I=\frac{V_1}{R_1}+\frac{V_2}{R_2}+\frac{V_3}{R_3}$$

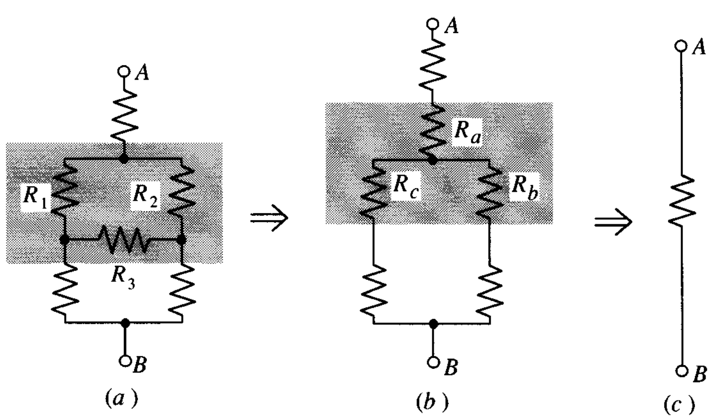
$$\frac{V_A}{R}=I$$

The above relations may be interpreted in terms of figs. 2.29(c) and (d).

Equation (2.46) in its general form, relating to any number of parallel sources, is sometimes referred to as *Millman's theorem.*

The process of circuit reduction using the star–delta transformation (or its inverse) may be illustrated with reference to the circuit of fig. 2.30(a). It will be appreciated that reduction of this circuit cannot be effected by simple series and parallel additions of resistances because of the bridging resistance R_3. This difficulty is overcome by recognizing that the three resistances within the shaded box form a delta configuration. Using the relations 2.23 the delta comprising, R_1, R_2, R_3 is transformed to the star

Fig. 2.30. Circuit reduction using the inverse star–delta transformation.



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comprising R_a, R_b, R_c (fig. 2.30(*b*)), after which series and parallel combination lead to a single resistance (fig. 2.30(*c*)).

†2.15.3 Ladder networks

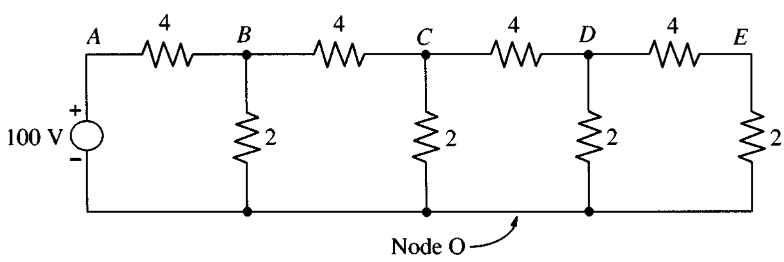
There are certain types of network that do not lend themselves readily to the standard methods of analysis so far discussed in this chapter, and for which special techniques are used. The *ladder* network falls into this category.

Suppose we wished to find the node voltages V_{EO} and V_{CO} at the end and mid-points of the ladder network shown in fig. 2.31(*a*). A mesh-node count at once indicates that a standard method of analysis would involve setting up and solving four simultaneous equations. The following step-by-step procedure is rather simpler and is particularly convenient to carry out numerically using a small calculator.

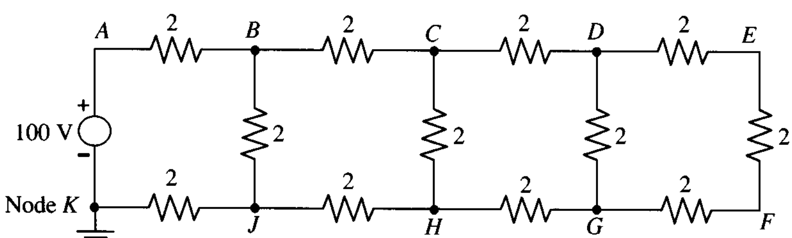
We start by assuming that the voltage across the end of the ladder V_{EO} is 1 V. The current I_{EO} in the end 2 Ω resistance is then 1/2 A, and the voltage V_{DO} is 3 V (terminals *EO* assumed open circuit). The calculation then proceeds as follows:

$$I_{DO}=\tfrac{3}{2}\,\mathrm{A}$$
$$I_{CD}=I_{DO}+I_{DE}=\tfrac{3}{2}+\tfrac{1}{2}=2\,\mathrm{A}$$

Fig. 2.31. Ladder networks.



(*a*) Asymmetrical (unbalanced)



(*b*) Symmetrical (balanced)

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$$V_{CD}=4I_{CD}=8\text{ V}$$
$$V_{CO}=V_{DO}+V_{CD}=3+8=11\text{ V}$$
$$I_{CO}=V_{CO}/2=\tfrac{11}{2}\text{ A}$$
$$I_{BC}=I_{CO}+I_{CD}=\tfrac{11}{2}+2=\tfrac{15}{2}\text{ A}$$
$$V_{BC}=4I_{BC}=30\text{ V}$$
$$V_{BO}=V_{CO}+V_{BC}=11+30=41\text{ V}$$
$$I_{BO}=V_{BO}/2=\tfrac{41}{2}\text{ A}$$
$$I_{AB}=I_{BO}+I_{BC}=\tfrac{41}{2}+\tfrac{15}{2}=\tfrac{56}{2}\text{ A}$$
$$V_{AB}=4I_{AB}=112\text{ V}$$
$$V_{AO}=V_{BO}+V_{AB}=41+112=153\text{ V}$$

This last figure is the voltage of the source assuming 1 V at the end of the ladder. But the actual value of V_{AO} is 100 V hence the true value of V_{EO} must be $1\times(100/153)$ V. Likewise all voltages and currents in the above calculation must be scaled in similar proportion to obtain true values. The voltage at the mid-point of the ladder is therefore $V_{CO}=11\times(100/153)$ V.

A similar procedure may be adopted in the case of a symmetrical ladder of the form shown in fig. 2.31(*b*). As far as the calculation of voltages across the rungs of a ladder network is concerned it is immaterial how the total resistance between rungs is distributed on the two sides of the ladder. It is convenient, therefore, in the analysis of a symmetrical ladder to first lump together the resistances between rungs and then proceed as for the analysis of an asymmetrical ladder. For the particular resistance values given in fig. 2.31 the voltages across each of the two forms of ladder are identical.

To determine node voltages with respect to ground (node K in fig. 2.31(*b*)) for the symmetrical ladder it is necessary only to apply the symmetry principle, once having found voltages across the ladder. For example, suppose the voltage V_{HK} is required. Using the procedure detailed above we first find that V_{CH} (corresponding to V_{CO} in fig. 2.31(*a*)) is equal to $11\times(100/153)$ V. We may then deduce, by symmetry, that V_{HK} is $\tfrac{1}{2}(100-11\times(100/153))$ V.

The ladder method described above is of value mainly for numerical calculation; extremely unwieldy expressions result if one attempts to apply the method in symbolic form. Other methods of dealing with ladder networks are discussed in chapter 8.

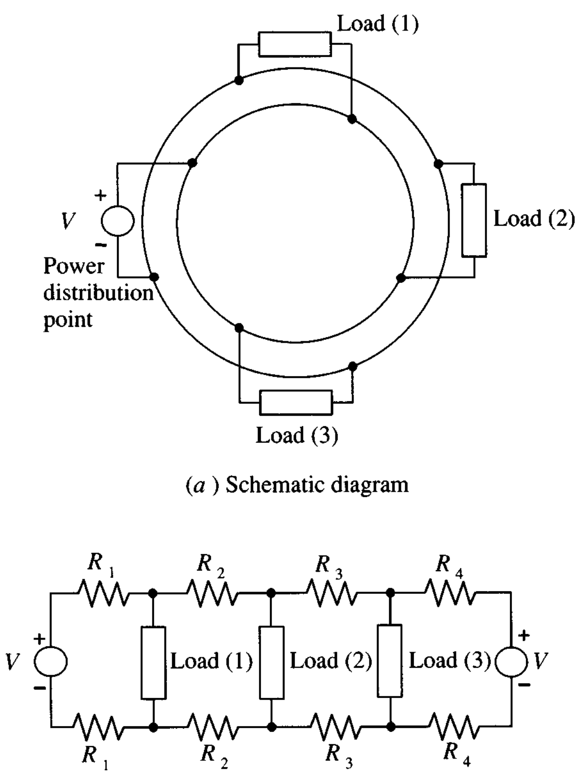
†2.15.4 Ring mains

Ring mains are employed extensively in power distribution systems. A number of loads are connected to a single distribution point via parallel conductors which form a closed loop or ring, as indicated schematically in fig. 2.32(*a*). This type of connection results in better utilization of the distribution conductors compared with a straight parallel connection. Calculations on ring mains are performed with the aim of determining voltages at loads and currents in ring conductors so that the correct conductor cross sections may be specified.

The lumped circuit model of a ring main with three loads is shown in fig. 2.32(*b*). R_1, R_2, R_3 and R_4 represent conductor resistances between points on the ring. By opening out the ring and treating the resulting circuit as if it were fed by identical sources, one at each end, the analysis is considerably simplified. The circuit in this form resembles somewhat the symmetrical ladder of fig. 2.31(*b*).

If the loads are specified as fixed resistances, and an exact analysis is required, then the ladder method described above can be used. Normally, however, the loads are specified in terms of the maximum currents to be drawn at particular points on the ring, in which case a mesh analysis is appropriate. In the following worked example both methods of analysis are illustrated.

Fig. 2.32. Ring main.



(*a*) Schematic diagram

(*b*) Lumped circuit model

†2.15.5 Worked example

A ring main is supplied from a power point at 240 V and has three loads. The total length of the ring is 12 m and the loads and the power point are spaced at equal intervals of 3 m. Each of the ring conductors has a resistance of 0.067 Ω/m. Find the currents in the ring conductors and the voltages at each of the loads: (a) if the loads are specified as three consecutive resistances of 24 Ω, 16 Ω, and 12 Ω; (b) if the loads are specified as three consecutive currents of 10 A; 15 A and 20 A.

Solution

(a) Each of the conductors in the ring main has a resistance of 0.067 Ω/m, therefore, the total resistance of two conductors (in series) over a 3 m length is $2 \times 3 \times 0.067 = 0.402\,\Omega$. The circuit model is shown in fig. 2.33(*a*).

We first find the contributions to the current I_1 due to each of the two voltage sources acting alone. The total current is then found from the superposition of the two separate contributions. To find the contribution due to the left-hand source, replace the right-hand source by a short-circuit and assume a current of 1 A to flow in this short-circuit (fig. 2.33(*b*)). Using the ladder method the calculation proceeds as follows:

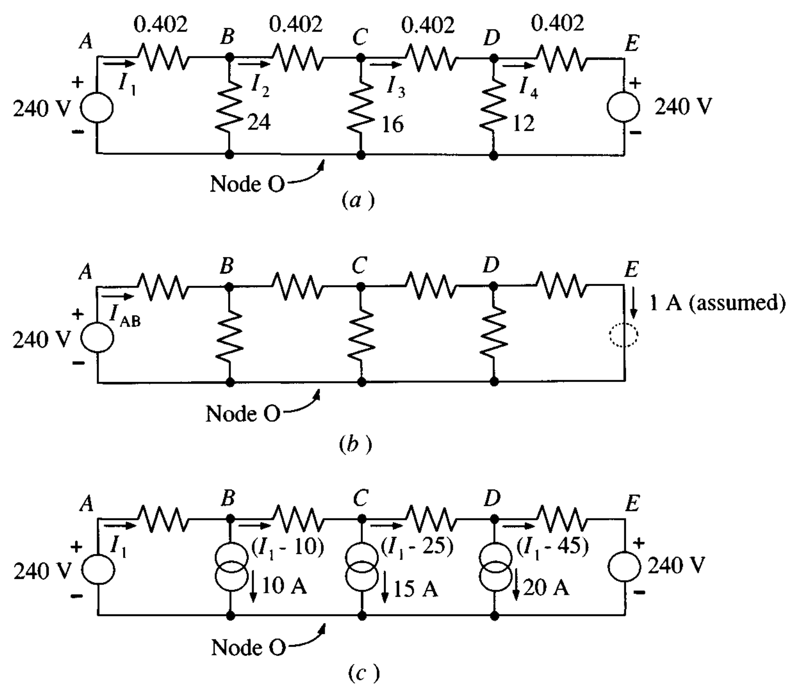
$$
\begin{aligned}
&I_{EO} = 1\text{ A (assumed)}\\
&V_{DO} = 1 \times 0.402 = 0.402\text{ V}\\
&I_{DO} = 0.402/12 = 0.0335\text{A}\\
&I_{CD} = I_{DO} + I_{EO} = 0.0335 + 1 = 1.0335\text{ A}\\
&V_{CD} = 1.0335 \times 0.042 = 0.4155\\
&V_{CO} = V_{CD} + V_{DO} = 0.4155 + 0.402 = 0.8175\\
&I_{CO} = 0.8175/16 = 0.0511\\
&I_{BC} = I_{CO} + I_{CD} = 0.0511 + 1.0335 = 1.0846\\
&V_{BC} = 1.0846 \times 0.402 = 0.4360\\
&V_{BO} = V_{BC} + V_{CO} = 0.4360 + 0.8175 = 1.2535\\
&I_{BO} = 1.2535/24 = 0.05223\\
&I_{AB} = I_{BO} + I_{BC} = 0.05\ 223 + 1.0846 = 1.1368\\
&V_{AB} = 1.1368 \times 0.402 = 0.4570\\
&V_{AO} = V_{AB} + V_{BO} = 0.4570 + 1.2535 = 1.7105
\end{aligned}
$$

But the actual value of V_{AO} is 240 V, therefore the true value of the current $I_{AB} = 1.1368 \times (240/1.7105) = 159.50$ A and the true value of the current $I_{EO} = 1 \times (240/1.7105) = 140.31$ A.

Now the right-hand source is restored and the left-hand source replaced by a short-circuit. The contribution to I_1 due to this source acting alone must, according to the reciprocity theorem, be -140.31 A. Therefore, when the two sources act together $I_1 = 159.50 - 140.31 = 19.19$ A. Having found the current I_1 we now employ a step-by-step procedure, working from left to right, in the circuit of fig. 2.33(a).

$$
\begin{aligned}
I_1 &= 19.19\ \text{A (calculated)}\\
V_{AB} &= 19.19 \times 0.402 = 7.7144\ \text{V}\\
V_{BO} &= 240 - 7.7144 = 232.29\ \text{V}\\
I_{BO} &= 232.29/24 = 9.6785\ \text{A}\\
I_2 &= 19.19 - 9.6785 = 9.5115\ \text{A}\\
V_{BC} &= 9.5115 \times 0.402 = 3.8236\ \text{V}\\
V_{CO} &= 232.29 - 3.8236 = 228.46\ \text{V}\\
I_{CO} &= 228.46/16 = 14.279\ \text{A}\\
I_3 &= 9.5115 - 14.279 = -4.7676\ \text{A}\\
V_{CD} &= -4.7676 \times 0.402 = -1.9166\ \text{V}\\
V_{DO} &= 228.46 - (-1.9166) = 230.37\ \text{V}\\
I_{DO} &= 230.37/12 = 19.198\ \text{A}\\
I_4 &= -4.7676 - 19.198 = -23.965\ \text{A}\\
V_{DE} &= -23.965 \times 0.402 = -9.6342\ \text{V}\\
V_{EO} &= 230.37 - (-9.6342) = 240\ \text{V}
\end{aligned}
$$

Fig. 2.33. Ladder circuits for worked example.



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This last figure, of course, provides a check on the accuracy of the calculations.

(b) With currents of 10 A, 15 A, and 20 A specified, the circuit model becomes that shown in fig. 2.33(c). Branch current I_1 is assigned and KCL is used to write down the other currents in the ring conductors as indicated. Then, using KVL round the loop AEO, we obtain

$$0.402[I_1+(I_1-10)+(I_1-25)+(I_1-45)]=240-240$$

or

$$\begin{aligned} 4I_1-80&=0 \\ I_1&=20\text{ A} \end{aligned}$$

(This is simply the average current drawn by the three loads.) Therefore

$$\begin{aligned} V_{BO}&=240-20\times 0.402=231.96 \\ V_{CO}&=231.96-(20-10)\times 0.402=227.94 \\ V_{DO}&=227.94-(20-25)\times 0.402=229.95 \\ V_{EO}&=229.95-(20-45)\times 0.402=240 \text{ (check)} \end{aligned}$$

It should be noted that the load currents specified here are those obtained by dividing the power point voltage, 240 V, by each of the three resistances specified in part (a) of the problem. The actual load currents calculated in part (a) are the same as these to within about 5%. We may, therefore, obtain an approximate solution to the ring main problem, when load resistances are specified, by first calculating approximate load currents (using the power point voltage) and then employing the method of solution outlined in part (b). Comparing the load voltages obtained in parts (a) and (b), we see that these are in agreement to within $\frac{1}{4}$%.

2.16 Summary

Two standard methods of analyzing linear circuits are available: mesh analysis and nodal analysis.

In mesh analysis currents are assigned to every branch in the circuit or, alternatively, currents are assigned to every mesh (Maxwell's cyclic current method). Kirchhoff's voltage law is then applied to set up the requisite number of simultaneous mesh equations, there being as many equations as assigned currents.

In nodal analysis, the dual of mesh analysis, voltages are assigned to nodes, one node being chosen as a reference. Kirchhoff's current law is then applied to set up the requisite number of nodal equations. Of the two methods, nodal analysis is usually easier to apply, and often results in fewer equations. Mesh analysis is generally unsuitable for circuits containing ideal current sources whereas nodal analysis may be used for circuits containing both voltage and current sources.

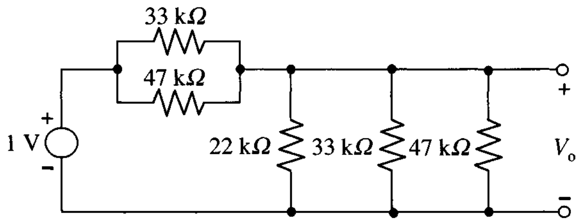
The labour of circuit analysis can sometimes be reduced by employing linear circuit theorems: superposition, reciprocity and Thévenin. The last mentioned, often used in conjunction with the potential divider circuit, is particularly useful for reducing parts of a circuit to a simpler form. Other methods of circuit reduction include Thévenin–Norton and star–delta transformations. The recognition of standard circuit building blocks – potential and current dividers, bridge circuits etc. – forms an important part of the art of circuit analysis.

For some circuits, the ladder circuit for example, special step-by-step methods are available which obviate the necessity of solving a large number of simultaneous equations.

2.17 Problems

1. A d.c. power supply has an output voltage of 5 V at its terminals on open circuit. A 2 Ω resistor connected across its terminals causes the output to fall by 0.1 V. Derive a linear circuit model for the supply.
2. A certain d.c. power supply has output potential differences of 600 V and 650 V when the output current is 0.4 A and 0.2 A respectively. What simple arrangement (a) of an ideal current generator in parallel with a resistance and (b) of an ideal current generator in parallel with a conductance will give the same relation between output p.d. and current?
3. For the voltage divider network shown in fig. 2.34, determine the output voltage V_o. If the output terminals are connected to a circuit having an input resistance of 10 kΩ, what then is the output voltage?
4. A device draws a constant current through a divider network as shown in fig. 2.35. A multi-range voltmeter V_m which draws a current of 1 mA at full-scale deflection, is used to measure the voltage supplying the device. When set to its 300 V range, it reads 90 V. What is the device voltage with the voltmeter removed?

Fig. 2.34. Circuit for problem 3.



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5. Show, by means of mesh analysis, that each of the mesh currents in the circuit of fig. 2.36 results from the superposition of two independent components, one proportional to V_1 the other proportional to V_2. If V_1 equals 1 V and V_2 is short-circuited, what will be the currents in the 3 Ω and 4 Ω resistances?

Apply this last result together with the reciprocity theorem to write down the new current in the 4 Ω resistance if V_1 equals 1 V and V_2 is made equal to 3 V.

Deduce, from the above results, the Thévenin equivalent circuit of the network across AB, as seen by the voltage source V_1, when V_2 equals 2 V. (Hint: to find the Thévenin e.m.f. consider the voltage V_1 required to reduce I_1 to zero; to find the equivalent resistance consider the current I_1 when AB is short circuit.)

6. In the circuit shown in fig. 2.37, additional generators are to be inserted into branches AB and BC so that the currents then flowing in the existing generators are each increased by 1 A from their original values. By means of

Fig. 2.35. Circuit for problem 4.

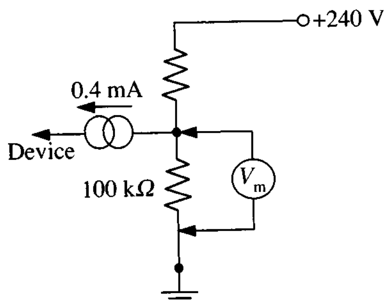
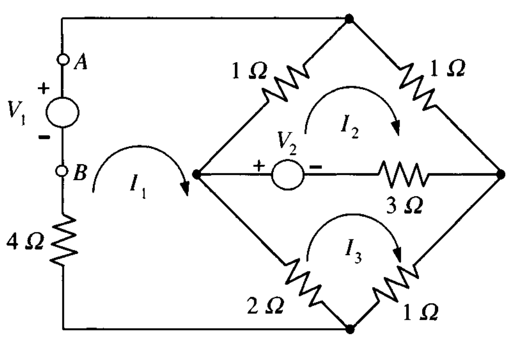


Fig. 2.36. Circuit for problem 5.



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the superposition and reciprocity theorems or otherwise, find the e.m.f.s of the additional generators.
(Cambridge University: First year)

7. In the circuit of fig. 2.38 each resistance is 0.5 Ω. How many current unknowns are involved in a mesh analysis? How many voltage unknowns are involved in a nodal analysis? Find the potential of the node P with respect to the node O.

8. Find the current flowing in the 4 Ω resistor in the circuit of fig. 2.39 giving reasons for the choice of method for conducting this calculation. Describe and compare at least two other methods which might have been used instead.
(Newcastle University: First year)

9. Determine the current I shown in the network represented by fig. 2.40. Also calculate the voltage of the point C with respect to ground.
(Cambridge University: Second year)

10. The resistance of each arm and of the detector of a Wheatstone Bridge is 1 kΩ. The bridge is driven by a 10 V battery of negligible internal

Fig. 2.37. Circuit for problem 6.

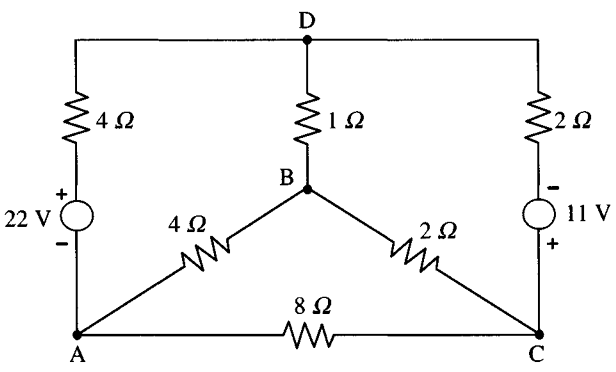
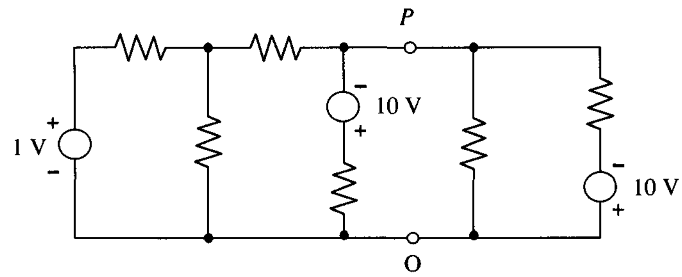


Fig. 2.38. Circuit for problem 7.



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resistance. Use Thévenin's theorem to find an expression for the detector current, accurate to within 1%, if the resistance of one arm of the bridge is increased by r ohms ($r \leqslant 5$).

11. Figure 2.41 shows a circuit which may be used for temperature measurement. R_T is a thermistor whose resistance is 20 kΩ at 0 °C and 2 kΩ at 100 °C. M is a sensitive ammeter whose internal resistance is 5 kΩ and gives full-scale deflection at a current of 25 μA.

Calculate the ohmic values to which R_1 and R_2 must be set so that M gives zero deflection at 0 °C and full-scale deflection at 100 °C.
(Newcastle University: First year)

12. In the circuit of fig. 2.42 each resistor has the ohmic value stated. Show that when viewed from the output terminals AB, the circuit is equivalent to a generator having an e.m.f. of 20 V and an internal resistance of 3 Ω.

What are the two possible ohmic values of resistor R which when connected across the output terminals, will absorb a power of 32 W?
(Newcastle University: Second year)

13. Twelve identical pieces of wire each of resistance 1 ohm are connected together to form a skeleton cube. Find the resistance between opposite ends of a diagonal of the cube.

14. Two 240 V generators of low internal resistance are connected together

Fig. 2.39. Circuit for problem 8.

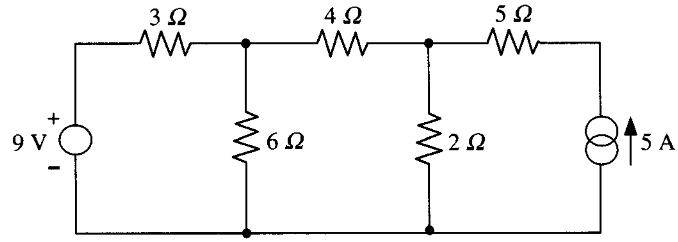
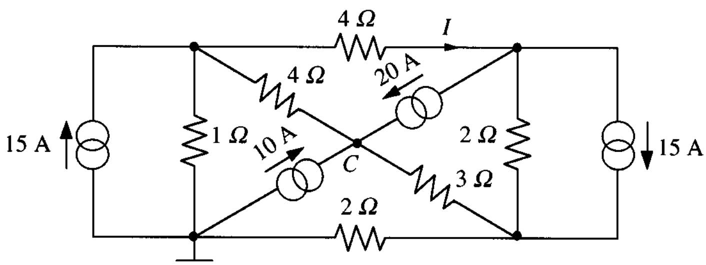


Fig. 2.40. Circuit for problem 9.



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in opposition by a cable 15 m long, and two loads are connected; one of 24 Ω at a distance of 6 m from one end of the cable, and one of 16 Ω at a distance of 4 m from the other end. If the voltage across the 24 Ω load is found to be 2% below the generator voltage, what is the resistance per unit length of each conductor of the cable?

If the voltage of one generator changes by 2%, what will be the current in the part of the cable between the loads?
(Cambridge University: Second year)

15. A ring main of total length 1 km has five load points distributed as shown in the Table. Find the point at which the voltage is minimum. If the voltage drop is nowhere to exceed 1 V plus 2% of the nominal supply voltage of 240 V, calculate the minimum cross-sectional area of copper required in each cable (resistivity of copper = 1.6×10^{-8} Ωm).

Distance (m)	200	300	400	600	800
Load (A)	30	20	40	20	40

Fig. 2.41. Circuit for problem 11.

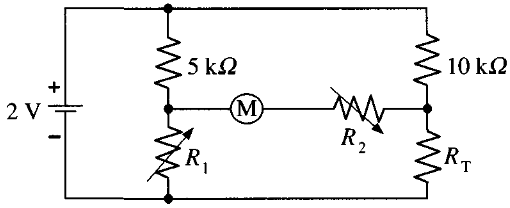

Fig. 2.42. Circuit for problem 12.