6

Transient and steady-state analysis

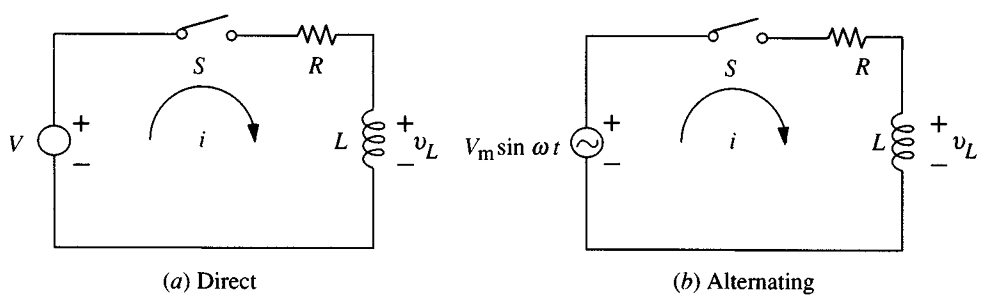
6.1 Introduction

In chapters 2 and 3 we found the steady-state response of linear circuits when they are driven by direct (d.c.) or sinusoidal (a.c.) voltages or currents. In this chapter we shall look at the conditions arising in a circuit during the time required for it to reach the steady state. What occurs is called the *transient* behaviour of the circuit.

Consider the simple series circuits of fig. 6.1, and assume that the switches have been closed for a long time so that the circuits have reached steady-state conditions. For the d.c. circuit (fig. 6.1(*a*)) the voltage across the inductance is zero; therefore, the steady state current is $i_{ss} = V/R$. For the a.c. circuit (fig. 6.1(*b*)) the inductive reactance is ωL and the steady-state current is

$$i_{ss} = \frac{V_m}{\sqrt{[R^2 + (\omega L)^2]}} \sin(\omega t - \theta) \text{ where } \theta = \tan^{-1}\frac{\omega L}{R} \qquad (6.1)$$

Fig. 6.1. Inductive (RL) circuits with direct and alternating voltage driving sources.



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Now consider again the circuits of fig. 6.1, but this time assume that the switches are initially open so there is no current in either circuit. At time $t=0$ let the switches be closed. What will be the current in each circuit the instant after the switch is closed? The answer is zero for both circuits. We have seen in section 1.9 that the energy stored in an inductance cannot change instantaneously. Since the initial stored energy is zero in both circuits, and since the stored energy depends upon the current in the inductance, it follows that for both circuits the current immediately after the switch is closed must be zero. It will be convenient to use $t=0^+$ to designate the time immediately after a switching operation has been completed and before there has been any change in energy storage in any circuit element.

6.2 Qualitative analysis of the *RL* circuit

Let us examine in detail how the current in the circuit of fig. 6.1(a) rises from zero to its final steady-state value. At $t>0$,

$$v_R + v_L = V \qquad \text{or} \qquad iR + L\frac{\mathrm{d}i}{\mathrm{d}t} = V$$

But at $t=0^+$, $i=0$, and so $v_R = iR = 0$, hence

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = V \qquad \text{or} \qquad \frac{\mathrm{d}i}{\mathrm{d}t} = \frac{V}{L}$$

and i is increasing. As i increases, v_R is no longer zero, so for $t>0$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{V}{L} - i\frac{R}{L}$$

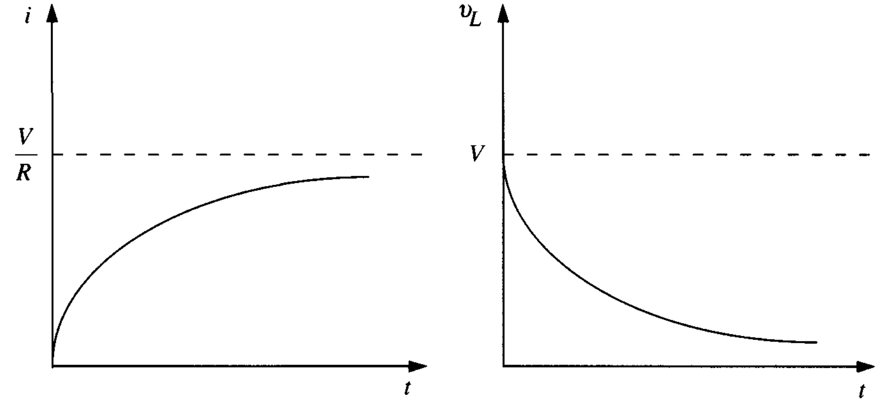
We see then that the rate of change of current depends upon the current already in the circuit. When $i = V/R$, $\mathrm{d}i/\mathrm{d}t = 0$ and the current is no longer changing, having reached its steady-state value. Thus there is an interval of time during which the current rises at a decreasing rate toward its final value. Because $\mathrm{d}i/\mathrm{d}t$ depend upon i, the current cannot reach the final value in a finite length of time; therefore, i approaches the value V/R asymptotically.

The information that we now have enables us to sketch qualitatively the curve i v. t. This is shown in fig. 6.2(a). The voltage across the resistance has exactly the same time dependence as i. At every instant $v_L = V - iR$. Therefore, the voltage across the inductance starts at V and approaches zero asymptotically as shown in fig. 6.2(b).

The currents in and the voltages across the circuit elements during the interval while the current rises from zero to V/R are referred to collectively as the *transient response* of the circuit. It represents the smooth transition from the initial state ($i=0$) to the final state ($i=V/R$). In some circuits, for example the lights in a building, or motors operating household appliances, the transient response is probably of no interest; in other circuits the transient response is the only feature of interest. Transient voltages and currents may, for example, generate useful waveforms or they may be used to provide precisely known time delays in circuits.

There is also a transient response if, after the steady state is reached, the switch in fig. 6.1(a) is opened. When the conducting path is completely broken, i must be zero. However, at $t=0^+$, $i=V/R$ because current in the inductance cannot change instantaneously. When L is large, the rapid decrease in current as the switch contacts part results in a large induced voltage in the coil, a voltage that may be high enough to make the air between the contacts become conducting. Thus, the current path is not broken in zero time, but in a time determined by the rate at which energy initially stored in the coil is dissipated in the resistance and in the conducting arc established between the switch contacts. Again, the transient response provides a smooth transition from the initial state of $i=V/R$ to the final state $i=0$.

Fig. 6.2. Qualitative analysis of the circuit of fig. 6.1(a).



(a) Current through inductance (b) Voltage across inductance

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6.3 Mathematical analysis of the *RL* circuit

Now let us find an explicit expression for the current in the circuit of fig. 6.1(*a*). Assume that the switch is initially open and is closed when $t=0$. Then for $t>0$,

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+iR=V \tag{6.2}$$

Obviously, the steady state current, $i_{ss}=V/R$ is a solution of (6.2). The complete solution, however, contains another term that goes to zero as t increases. The complete solution describes the transient and reduces to the steady state solution as $t\to\infty$.

It is easy to write the solution of (6.2) by separating the variables, integrating, and using the initial condition $i=0$ at $t=0^+$. The result is

$$i=\frac{V}{R}-\frac{V}{R}\mathrm{e}^{-Rt/L} \tag{6.3}$$

The current then is the sum of two terms. The first term is the steady state current that is independent of time. The second term represents an exponentially decaying current. The two terms and their sum are shown in fig. 6.3. We see that the total current has the type of time dependence that was predicted in the qualitative analysis of the circuit.

When there is a sinusoidal driving voltage as shown in fig. 6.1(*b*), the differential equation is

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+iR=V_{\mathrm{m}}\sin\omega t \tag{6.4}$$

Before finding a solution for (6.4) we add a phase angle to the driving voltage. This is convenient because the solution will depend upon the value of the driving voltage at the instant the switch is closed. With the phase angle included in the voltage, switching can always occur at $t=0$. The phase angle λ may then be used to specify the value of the applied voltage at $t=0$. With this addition, and putting $R/L=\alpha$, (6.4) becomes

$$\frac{\mathrm{d}i}{\mathrm{d}t}+\alpha i=\frac{V_{\mathrm{m}}}{L}\sin(\omega t+\lambda) \tag{6.5}$$

To solve this equation we multiply by the integrating factor

$$\mathrm{e}^{\int\alpha\mathrm{d}t}=\mathrm{e}^{\alpha t}$$

Then,

$$e^{\alpha t}\frac{di}{dt}+e^{\alpha t}\alpha i=e^{\alpha t}\frac{V_m}{L}\sin(\omega t+\lambda)$$

Integration gives

$$ie^{\alpha t}=\frac{V_m}{L}\int e^{\alpha t}\sin(\omega t+\lambda)\,dt$$

The right-hand side of the equation may be integrated by parts or by consulting tables.

$$ie^{\alpha t}=\frac{V_m e^{\alpha t}}{L(\alpha^2+\omega^2)}[\alpha\sin(\omega t+\lambda)-\omega\cos(\omega t+\lambda)]+K$$

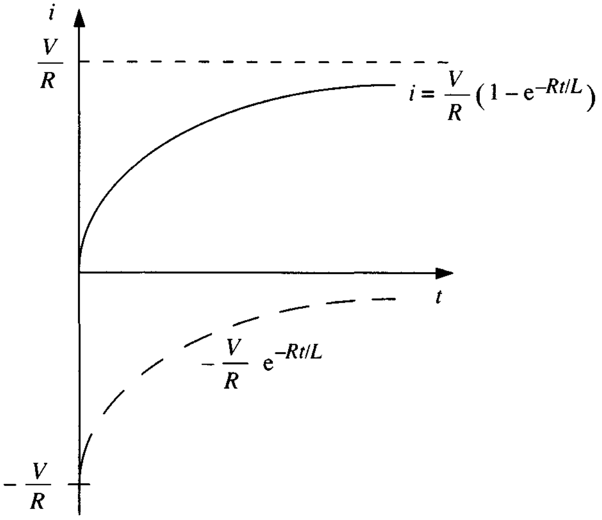
Now multiply through by $e^{-\alpha t}$ and simplify the trigonometric expression in brackets by using the identity

$$A\sin\beta-B\cos\beta=\sqrt{(A^2+B^2)}\sin(\beta-\theta)\text{ where }\theta=\tan^{-1}\frac{B}{A}$$

Then

$$i=\frac{V_m}{\sqrt{[R^2+(\omega L)^2]}}\sin(\omega t+\lambda-\theta)+Ke^{-\alpha t}$$

Fig. 6.3. Quantitative analysis of the circuit of fig. 6.1(*a*): equation (6.3). The current is the sum of a steady-state term V/R and a transient term $-(V/R)e^{-Rt/L}$.



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To evaluate K, use the condition $i=0$ at $t=0^+$. Then

$$K=-\frac{V_m}{\sqrt{[R^2+(\omega L)^2]}}\sin(\lambda-\theta)$$

and

$$i=\frac{V_m}{\sqrt{[R^2+(\omega L)^2]}}\sin(\omega t+\lambda-\theta)-\frac{V_m e^{-\alpha t}}{\sqrt{[R^2+(\omega L)^2]}}\sin(\lambda-\theta) \quad (6.6)$$

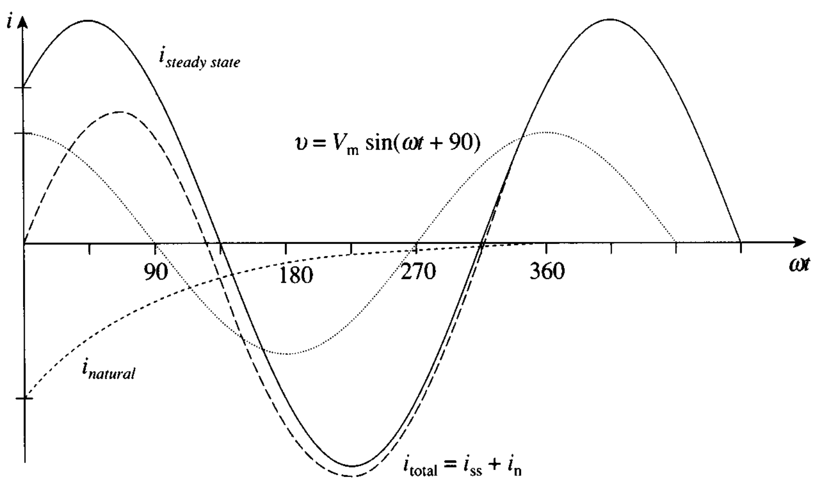
As in the case with d.c. driving voltage, i is the sum of a steady-state term and a transient term that decreases exponentially and eventually becomes zero. The two terms on the right of (6.6) and their sum are shown in fig. 6.4. Except for the arbitrary phase angle λ the first of these terms is identical to (6.1).

As the circuits under consideration become more complex it is useful to take advantage of the fact that the complete solution of the differential equation for a linear circuit is the sum of two responses. That is,

$$i=i_{ss}+i_n \quad (6.7)$$

where i_{ss} is the *steady-state response* or *forced response*, which we know how to find for d.c. and sinusoidal driving sources, and i_n is the *transient response* or *natural response* of the circuit. (The steady-state response and the natural response correspond respectively to the particular integral and the complementary function in the mathematical solution of the circuit

Fig. 6.4. Response of an RL circuit to a sinusoidal driving source: plot of equation 6.6 with $\lambda=90°$, $\theta=45°$.



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differential equation.) The natural response describes the behaviour of the circuit as energy initially stored in one or more elements is dissipated in the resistive elements of the circuit. To find the natural response one simply finds the solution to the differential equation for the circuit when the driving force is set equal to zero. For linear circuit elements the forced response has the same time dependence as the forcing function, and its amplitude (and in the case of a.c., its phase) is completely determined by the circuit parameters. In contrast, the time dependence of the natural response is independent of the forcing function; all currents and voltages have the same time dependence, which is always of the form $r_n = Ae^{st}$. In this expression, s is determined by the circuit configuration and the values of the circuit elements; the constant A depends upon the conditions obtaining at the instant the change occurs (e.g. the throwing of a switch) that initiates the transient behaviour.

Our procedure will be first to find the natural response of some simple circuits. The natural response will then be added to the forced response to obtain the total response. Initial conditions are applied to the total response in order to evaluate the constants that appear in the natural response. For any circuit, then, the natural response provides the smooth transition between the initial state of the circuit and the steady state response to a time dependent driving function.

6.4 Time constant

Consider time dependence of the form

$$y = Ae^{-\alpha t} \tag{6.8}$$

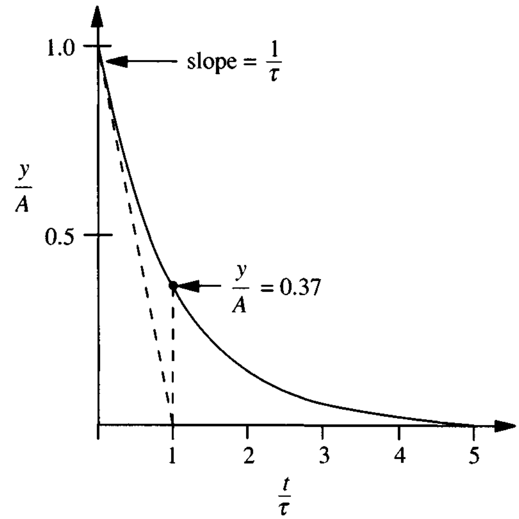
Here, y has value A at $t=0$ and decreases exponentially approaching zero asymptotically as $t \to \infty$. The constant α is a measure of how rapidly y decreases from its initial value. When $t = 1/\alpha$, $y = Ae^{-1} = 0.368\,A$. $1/\alpha$ is called the *time constant* and represents the time required for y to fall to 36.8% of its initial value, A. The time constant usually is designated by τ and is commonly expressed in seconds. (There are however some systems for which the time constant is more appropriately expressed in minutes or hours.) If for (6.8) we plot the ratio y/A against time, expressed as multiples of τ, we have a relation between two dimensionless quantities that is applicable to any equation of the form of (6.8). This is shown in fig. 6.5(*a*).

The concept of time constant is applicable also to

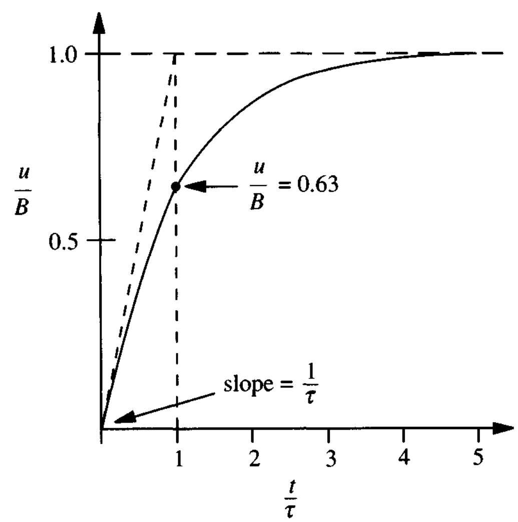
$$u = B(1 - e^{-t/\tau}) \tag{6.9}$$

when $t = \tau$, $u = B(1 - 0.368) = 0.632B$. So for this time dependence, the time constant represents the time required for u to reach 63.2% of its final value.

Fig. 6.5. Illustrating the time constant τ: dimensionless plots of exponential waveforms.



(*a*) Exponential decay



(*b*) Exponential rise

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The curve of u/B v. t/τ is shown in fig. 6.5(*b*).

The initial slope of the curve represented by (6.8) is

$$\frac{\mathrm{d}(y/A)}{\mathrm{d}t}=-\alpha \mathrm{e}^{-\alpha t}\Big|_{t=0}=-\frac{1}{\tau}$$

If y/A continued to decrease linearly at the initial rate it would reach zero in time equal to τ. This is shown in fig. 6.5(*a*) where the tangent drawn at $t=0$ is extended to intersect the t-axis. Similarly, in fig. 6.5(*b*) the initial slope is $+1/\tau$. Then the tangent drawn at $t=0$ intersects the line $u/B=1$ at time $t=\tau$.

For both (6.8) and (6.9), when $t=5\tau$, the dependent variable is within less than 1% of its final value. Therefore for practical purposes one may assume the final value has been achieved when $t>5\tau$.

6.5 Natural response of some basic series circuits

6.5.1 *RL* circuit

In fig. 6.6 switch S_1 has been closed for a long time and S_2 is open. The current through R and L in series is, therefore, $I_0=V/(R_1+R)$.

At $t=0$, S_2 is closed and, simultaneously, S_1 is opened. Current now flows in the part of the circuit completed by S_2, and the energy initially stored in the inductance $(=\frac{1}{2}LI_0{}^2)$ is dissipated over a period of time in the resistance. Since the driving source voltage V is disconnected, the forced response in the part of the circuit which is active for $t>0$ must be zero.

For $t>0$, Kirchhoff's voltage law gives

$$v_L+v_R=0$$

or

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri=0 \tag{6.10}$$

The solution must be such that i and $\mathrm{d}i/\mathrm{d}t$ have the same time dependence; the only appropriate function is the exponential.

Let $i=A\mathrm{e}^{st}$; substitution in (6.10) then gives:

$$sLA\mathrm{e}^{st}+RA\mathrm{e}^{st}=0$$

or

$$sL+R=0 \tag{6.11}$$

(In the mathematical theory of differential equations this equation is referred to as the *auxiliary* equation.)

From (6.11) we obtain

$$s = -R/L, \text{ hence}$$
$$i = A\mathrm{e}^{-Rt/L}$$

The current therefore decays exponentially with time constant L/R.

The constant A is evaluated from the initial condition

$$i = -I_0 = -\frac{V}{R_1 + R} \text{ at } t = 0^+$$

Note that the negative sign appearing, in this expression, arises because of the assignment of current in a clockwise direction in fig. 6.6.

The solution is then

$$i = -\frac{V}{R_1 + R}\mathrm{e}^{-Rt/L} \tag{6.12}$$

This equation is represented by the dimensionless plot of fig. 6.5(a) with $A = |-V/(\mathrm{R}_1 + R)|$ and $\tau = L/R$.

Equation (6.12) gives the natural *current* response of the circuit. The natural *voltage* response across either R or L may be written immediately using (6.12). For the voltage across R we have

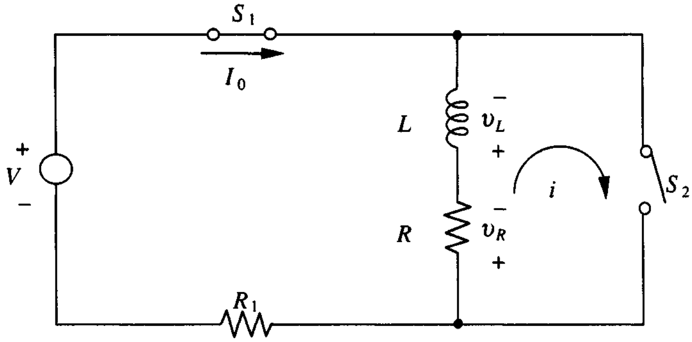
$$v_R = iR = -\frac{VR}{R_1 + R}\mathrm{e}^{-Rt/L} \tag{6.13}$$

and for the voltage across L we have

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = L\left[\frac{R}{L}\frac{V}{(R_1 + R)}\mathrm{e}^{-Rt/L}\right] = \frac{VR}{R_1 + R}\mathrm{e}^{-Rt/L} \tag{6.14}$$

This expression also follows from the fact that $v_L = -v_R$.

Fig. 6.6. Circuit for calculating natural RL response. I_0 is the magnitude of the initial current through L (S_1 closed, S_2 open).



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6.5.2 *RC* circuit

Referring to fig. 6.7, the voltage source V_0 is connected to the capacitance C and S_2 is open. At $t=0$, S_2 is closed and S_1 is opened. At this instant the voltage across the capacitance remains unchanged at V_0 since the stored energy cannot change instantaneously. If we assign current i in a clockwise direction, then v_C will have the polarity indicated and initially (at $t=0^+$) $v_C=-V_0$.

From Kirchhoff's voltage law,

$$v_C+v_R=0$$

or

$$\frac{1}{C}\int i\,\mathrm{d}t+Ri=0 \tag{6.15}$$

Note that, after S_1 is opened, there is no driving source in the circuit so the right-hand side of this equation is zero. (The initial voltage V_0 on the capacitor is not to be confused with a driving source voltage.)

Differentiating (6.15) we obtain

$$\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{1}{CR}i=0$$

Following a procedure similar to that in the previous section for the *RL* circuit, the solution of this equation is found to be:

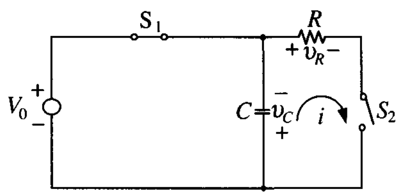
$$i=A\mathrm{e}^{-t/RC} \tag{6.16}$$

Now, referring to the directions of current and voltage shown in fig. 6.7, it is seen that

$$i=\frac{v_R}{R}=-\frac{v_C}{R}=\frac{V_0}{R} \text{ at } t=0^+$$

giving $A=V_0/R$. The natural current response is therefore

Fig. 6.7. Circuit for calculating natural *RC* response. V_0 is the magnitude of the initial voltage on C (S_1 closed, S_2 open).



$$i = \frac{V_0}{R} e^{-t/RC} \tag{6.17}$$

This equation is also represented by the dimensionless plot of fig. 6.5(*a*) with the time constant $\tau = RC$.

The natural voltage response is given by

$$v_R = -v_C = iR = V_0 e^{-t/RC} \tag{6.18}$$

The *RL* circuit and the *RC* circuit each contain one energy storage element and, for this reason, are called *single-energy* circuits. They are also referred to as *first-order* circuits because their behaviour can be described by a first-order differential equation.

Commencing at $t = 0^+$ there is in each circuit a unidirectional current which continues, decreasing exponentially in amplitude, until all the energy that was initially stored is transformed into heat in the resistance. In the *RC* circuit, for example, the energy dissipated in the resistance is

$$W_R = \int_0^\infty i^2 R \, \mathrm{d}t = \frac{V_0^2}{R} \int_0^\infty e^{-2t/RC} \mathrm{d}t = \frac{V_0^2}{R} (-RC/2)[e^{-2t/RC}]_0^\infty$$
$$= \tfrac{1}{2} C V_0^2$$

which is just equal to the energy initially stored in the capacitance.

6.5.3 RLC circuit

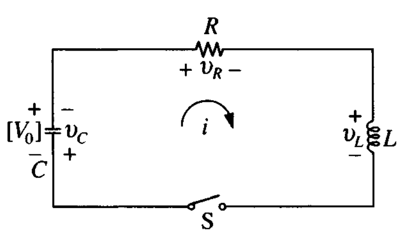
The circuit shown in fig. 6.8 has two energy storage elements and is referred to as a *double-energy* or *second-order* circuit; a second-order differential equation is required to describe its behaviour.

Initially the switch is open and we assume that the capacitance is charged (by means of a circuit similar to that shown in fig. 6.7) to some voltage V_0. For the polarity of V_0 indicated, $v_C = -V_0$ initially.

At $t = 0$ the switch is closed, then for $t > 0$

$$v_L + v_C + v_R = 0$$

Fig. 6.8. *RLC* circuit; V_0 is the magnitude of the initial voltage on *C*.



or

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{1}{C}\int i\,\mathrm{d}t+Ri=0 \tag{6.19}$$

Differentiating removes the integral sign and produces a second-order differential equation of *homogeneous form* (RHS of equation identically zero):

$$\frac{\mathrm{d}^2i}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{1}{CL}i=0$$

Assume, as before, a solution of the form $i=Ae^{st}$; then the auxiliary equation becomes

$$s^2+\frac{R}{L}s+\frac{1}{CL}=0 \tag{6.20}$$

Solving this quadratic equation gives

$$s=-\frac{R}{2L}\pm\sqrt{\left[\left(\frac{R}{2L}\right)^2-\frac{1}{LC}\right]} \tag{6.21}$$

In general, there will be two distinct values of s so

$$i=A_1e^{s_1t}+A_2e^{s_2t} \tag{6.22}$$

giving the two arbitrary constants required by the original second-order differential equation. (The special case where $s_1=s_2$ will be considered later.)

Two initial conditions are required for the evaluation of these constants. Appropriate conditions are:

(1) at $t=0^+$, $i=0$.
This follows from the fact that current through the inductance cannot change instantaneously.

(2) at $t=0^+$, $\mathrm{d}i/\mathrm{d}t=V_0/L$.
This follows from the fact that when $i=0$, then $v_R=0$ and $v_L=-v_C$. But $v_L=L\,\mathrm{d}i/\mathrm{d}t$ and $v_C=-V_0$, so $\mathrm{d}i/\mathrm{d}t=V_0/L$.

Use of these conditions in (6.22) enables us to evaluate A_1 and A_2. Then the solution of (6.19) is

$$i=\frac{V_0}{L}\frac{1}{(s_1-s_2)}(e^{s_1t}-e^{s_2t}) \tag{6.23}$$

Now, referring to (6.21), the quantity

$$\left(\frac{R}{2L}\right)^2-\frac{1}{LC} \tag{6.24}$$

called the *discriminant*, determines which of three particular forms the solution (6.23) takes.

If the discriminant is positive, that is, if $(R/2L)^2 > 1/LC$, s_1 and s_2 are real, negative and unequal. (Remember that R, L and C are intrinsically positive.)

Let $s_1 = -m$, $s_2 = -n$, and let $|n| > |m|$. Then (6.23) becomes

$$i = \frac{V_0}{L}\frac{1}{|n-m|}(e^{-mt} - e^{-nt}) \tag{6.25}$$

The solution is then the sum of two decaying exponentials, as shown in fig. 6.9(*a*).

If the discriminant is negative ($(R/2L)^2 < 1/LC$), s_1 and s_2 are complex conjugate numbers.

Let

$$\frac{R}{2L} = \alpha \text{ and } \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 = \omega_n^{\,2} \tag{6.26}$$

then,

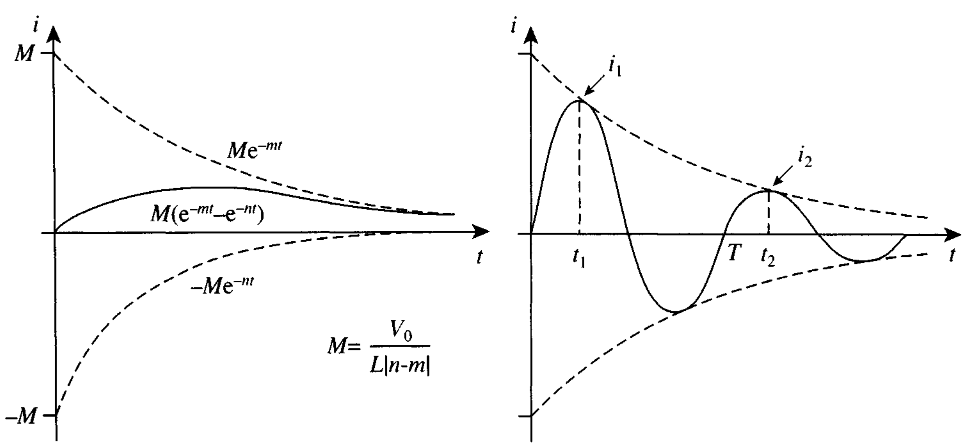
$$s_1 = -\alpha + j\omega_n; \qquad s_2 = -\alpha - j\omega_n \tag{6.27}$$

and, using Euler's identity, (6.23) takes the form:

$$i = \frac{V_0 e^{-\alpha t}}{2j\omega_n L}(e^{j\omega_n t} - e^{-j\omega_n t}) = \frac{V_0 e^{-\alpha t}}{\omega_n L}\sin\omega_n t \tag{6.28}$$

which is an exponentially damped sine wave as shown in fig. 6.9(*b*).

Fig. 6.9. Natural current response for the *RLC* circuit of fig. 6.8.



(*a*) Overdamped response: equation (6.25) (*b*) Underdamped response: equation (6.28)

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When s_1 and s_2 are real, the current starts at zero, increases in magnitude, and then decreases to zero, but it is always in the same direction.

With complex values of s the current is oscillatory at a frequency

$$\omega_n = \sqrt{\left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]} = \sqrt{(\omega_0{}^2 - \alpha^2)} \tag{6.29}$$

where $\omega_0{}^2 = 1/LC$.

ω_n is called the *damped natural frequency* and it represents the frequency of the periodic transfer of energy between the two storage elements. ω_0 is the resonant frequency of the circuit (discussed in section 3.13.2). The *damping constant* α governs the rate at which the amplitude of the oscillations approaches zero as the initial stored energy is dissipated as heat in the resistance. For small values of α, $\omega_n \simeq \omega_0$ and the oscillations last for many cycles. A circuit that has an oscillatory natural response is said to be *underdamped.*

When s_1 and s_2 are real, the circuit is said to be *overdamped.* The condition $s_1 = s_2$ $((R/2L)^2 = 1/LC)$ represents the transition between the overdamped and the underdamped states, and it is called the condition of *critical damping.* To get a complete solution of (6.19) for this case one must use

$$i = (A_1 + tA_2)e^{st} \tag{6.30}$$

in order to have the required two constants of integration. Although it represents the condition for which the current reaches zero in minimum time, critical damping is of no special practical significance in electrical circuits. It usually is not worthwhile to select circuit components carefully enough to achieve exact critical damping. Moreover, there is the distinct possibility that ageing of carefully chosen circuit components may cause their values to change in such a way as to make the circuit oscillatory.

6.5.4 Q-factor and logarithmic decrement

For the oscillatory RLC circuit whose current is given by (6.28) it is useful to have a number that relates the damping to the natural frequency ω_n of the circuit. We denote this number by Q_n (the Q-factor) and we define it as*

* This quantity is different from Q_0 the Q-factor defined in the discussion of resonance in section 3.13.2. There we defined $Q_0 = \omega_0 L/R$ where $\omega_0 = 1/(LC)^{\frac{1}{2}}$. Since large values of Q_n are associated with small values of R, it follows from (6.29) that when Q_n is large α is small, $\omega_n \simeq \omega_0$, and $Q_n \simeq Q_0$.

$$Q_n = \frac{\omega_n}{2\alpha} = \frac{\omega_n L}{R} \tag{6.31}$$

$$\alpha = \frac{R}{2L} = \frac{\omega_n}{2Q_n} = \frac{\pi}{Q_n T} \tag{6.32}$$

where T is the period of the natural response as indicated on fig. 6.9. Then, putting $I_0 = V_0/\omega_n L$, (6.28) becomes

$$i = I_0 e^{-(\pi/Q_n T)t} \sin\omega_n t \tag{6.33}$$

We see from (6.33) that when t increases by one period, the amplitude decreases by the factor $e^{-\pi/Q_n}$. So Q_n is a measure of the damping per cycle. Furthermore, the time required for i to decrease by a factor $1/e$ is equal to $Q_n T/\pi$. Since Q_n is inversely proportional to R we expect that a large value of Q_n is characteristic of a circuit that requires a long time for the oscillations to die out.

We can determine the Q of the circuit by examining the oscillatory decay and measuring the amplitude of two successive peaks. In fig. 6.9

$$i_1 = I_0 e^{-(\pi/Q_n T)t_1}, \qquad i_2 = I_0 e^{-(\pi/Q_n T)t_2}$$

and

$$\frac{i_1}{i_2} = e^{(\pi/Q_n T)(t_2 - t_1)} \tag{6.34}$$

If $(t_2 - t_1) = T$, then $i_1/i_2 = e^{\pi/Q_n}$

and

$$\ln(i_1/i_2) = \pi/Q_n.$$

The quantity π/Q_n is the *logarithmic decrement.*

6.6 Total response

The natural responses of the circuits that we have considered so far are examples of transient behaviour for the special situation where the steady-state response is zero because there are no driving currents or voltages present. When such energy sources are part of the circuit the constants that appear in the natural response must be evaluated by applying the initial conditions to the *complete* solution. Depending upon the driving function and upon conditions that exist immediately after the switching operation (that is, at $t = 0^+$) the constants assume the values necessary to provide a smooth transition from the initial to the final state of the circuit.

6.6.1 RL circuit with sinusoidal driving voltage

The usefulness of writing

$$i = i_{ss} + i_n$$

is well illustrated by a reconsideration of the circuit of fig. 6.1(*b*). The differential equation is (6.5)

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V_m}{L} \sin(\omega t + \lambda)$$

The steady state solution is

$$i_{ss} = \frac{V_m}{\sqrt{[R^2 + (\omega L)^2]}} \sin(\omega t + \lambda - \theta); \qquad \theta = \tan^{-1} \frac{\omega L}{R} \tag{6.35}$$

The natural response is

$$i_n = A e^{-Rt/L} \tag{6.36}$$

The total response then is the sum of (6.35) and (6.36)

$$i = i_{ss} + i_n = \frac{V_m}{\sqrt{[R^2 + (\omega L)^2]}} \sin(\omega t + \lambda - \theta) + A e^{-Rt/L} \tag{6.37}$$

Applying the initial condition ($i = 0$ at $t = 0^+$),

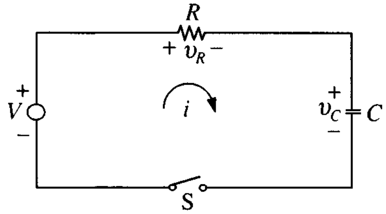
$$A = -\frac{V_m}{\sqrt{[R^2 + (\omega L)^2]}} \sin(\lambda - \theta)$$

and

$$i = \frac{V_m}{\sqrt{[R^2 + (\omega L)^2]}} \sin(\omega t + \lambda - \theta) - \frac{V_m}{\sqrt{[R^2 + (\omega L)^2]}} e^{-Rt/L} \sin(\lambda - \theta) \tag{6.38}$$

This is identical to (6.6). It is apparent that the approach just employed is more direct than that followed in deriving (6.6).

Fig. 6.10. *RC* circuit with constant voltage driving source.



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6.6.2 RC circuit with constant voltage source

In the circuit of fig. 6.10 we are interested in the voltage v_C on the capacitance after the switch is closed at $t=0$.

The natural response will, from the theory given in section 6.5.2, be of the form:

$$v_{Cn} = A\mathrm{e}^{-t/RC} \tag{6.39}$$

For the steady state,

$$v_{Css} = V \tag{6.40}$$

The total response is the sum of (6.39) and (6.40).

$$v_C = A\mathrm{e}^{-t/RC} + V$$

Assuming no initial charge on the capacitance, the constant A is evaluated from the initial condition:

$$v_C = 0 \text{ at } \mathrm{t} = 0^+,$$

hence,

$$A = -V$$

and

$$v_C = V(1 - \mathrm{e}^{-t/RC}) \tag{6.41}$$

This expression is represented by the dimensionless plot of fig. 6.5(*b*) with $B=V$ and $\tau = RC$.

The current is given by

$$i = C\frac{\mathrm{d}v_C}{\mathrm{d}t} = \frac{V}{R}\mathrm{e}^{-t/RC} \tag{6.42}$$

6.6.3 Worked example

A circuit designed to fire a laser flash tube consists of the following (see fig. 6.11): a 12 V battery of internal resistance 10 Ω is connected via a switch s_1 to a resistance of 80 Ω in series with a relay coil of resistance 10 Ω and inductance 2 H. The relay operates when the current in the circuit reaches 50 mA. The operation of the relay closes a switch s_2 in another circuit so that a capacitor bank of 100 μF is charged up via a resistor of 1 kΩ in series with a 2 kV supply.

If the laser fires when the capacitor bank is charged up to 1 kV, find the time taken from the closing of s_1 to firing of the laser. Neglect the time required for the relay to operate.

Solution.

The diagrams for the two parts of the circuit are shown in fig. 6.11.

Relay circuit (fig. 6.11(a)): let R be the total resistance in the circuit and L be the inductance of the relay coil. The natural current response of the circuit is then, from section 6.5.1,

$$i_{\text{n}} = A\text{e}^{-Rt/L}$$

The steady state, or forced response, is obviously

$$i_{\text{ss}} = \frac{V}{R}$$

Hence, the total response is

$$i = A\text{e}^{-Rt/L} + \frac{V}{R}$$

At $t = 0^+$, $i = 0$ therefore $A = -V/R$ so the current in the relay circuit is given by

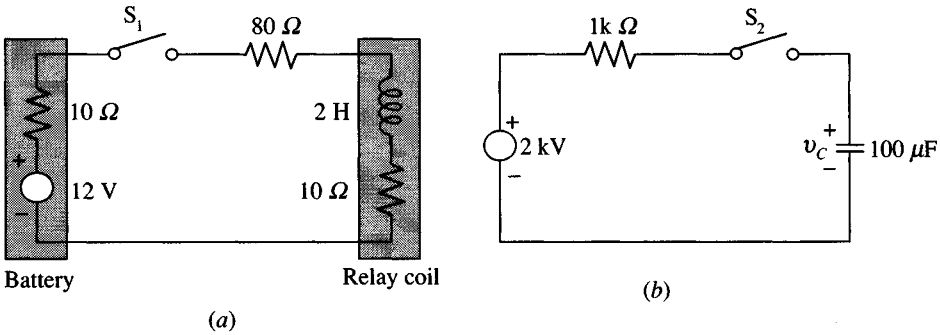
$$i = \frac{V}{R}(1 - \text{e}^{-Rt/L})$$

Inserting numerical values ($R = 100\,\Omega$; $L = 2\text{H}$) we find the time t_1 for the current to reach 50 mA

$$50 \times 10^{-3} = \frac{12}{100}(1 - \text{e}^{-50t_1})$$

giving $\text{e}^{-50t_1} = 0.583$ or $t_1 = 10.8$ ms.

Fig. 6.11. Circuits for worked example (section 6.6.3).



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Capacitor circuit (fig. 6.11(*b*)): after the switch s_2 closes, the voltage on the capacitor is, from (6.41),

$$v_C = V(1 - e^{-t/RC})$$

If t_2 is the time for the voltage to reach 1 kV, then

$$1 \times 10^3 = (2 \times 10^3)(1 - e^{-10t_2})$$

giving $e^{-10t_2} = 0.5$ or $t_2 = 69.3$ ms.

So the total time from closing the first switch to the firing of the laser is $10.8 + 69.3 = 80.1$ ms.

6.6.4 RLC circuit with constant voltage source

In the circuit of fig. 6.12(*a*) the capacitor is initially uncharged. At $t=0$, S is closed. We require an expression for the current for $t>0$. The steady-state current i_{ss} is zero and the natural response is, from (6.22),

$$i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The initial conditions are:

$$i = 0 \text{ and } v_C = v_R = 0, \text{ so } v_L = V, \text{ or } di/dt = V/L$$

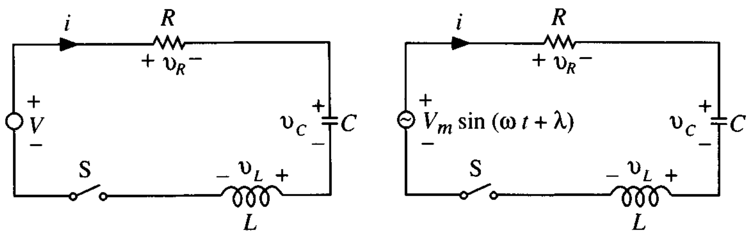
When these conditions are used to evaluate A_1 and A_2, we obtain

$$i = \frac{V}{L} \frac{1}{s_1 - s_2}(e^{s_1 t} - e^{s_2 t}) \tag{6.43}$$

This is identical with the expression contained in section 6.5.3 for the natural response of the circuit; a result which is to be expected since the forced response is zero in the present case.

As discussed previously in section 6.5.3, the current will be either the sum of two decreasing exponentials or oscillatory with exponentially decreasing amplitude, depending upon the relative magnitudes of $(R/2L)^2$ and $1/LC$.

Fig. 6.12. *RLC* circuit with constant and sinusoidal driving voltages.



(*a*) Constant (d.c.) driving voltage (*b*) Sinusoidal driving voltage

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6.6.5 RLC circuit with sinusoidal driving voltage

When the *RLC* circuit is driven by a sinusoidal voltage, as in fig. 6.12(*b*), the forced response is (from our a.c. theory)

$$i_{ss}=\frac{V_m}{Z}\sin(\omega t+\lambda-\theta) \tag{6.44}$$

where

$$Z=\sqrt{[R^2+(X_L-X_C)^2]} \text{ and } \tan\theta=\frac{X_L-X_C}{R}$$

Because the natural response can have different forms depending upon circuit constants, the transient response may exhibit wide variations. In every case, however, the *form* of the transient is determined by the circuit, and the *amplitude* is whatever is required to satisfy the initial conditions. These conditions depend upon the initial energy stored (if any) and upon the instant in the cycle of the applied voltage at which the switch is closed.

If the *RLC* circuit is overdamped, the natural response is given by (6.22)

$$i_n=A_1e^{-mt}+A_2e^{-nt} \tag{6.45}$$

where $-m$ and $-n$ are the two appropriate values of s in (6.22). The complete solution is then

$$i=i_{ss}+i_n=\frac{V_m}{Z}\sin(\omega t+\lambda-\theta)+A_1e^{-mt}+A_2e^{-nt} \tag{6.46}$$

Two initial conditions are required for evaluation of the constants A_1 and A_2.

For the underdamped case the natural response is, from (6.22), (6.26) and (6.27),

$$i_n=A_1e^{s_1t}+A_2e^{s_2t}=e^{-\alpha t}[A_1e^{j\omega_nt}+A_2e^{-j\omega_nt}] \tag{6.47}$$

Using Euler's identity this can be written

$$i_n=e^{-\alpha t}[B_1\sin\omega_nt+B_2\cos\omega_nt]$$

or

$$i_n=e^{-\alpha t}M\sin(\omega_nt+\phi) \tag{6.48}$$

where

$$M=\sqrt{(B_1{}^2+B_2{}^2)};\qquad \phi=\tan^{-1}\frac{B_2}{B_1}$$

The complete solution is then

$$i=\frac{V_m}{Z}\sin(\omega t+\lambda-\theta)+e^{-\alpha t}M\sin(\omega_n t+\phi) \tag{6.49}$$

Note that the evaluation of the constants B_1 and B_2 is often easier than the direct evaluation of constants M and ϕ.

6.6.6. RLC circuit with sinusoidal drive and $\omega_o \simeq \omega_n$

Referring to fig. 6.12(*b*), if $1/LC \gg (R/2L)^2$, the circuit is lightly damped, and the damped natural frequency ω_n and the resonant frequency ω_0 are very nearly equal. If there is no initial stored energy in the circuit, the transient response depends upon the phase angle λ of the applied voltage and also upon how the applied frequency ω compares with ω_n.

Case 1. Let $\lambda=0$. The initial conditions are then:

(1) $i=0$ (current in inductor cannot change instantaneously);

(2) $di/dt=0$ ($v=v_R+v_C+v_L$, but $v=0$ and also $v_R=0$ and $v_C=0$, therefore $v_L=L di/dt=0$).

Case 1a: $\omega=\omega_n$. The circuit is resonant, therefore in (6.44) $X_L=X_C$, $Z=R$, and $\theta=0$.

So, (6.44) becomes

$$i_{ss}=\frac{V_m}{R}\sin\omega t$$

and (6.49) becomes

$$i=\frac{V_m}{R}\sin\omega t+e^{-\alpha t}M\sin(\omega t+\phi) \tag{6.50}$$

Differentiating (6.50) gives

$$\frac{di}{dt}=\frac{\omega V_m}{R}\cos\omega t+e^{-\alpha t}M\omega\cos(\omega t+\phi)-\alpha e^{-\alpha t}M\sin(\omega t+\phi) \tag{6.51}$$

The first initial condition gives, by (6.50),

$$M\sin\phi=0$$

If $M=0$, there is no transient, so we take this condition to mean $\sin\phi=0$. Hence, $\sin\phi=0$ and so $\phi=0$.

The second initial conditions gives, by (6.51),

$$0=\omega\frac{V_m}{R}+M\omega, \text{ hence } M=-\frac{V_m}{R}$$

Then

$$i=\frac{V_{\rm m}}{R}(1-{\rm e}^{-\alpha t})\sin\omega t \tag{6.52}$$

The current is sinusoidal inside an envelope that starts at zero and approaches asymptotically the values $\pm V_{\rm m}/R$ (see fig. 6.13).

Case 1b: $\omega \gg \omega_{\rm n}$. The circuit is predominantly inductive and so to a close approximation we may write $Z=\omega L$ and $\theta=\pi/2$ rad. So (6.44) becomes,

$$i_{\rm ss}=-\frac{V_{\rm m}}{\omega L}\cos\omega t$$

Then

$$i=-\frac{V_{\rm m}}{\omega L}\cos\omega t+{\rm e}^{-\alpha t}M\sin(\omega_{\rm n}t+\phi)$$

and

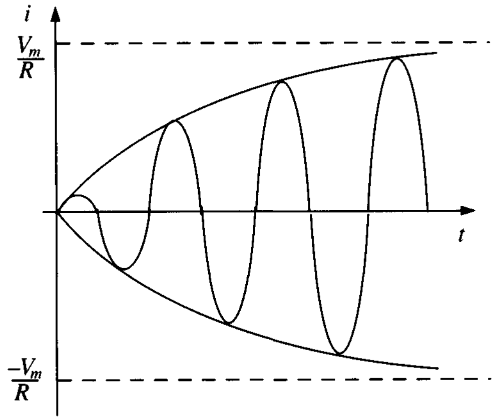
$$\frac{{\rm d}i}{{\rm d}t}=\frac{V_{\rm m}}{L}\sin\omega t-\alpha{\rm e}^{-\alpha t}M\sin(\omega_{\rm n}t+\phi)+{\rm e}^{-\alpha t}\omega_{\rm n}M\cos(\omega_{\rm n}t+\phi)$$

Substitution of the initial conditions yields

$$\tan\phi=\frac{\omega_{\rm n}}{\alpha}, \text{ and since } \omega_{\rm n}\gg\alpha, \phi\simeq\pi/2$$

$$M=\frac{V_{\rm m}}{\omega L}$$

Fig. 6.13. Response of *RLC* circuit driven at its natural frequency.



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Therefore,

$$i = -\frac{V_m}{\omega L}(\cos\omega t - e^{-\alpha t}\cos\omega_n t) \tag{6.53}$$

Case 1c: $\omega \ll \omega_n$. In the steady state this circuit is capacitive with current leading the voltage by 90°. So (6.44) becomes:

$$i_{ss} = \omega C V_m \cos\omega t$$

Following the same procedure as that used for Case 1b, we obtain

$$i = \omega C V_m(\cos\omega t - e^{-\alpha t}\cos\omega_n t) \tag{6.54}$$

In neither Case 1b nor Case 1c does the current reach exceptionally high values; under no circumstances will it be greater than twice the steady-state value.

Case 2. Let $\lambda = \pi/2$ radians. Now the driving voltage has maximum value at $t = 0^+$. Initial conditions are:

(1) $i = 0$
(2) $di/dt = V_m/L$

These lead to the following expressions for the current

Case 2a: $\omega = \omega_n$.

$$i = \frac{V_m}{R}(1 - e^{-\alpha t})\cos\omega t + \frac{V_m}{2\omega L}e^{-\alpha t}\sin\omega t \tag{6.55}$$

Case 2b: $\omega \gg \omega_n$.

$$i = \frac{V_m}{\omega L}\left(\sin\omega t - \frac{\omega_n}{\omega}e^{-\alpha t}\sin\omega_n t\right) \tag{6.56}$$

The transient term is negligibly small because of the multiplying factor (ω_n/ω).

Case 2c: $\omega \ll \omega_n$.

$$i = -\omega C V_m\left(\sin\omega t - \frac{\omega_n}{\omega}e^{-\alpha t}\sin\omega_n t\right) \tag{6.57}$$

Now the ratio (ω_n/ω) is large and so the amplitude of the transient component may be many times the steady-state amplitude.

6.7 The D-operator

In our study of transient analysis so far we have considered circuits mainly of a simple series form containing not more than two storage elements, and driven by constant (d.c.) or sinusoidal (a.c.) driving sources.

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For more complicated circuits with arbitrary driving sources, the circuit integro-differential equations can be complex, and their solution correspondingly difficult. For this reason 'operational' methods have been devised which greatly simplify the process of formulating the circuit equations and which, for some important practical cases, provide elegant methods of solution. One such operational method is presented in this section. Our procedure will be to describe the method and illustrate its use with examples. A more complete description and mathematical justification of the method will be found in ref. 15.

6.7.1 The operators D and D^{-1}

We define the 'differential operator' D by

$$\mathrm{D}=\frac{\mathrm{d}}{\mathrm{d}t} \tag{6.58}$$

so that we may write

$$\mathrm{D}x=\frac{\mathrm{d}x}{\mathrm{d}t} \tag{6.59}$$

and we interpret $\mathrm{D}^n x$ to mean $\mathrm{d}^n x/\mathrm{d}t^n$, that is the symbol D^n operating on x signifies the process of differentiating n times.

Extending the notation further we interpret $1/\mathrm{D}=\mathrm{D}^{-1}$ as signifying the process of integration, that is,

$$\frac{1}{D}x=\mathrm{D}^{-1}x=\int x\,\mathrm{d}t \tag{6.60}$$

so that

$$\mathrm{D}\frac{1}{\mathrm{D}}x=\frac{\mathrm{d}}{\mathrm{d}t}\int x\,\mathrm{d}t=x$$

Using this notation the voltage–current relationships for inductance and capacitance,

$$v_L=L\frac{\mathrm{d}i}{\mathrm{d}t} \qquad v_C=\frac{1}{C}\int i\,\mathrm{d}t$$

become,

$$v_L=L\,\mathrm{D}i \qquad v_C=\frac{1}{C\mathrm{D}}i \tag{6.61}$$

Similarly, the circuit equation appertaining to the general branch with a

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sinusoidal driving voltage (fig. 6.12(*b*)), namely,

$$L\frac{di}{dt}+\frac{1}{C}\int i\,dt+Ri=V_m\sin(\omega t+\lambda)$$

becomes

$$LDi+\frac{1}{CD}i+Ri=V_m\sin(\omega t+\lambda) \tag{6.62}$$

The use of D and D^{-1} to indicate respectively differentiation and integration, would appear to be but a modest extension of the process of symbolic representation, however, it can be shown that for linear differential equations with constant coefficients the D-operator can be treated like a coefficient in an algebraic equation. Specifically, the operator obeys the distributive, commutative and associative laws of algebra. This implies, for example, that

$$D(x+y)=Dx+Dy$$

and

$$(D-m_1)(D-m_2)=(D-m_2)(D-m_1)=D^2-(m_1+m_2)D+m_1m_2$$

Functions of D also obey the laws of algebra; for example,

$$F_1(D)\times F_2(D)=F_2(D)\times F_1(D) \text{ and } F(D)(x+y)=F(D)x+F(D)y$$

These algebraic properties allow us to multiply both sides of (6.62) by D (corresponding to differentiation term by term) to obtain

$$LD^2i+\frac{1}{C}i+DRi=DV_m\sin(\omega t+\lambda)$$

or

$$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)i=\frac{D}{L}V_m\sin(\omega t+\lambda) \tag{6.63}$$

6.7.2 Solution of differential equations by D-operator

In general, the differential equations that characterize linear circuits are of the form:

$$F(D)y=X \tag{6.64}$$

where the function $F(D)$ is a polynomial, y represents voltage or current, and X represents a time-dependent driving source (voltage or current). We have already seen that such an equation may be solved in two stages: first, the complementary function (natural response of the circuit) is found from the homogeneous equation

$$F(\mathrm{D})y=0 \tag{6.65}$$

Substitution of $y=\mathrm{A}e^{st}$ leads directly to the auxiliary equation $F(s)=0$. For example, putting $i=Ae^{st}$ in (6.63) and using (6.65) we obtain

$$\left(\mathrm{D}^2+\frac{R}{L}\mathrm{D}+\frac{1}{LC}\right)Ae^{st}=0 \tag{6.66}$$

$$As^2e^{st}+\frac{RA}{L}se^{st}+\frac{A}{LC}e^{st}=0$$

or

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0 \tag{6.67}$$

Comparing (6.66) and (6.67) we see that the auxiliary equation may be written directly merely by substitution of s for D.

The second stage in the solution of (6.64) is to find the particular integral (forced or steady-state response of the circuit). Now it can be shown that the D-operator method enables one to obtain the particular integral from

$$y=\frac{1}{F(\mathrm{D})}X \tag{6.68}$$

A variety of methods exist for solving this equation, depending upon the particular form of X; we consider three important cases.

Case 1. $X=x^n$ (n=positive integer)

In this case

$$y=[F(\mathrm{D})]^{-1}x^n \tag{6.69}$$

and $[F(\mathrm{D})]^{-1}$ is expressed as a polynomial in rising powers of D as far as D^n. (Any higher powers of D will yield zero.)

Example: $(\mathrm{D}^2-4\mathrm{D}+4)y=x^2$

The P.I. is

$$y=\frac{1}{\mathrm{D}^2-4\mathrm{D}+4}x^2=\frac{1}{4(1-\mathrm{D}+\mathrm{D}^2/4)}x^2$$

$$=\tfrac{1}{4}\left(1-\mathrm{D}+\frac{\mathrm{D}^2}{4}\right)^{-1}x^2$$

Expanding by the binomial theorem:

$$y=\tfrac{1}{4}\left[1+\left(D-\frac{D^2}{4}\right)+\left(D-\frac{D^2}{4}\right)^2\ldots\right]x^2$$

$$=\tfrac{1}{4}\left(1+D-\frac{D^2}{4}+D^2\ldots\right)x^2$$

$$y=\tfrac{1}{4}(x^2+2x+\tfrac{3}{2})$$

Case 2. $X=e^{ax}$

We have $De^{ax}=ae^{ax}$; $D^2e^{ax}=a^2e^{ax}$; $D^ne^{ax}=a^ne^{ax}$ hence, if $F(D)$ is a polynomial,

$$F(D)e^{ax}=F(a)e^{ax} \tag{6.70}$$

Now

$$\frac{1}{F(D)}F(D)e^{ax}=\frac{1}{F(D)}F(a)e^{ax}=F(a)\frac{1}{F(D)}e^{ax}$$

but

$$\frac{1}{F(D)}F(D)e^{ax}=e^{ax}$$

therefore

$$y=\frac{1}{F(D)}e^{ax}=\frac{1}{F(a)}e^{ax} \qquad (F(a)\neq 0) \tag{6.71}$$

Example: $(3D^2-2D+4)y=36e^{-x}$

The P.I. is

$$y=\frac{36e^{-x}}{3D^2-2D+4}=\frac{36e^{-x}}{3(-1)^2-2(-1)+4}=4e^{-x}$$

Case 3. $X=e^{ax}V(x)$ where $V(x)$ is a function of x only.

For this case it can be shown that

$$y=\frac{1}{F(D)}e^{ax}V(x)=e^{ax}\frac{1}{F(D+a)}V(x) \tag{6.72}$$

Example: $(D^2+D-2)y=xe^x$

The P.I. is

$$y=\frac{1}{D^2+D-2}xe^x$$

Using theorem (6.72), the exponential is shifted to the left of the operator and D becomes $D+1$:

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$$y = e^x \frac{1}{(D+1)^2 + (D+1) - 2} x = e^x \frac{1}{D(D+3)} x$$

$$= \frac{e^x}{3D}\left(1 + \frac{D}{3}\right)^{-1} x = \frac{e^x}{3D}\left(1 - \frac{D}{3} \ldots\right) x$$

$$y = \frac{e^x}{3D}(x - \tfrac{1}{3})$$

The D in the denominator means that the expression to its right is integrated once, thus

$$y = \frac{e^x}{3}\left(\frac{x^2}{2} - \frac{x}{3}\right)$$

Theorem (6.72) allows us to deal with situations for which theorem (6.71) of Case 2 breaks down, for example

$$y = \frac{1}{D^2 + 4D + 4} e^{-2x}$$

Simply replacing D by -2 as required by (6.71), gives zero in the denominator of this expression. However, if we take $V(x) = 1$ in (6.72) we obtain:

$$y = e^{-2x} \frac{1}{(D-2)^2 + 4(D-2) + 4}(1) = e^{-2x} \frac{1}{D^2}(1) = e^{-2x} \frac{x^2}{2}$$

Returning now to (6.62), appertaining to the general branch with sinusoidal excitation, we may use theorem (6.71) of Case 2 to derive the particular integral. Representing the RHS of (6.62) by the imaginary part of the complex exponential we may write

$$\left(LD + \frac{1}{CD} + R\right) i = \mathrm{Im}\, V_m e^{j(\omega t + \lambda)}$$

Now, according to (6.68), the particular integral is given by

$$i = \frac{V_m}{LD + 1/CD + R} \mathrm{Im} e^{j(\omega t + \lambda)} \tag{6.73}$$

and, by (6.71), D may be replaced by $j\omega$ to give

$$i = \mathrm{Im} \frac{V_m}{j\omega L + 1/j\omega C + R} e^{j(\omega t + \lambda)} \tag{6.74}$$

The denominator of this expression is recognized as the complex impedance, which may be written $Ze^{j\theta}$ where $Z = [R^2 + (\omega L - 1/\omega C)^2]^{\frac{1}{2}}$ and

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$\theta = \tan^{-1}(\omega L - 1/\omega C)/R$, hence

$$i = \mathrm{Im}\frac{V_{\mathrm{m}}}{Z\mathrm{e}^{\mathrm{j}\theta}}\mathrm{e}^{\mathrm{j}(\omega t+\lambda)} = \mathrm{Im}\frac{V_{\mathrm{m}}}{Z}\mathrm{e}^{\mathrm{j}(\omega t+\lambda-\theta)}$$

or

$$i = \frac{V_{\mathrm{m}}}{Z}\sin(\omega t + \lambda - \theta)$$

The derivation of this result should be compared with the methods used in section 3.4.

6.7.3 D-impedance

The concepts of complex exponential and complex impedance were introduced in sections 3.3 and 3.4 in connection with steady-state a.c. circuit analysis. A related concept which is useful in the context of transient analysis will now be considered.

We observe that the 'jω' in the denominator of (6.74), which is the complex impedance, arise directly as a result of the 'D' in (6.73). This result follows whatever the combination of circuit elements under consideration, and by analogy we call the function of D appearing in the denominator of equations such as (6.73) the D-impedance. This concept offers a convenient approach to the setting up of circuit differential equations, which is exactly analogous to that used for setting up the steady-state a.c. circuit equations.

First, write down the a.c. impedance (or reactance) of each circuit element but with D in place of jω. Then combine the impedances and derive the circuit equations in the usual way. The ratio of voltage $v(t)$ to current $i(t)$ at any terminal pair of a network will be the D-impedance at that terminal pair. For pure inductive and capacitive elements the relations analogous to jωL and $1/\mathrm{j}\omega C$ are LD and $1/C$D. Notice that we place the operator D after the constant since in the full equations in which they occur, for instance (6.62), the D or 1/D will be operating upon a variable, either i or v, situated to the right-hand side of the operator. This convention need not be strictly adhered to in the course of algebraic manipulation but it makes for clarity in the interpretation of the end formulation of the circuit differential equations. This point will become apparent in the following worked example.

6.7.4 Worked example

For the circuit of fig. 6.14 derive differential (D-operator) equations for the current $i(t)$ and the voltage $v(t)$. If $R_1 = R_2 = 2\,\mathrm{M}\Omega$ and $C = 0.5\mu\mathrm{F}$, find explicit expressions for the steady state components of $i(t)$

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and $v(t)$ given that the driving function $v_1(t)$ is: (a) V_0(constant); (b) t; (c) e^{-t}; (d) te^{-t}.

Solution:
Let $Z(\mathrm{D})$ be the impedance of $R_2//(1/C\mathrm{D})$ then

$$Z(\mathrm{D})=\frac{R_2(1/C\mathrm{D})}{R_2+1/C\mathrm{D}}=\frac{R_2}{1+R_2C\mathrm{D}}$$

R_1 and $Z(\mathrm{D})$ form a voltage divider, hence,

$$v=\frac{Z(\mathrm{D})}{R_1+Z(\mathrm{D})}v_1=\frac{R_2}{(1+R_2C\mathrm{D})R_1+R_2}v_1$$

or

$$v=\frac{R_2}{R_1R_2C\mathrm{D}+R_1+R_2}v_1$$

A differential equation for v is then

$$(R_1R_2C\mathrm{D}+R_1+R_2)v=R_2v_1$$

or

$$\left[\mathrm{D}+\frac{R_1+R_2}{CR_1R_2}\right]v=\frac{1}{CR_1}v_1$$

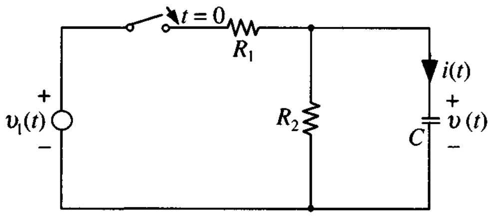
Current i and voltage v are related by $v=(1/C\mathrm{D})i$, hence substituting in the above expression gives a differential equation for i:

$$\left[\mathrm{D}+\frac{R_1+R_2}{CR_1R_2}\right]i=\frac{\mathrm{D}}{R_1}v_1$$

If $R_1=R_2=2\ \mathrm{M}\Omega$ and $C=0.5\ \mu\mathrm{F}$, the above equations for i and v reduce to:

$$(\mathrm{D}+2)v=v_1 \qquad \text{and} \qquad (\mathrm{D}+2)i=\frac{10^{-6}}{2}\mathrm{D}v_1$$

Fig. 6.14. Circuit for worked example (section 6.7.4).



The steady state responses are then given by

$$v_{ss}=\frac{1}{D+2}v_1 \qquad \text{and} \qquad i_{ss}=\frac{10^{-6}}{2(D+2)}Dv_1$$

(a) $v_1=V_0$

$$v_{ss}=\frac{1}{2(1+D/2)}V_0=\tfrac{1}{2}\left(1+\frac{D}{2}\right)^{-1}V_0=\tfrac{1}{2}\left(1-\frac{D}{2}\dots\right)V_0=\frac{V_0}{2}$$

$$i_{ss}=\frac{10^{-6}}{2(D+2)}DV_0=0 \quad [\text{Alternatively } i_{ss}=CDv_{ss}=0]$$

(b) $v_1=t$

$$v_{ss}=\frac{1}{2}\left(1-\frac{D}{2}\cdots\right)t=\frac{1}{2}\left(t-\frac{1}{2}\right)=\left(\frac{t}{2}-\frac{1}{4}\right)$$

$$i_{ss}=\frac{10^{-6}}{2(D+2)}Dt=\frac{10^{-6}}{4\left(1+\frac{D}{2}\right)}=\frac{10^{-6}}{4}\left(1+\frac{D}{2}\right)^{-1}$$

$$=\frac{10^{-6}}{4}\left(1-\frac{D}{2}\dots\right)=\frac{10^{-6}}{4} \quad [\text{Alternatively } i_{ss}=CDv_{ss}=\frac{10^{-6}}{4}.]$$

(c) $v_1=e^{-t}$

$$v_{ss}=\frac{1}{D+2}e^{-t}=\frac{1}{-1+2}e^{-t}=e^{-t}$$

$$i_{ss}=\frac{10^{-6}}{2(D+2)}De^{-t}=\frac{10^{-6}}{2(D+2)}(-e^{-t})=-\frac{10^{-6}}{2}e^{-t}$$

(d) $v_1=te^{-t}$

$$v_{ss}=\frac{1}{D+2}te^{-t}=e^{-t}\frac{1}{D-1+2}t=e^{-t}(1+D)^{-1}t=e^{-t}(1-D\dots)t$$
$$=e^{-t}(t-1)$$

$$i_{ss}=\frac{10^{-6}}{2(D+2)}Dte^{-t}=\frac{10^{-6}}{2(D+2)}(e^{-t}-te^{-t})$$

$$=\frac{10^{-6}}{2}\left[\frac{e^{-t}}{-1+2}-e^{-t}\frac{1}{D-1+2}t\right]=\frac{10^{-6}}{2}e^{-t}(2-t)$$

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Case (b) in this example may be used to illustrate an important point in connection with the response of circuits containing storage elements. For the component values given, the time constant of the circuit is $\frac{1}{2}$ second, and the complete solution for the voltage is

$$v = Ae^{-2t} + \frac{t}{2} - \frac{1}{4}$$

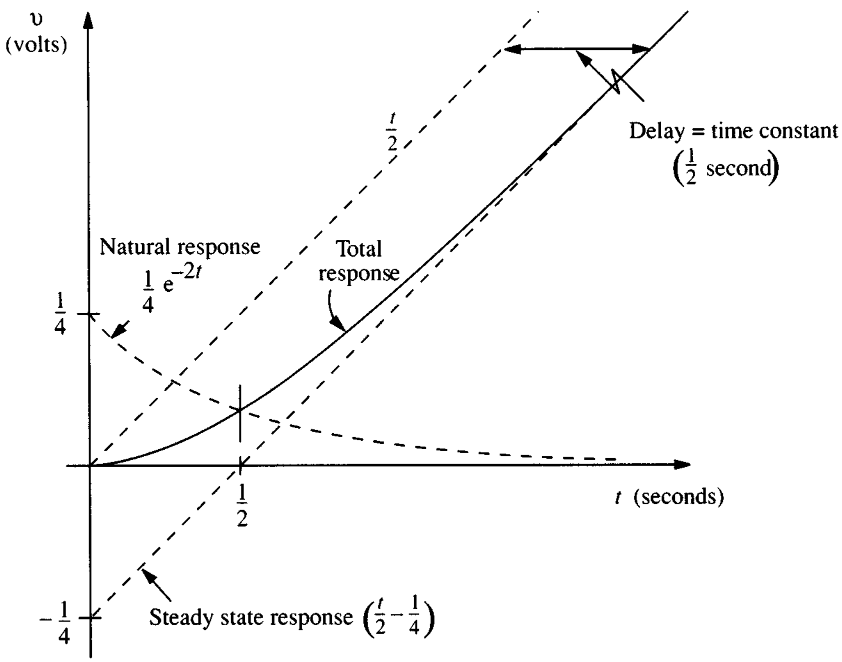
The initial condition is $v=0$ at $t=0^+$, therefore $A=\frac{1}{4}$ giving

$$v = \frac{1}{4}e^{-2t} + \frac{t}{2} - \frac{1}{4} \tag{6.75}$$

The curve of v versus t, shown in fig. 6.15, is asymptotic to the line $\frac{t}{2}-\frac{1}{4}$ which is, of course, the steady state solution.

Without the capacitance the response of the circuit would simply be $\frac{t}{2}$ since $R_1 = R_2$. We see then that the addition of the capacitance has the effect of shifting the steady-state response bodily to the right by one time constant. This result is true in general; a circuit containing a single storage element will introduce a time delay between excitation and response equal

Fig. 6.15. Illustrating delay in the response of a circuit containing a single storage element.



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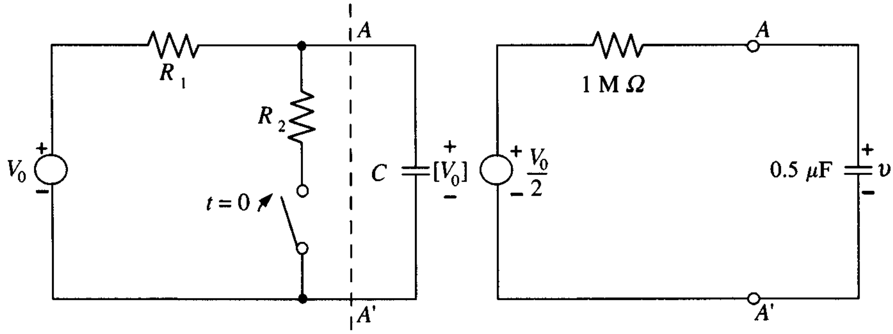
to the time constant of the circuit. Circuits containing multiple storage elements will introduce time delays of magnitude depending on the number and type of storage elements and on the circuit configuration.

6.7.5 Thévenin's theorem in transient analysis

We have seen that in the solution of the circuit differential equations the superposition theorem plays a central role; it allows us to find the natural and forced response separately which can be then added together to give the complete solution. The other linear circuit theorems discussed in chapter 2 are occasionally useful in the transient analysis of circuits, particularly Thévenin's theorem. The circuit of fig. 6.16(*a*) illustrates how this theorem can be used to simplify a circuit problem. In this circuit the capacitance is charged fully to the source voltage V_0 (constant); the switch is then closed at $t=0$. We wish to find the current $i(t)$ through R_2.

To apply Thévenin's theorem the circuit is broken at AA' in fig. 6.16(*a*) and the equivalent circuit to the left of AA' is found. The Thévenin equivalent e.m.f. (equal to the open-circuit voltage across AA') will be $V_0R_2/(R_1+R_2)$, and the resistance looking into AA' (with V_0 reduced to a short circuit) is $R_1R_2/(R_1+R_2)$. Hence, for the component values shown, the circuit of fig. 6.16(*b*) is obtained. The circuit is now reduced to a simple series form and it will be obvious that the steady-state value of v (voltage across AA') is $V_0/2$. With a circuit time constant of 0.5 seconds the voltage v is given by

Fig. 6.16. The application of Thévenin's theorem to a transient problem.



$R_1=R_2=2\,\mathrm{M\Omega}$ $C=0.5\,\mu\mathrm{F}$

(*a*) Original circuit (*b*) Thévenin circuit

$$v=Ae^{-2t}+\frac{V_0}{2}$$

We have not changed the circuit to the right of terminals AA' so the initial condition is still $v=V_0$ at $t=0^+$, which gives $A=V_0/2$, hence

$$v=\frac{V_0}{2}(e^{-2t}+1)$$

Now the voltage v, which is that across C, is unchanged between circuits (a) and (b) in fig. 6.16 so that the current through R_2 will be v/R_2 giving finally,

$$i=\frac{V_0}{4\times 10^6}(e^{-2t}+1)$$

The circuits of fig. 6.14 and fig. 6.16 are similar; the reader should compare the above approach with that adopted in section 6.14.

Thévenin's theorem is also useful if we wish to determine the effect of some modification to a circuit upon which an analysis has already been carried out. For example, suppose an additional resistance R_3 is switched into the circuit of fig. 6.14 at some instant $t=t_1$, as shown in fig. 6.17(a). We wish to find the voltage v across the circuit for $t\geqslant t_1$. With both switches closed, the circuit becomes as shown in fig. 6.17(b). To apply Thévenin's theorem the circuit is broken at AA' and the equivalent circuit to the left of AA' is found. The Thévenin equivalent e.m.f. e_T, that is, the open circuit voltage, is given by the solution (6.75) previously obtained for the unmodified circuit:

$$e_T=\frac{1}{4}e^{-2t}+\frac{t}{2}-\frac{1}{4}$$

The equivalent impedance, in terms of the D-operator notation, is

$$Z_T=\frac{R(1/CD)}{R+1/CD}=\frac{R}{RCD+1}$$

where $R=R_1//R_2$.

The circuit is thus reduced to the form shown in fig. 6.17(c) and the voltage is given by

$$v=\frac{R_3}{R_3+Z_T}\,e_T=\frac{RCD+1}{RCD+1+R/R_3}\,e_T$$

For the component values shown the differential equation for v is

$$(D+3)v=(D+2)e_T=(D+2)\left(\frac{1}{4}e^{-2t}+\frac{t}{2}-\frac{1}{4}\right) \tag{6.76}$$

Performing the operation on the RHS we obtain

$$(\mathrm{D}+3)v=t$$

The steady-state solution is given by

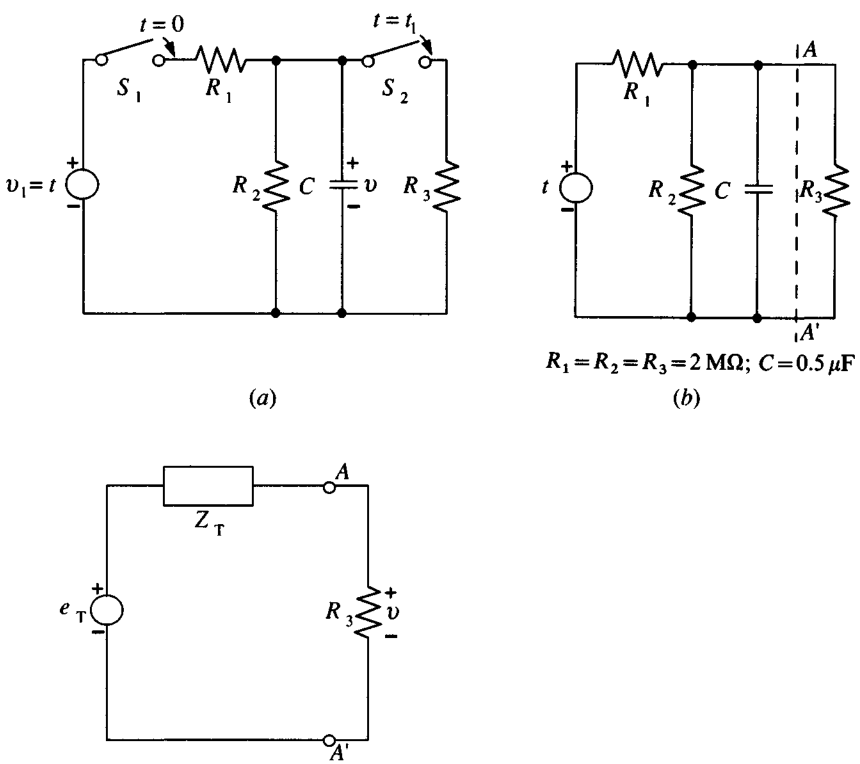
$$v=\frac{1}{\mathrm{D}+3}t=\frac{1}{3}\left(1+\frac{\mathrm{D}}{3}\right)^{-1}t=\frac{1}{3}\left(1-\frac{\mathrm{D}}{3}\cdots\right)t$$

or

$$v=\frac{t}{3}-\frac{1}{9}$$

We are here interested in the transient conditions after closing the switch S_2, that is, for $t>t_1$. Clearly, from the LHS of (6.76), the effective time constant is $\frac{1}{3}$ second so the complete solution may be written

Fig. 6.17. Application of Thévenin's theorem to a double switching problem.



(*a*) (*b*)

(*c*)

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$$v = A\mathrm{e}^{-3(t-t_1)} + \frac{t}{3} - \frac{1}{9} \quad (t \geqslant t_1)$$

The constant A is found from the condition of v at the instant t_1

$$v\big|_{t=t_1} = \frac{1}{4}\mathrm{e}^{-2t_1} + \frac{t_1}{2} - \frac{1}{4} = A + \frac{t_1}{3} - \frac{1}{9}$$

hence

$$A = \frac{1}{4}\mathrm{e}^{-2t_1} + \frac{t_1}{6} - \frac{5}{36}$$

and

$$v = \left(\frac{1}{4}\mathrm{e}^{-2t_1} + \frac{t_1}{6} - \frac{5}{36}\right)\mathrm{e}^{-3(t-t_1)} + \frac{t}{3} - \frac{1}{9} \quad (t \geqslant t_1)$$

For this particular example the use of Thévenin's theorem does not effect a great saving in the amount of algebraic manipulation involved. This is because it would be particularly simple in this instance to incorporate the additional resistance R_3 within the formulation of the original circuit equation. However, for more complicated situations the Thévenin approach can offer significant advantages.

It may be remarked, finally, that for the simple circuits discussed in this and previous sections containing a single source, mesh analysis is *not* an efficient approach to the formulation of the circuit differential equations. The reader may care to consider, for example, the use of mesh analysis (rather than the 'voltage divider' approach) for the worked example of section 6.7.4 (fig. 6.14).

6.7.6 Differentiating and integrating circuits

The RC circuits of fig. 6.18 are frequently used to perform simple signal differentiation and integration. Consider the circuit of fig. 6.18(a); with voltage v_1 applied at its input. The output v_2 is

$$v_2 = \frac{R}{R + 1/C\mathrm{D}}\, v_1 = \frac{RC\mathrm{D}}{RC\mathrm{D} + 1}\, v_1$$

or

$$(RC\mathrm{D} + 1)v_2 = RC\mathrm{D}v_1$$

For RC sufficiently small ($RC\mathrm{D}v_2 \ll v_2$) we may write:

$$v_2 \simeq RC\mathrm{D}v_1 \simeq RC\frac{\mathrm{d}}{\mathrm{d}t}(v_1)$$

Thus, the output is approximately equal to the derivative of the input. Note that the accuracy of differentiation will depend both on the magnitude of RC and upon v_1 and its rate of change, which implies that the accuracy is signal dependent.

For the circuit of fig. 6.18(b) we may show that, if RC is sufficiently large,

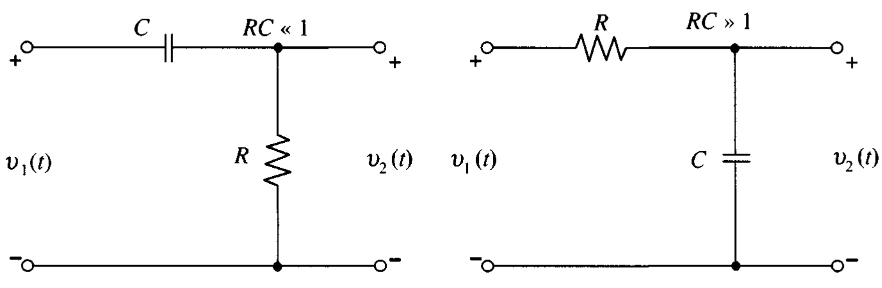
$$v_2 \simeq \frac{1}{CR}\int v_1 \, \mathrm{d}t$$

Because of the requirement in the integrating circuit that CR should be large, the output is generally small, and for this reason the circuit is usually used in conjunction with an active device which amplifies the signal and improves the accuracy of integration by ensuring that the capacitance receives, effectively, a constant charging current through the resistance (see page 131 of reference 5).

6.8 The unit step and related driving functions

In this section we introduce the concept of the *step* function and its relatives, the *impulse* function and the *ramp* function. These are members of a class of functions, called *singularity* functions, that are of fundamental importance to the development of more advanced aspects of circuit theory. The singularity functions allow us to describe the behaviour of circuits subject to driving waveforms of arbitrary shape and of a discontinuous nature. Examples of the latter have already been encountered in which the action of a switch impresses a driving voltage on a circuit at the instant $t=0$. It is convenient to introduce the step function as a mathematical description of this discontinuous process although, as will be seen later, the concepts embodied in the step and its related functions extend far beyond this simple application.

Fig. 6.18. RC circuits used for signal differentiation and integration.



(a) Differentiating circuit

(b) Integrating circuit

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6.8.1 Step function

The unit step function (fig. 6.19) is defined by

$$\left.\begin{aligned} u(t) &= 1 \qquad t \geq 0 \\ &= 0 \qquad t < 0 \end{aligned}\right\} \tag{6.77}$$

It should be noted that according to this definition the function is zero at $t=0^-$ and unity at both $t=0$ and $t=0^+$.

Fig. 6.20 illustrates the way in which the step function may be used to describe the action of a switch. The voltage source V and switch S of fig. 6.20(a) are replaced by an ideal voltage source, permanently connected to the circuit as shown in fig. 6.20(b), producing a driving voltage:

$$v(t) = Vu(t) \tag{6.78}$$

This expression signifies that for $t<0$ the voltage impressed on the circuit is zero, for $t \geq 0$ the voltage is V. It will be appreciated that the representation shown in fig. 6.20 refers to an ideal switch; that is, one which has infinite resistance before closure, zero resistance after closure, and for which the transition between these states is of infinitesimally short duration. An absence of inductive and capacitive effects is also implied.

Fig. 6.19. The unit step function.

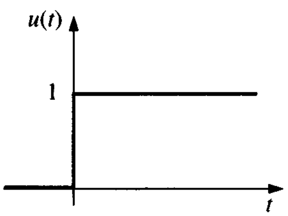
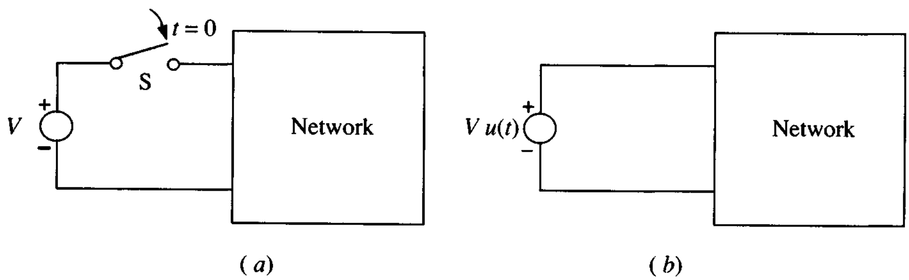


Fig. 6.20. Representation of switching action by means of the step function. (a) Original circuit with voltage source and switch. (b) Representation of circuit (a) by step-function source.



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The representation of fig. 6.20 also assumes that the circuit is in the zero energy state at $t=0$; otherwise the voltage across the terminals to which V is applied may not be zero at $t=0^-$, thus invalidating the definition (6.77).

A current source switched into a circuit at $t=0$ can be represented in like manner by the unit step function. If I is the amplitude of the constant current source, then the switching action may be represented by:

$$i(t)=Iu(t) \tag{6.79}$$

Voltages or currents which arise in a circuit subject to a unit step driving function are called the *step response.*

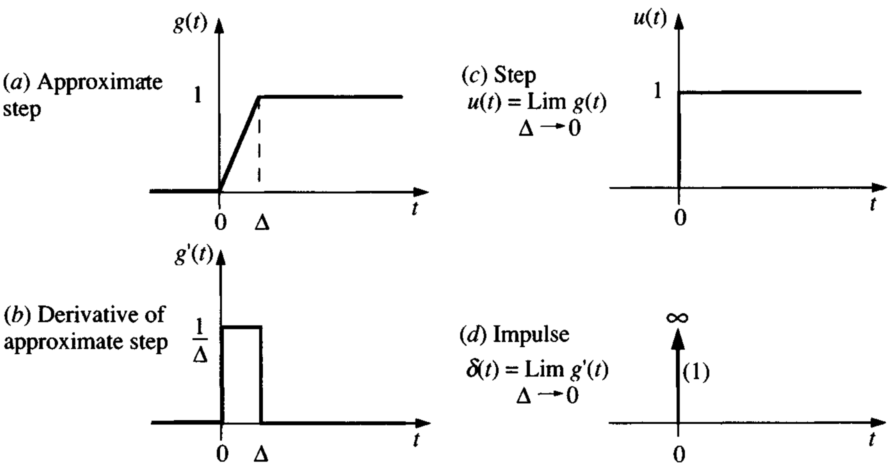
6.8.2 Impulse function

The unit impulse function (also known as the *Dirac* or *delta* function) is the derivative of the step function, and is defined by

$$\delta(t)=\frac{\mathrm{d}}{\mathrm{d}t}[u(t)]=u'(t) \tag{6.80}$$

Now the slope of the step function is infinite at $t=0$, in other words the function is not, in the usual mathematical sense, differentiable at this point and is therefore singular. However, the meaning of (6.80) will become clear if we consider the approximate step function $g(t)$ shown in fig. 6.21(a). This is zero at $t=0$ and rises linearly to unit amplitude at $t=\Delta$. The derivative of $g(t)$ is the rectangular pulse shown in fig. 6.21(b). We see that as Δ is made smaller and allowed to approach zero, $g(t)\to u(t)$ (fig. 6.21(c)) while the

Fig. 6.21. Illustrating relationship between unit step and unit impulse functions.



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amplitude of its derivative $1/\Delta \to \infty$ (fig. 6.21(*d*)). However, although $g'(t)$ becomes infinite for $\Delta \to 0$, the area under the derivative curve remains finite and independent of Δ since

$$\frac{1}{\Delta} \times \Delta = 1$$

So, the unit impulse function is infinite at $t = 0$ and zero elsewhere, while its area is unity. The symbolism of fig. 6.21(*d*) is used to indicate these properties of the unit impulse.

Since the impulse is the derivative of the step, the step must be the integral of the impulse. That this is so will be appreciated if we consider the area under the impulse function shown in fig. 6.21(*d*). To the left of the origin the function is zero so there is no contribution to the area. As we pass from $t = 0^-$ through to $t = 0^+$ we include unit area and the integral jumps to unity. There is no further contribution to the right of the origin so the value of the integral remains at unity. From this point of view the unit impulse may be defined by:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\delta(t) = 0, t \neq 0 \tag{6.81}$$

It follows from the definitions of the unit step and unit impulse functions, that the derivative of a step of amplitude E is an impulse of area E; that is

$$\frac{d}{dt}[Eu(t)] = E\delta(t) \tag{6.82}$$

The impulse function of voltage has dimensions of volt seconds; the impulse function of current has dimensions of amp seconds (coulombs). Voltages or currents which arise in a circuit subject to a unit impulse driving function are called the *impulse response*.

The importance of the impulse function lies in the fact that it can be used to represent functions and transforms of widely differing form. This will emerge fully when we deal with the theory of the convolution integral in the final section of this chapter. For the present we establish the circumstances under which the impulse function may be used to represent a single, short pulse of arbitrary shape. We begin by considering the response of a simple RC circuit (fig. 6.22) to: (*a*) a rectangular pulse, of duration Δ and unit area; (*b*) a unit impulse.

To find the response to the pulse we make use of the results obtained in section 6.6.2. There we found that the response of an RC circuit to an impressed voltage V was

$$v_2(t) = V(1 - e^{-t/RC}) \tag{6.41}$$

For the case considered here the pulse has amplitude $1/\Delta$ so, putting $V = 1/\Delta$ in (6.41) we obtain

$$v_2(t)_{\text{pulse}} = \frac{1}{\Delta}(1 - e^{-t/\tau}) \qquad 0 < t < \Delta \tag{6.83}$$

where $\tau = RC$.

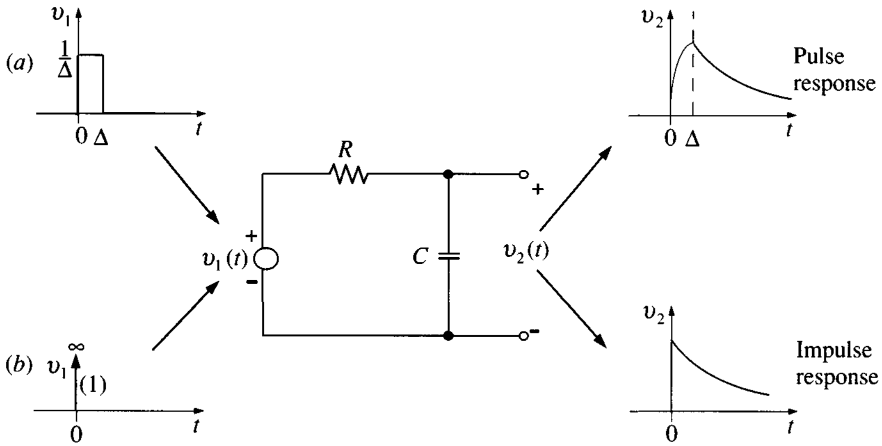
We may make use of (6.41) also to find the response to the unit impulse (see also section 6.9.6). This equation, with $V = u(t)$, gives the response to unit step; by differentiating this we obtain the response to unit impulse since the impulse is the derivative of the step. We have

$$v_2(t)_{\text{step}} = u(t)(1 - e^{-t/\tau}) \tag{6.84}$$

hence,

$$\begin{aligned} v_2(t)_{\text{impulse}} &= \frac{d}{dt}[u(t)(1 - e^{-t/\tau})] \\ &= \frac{d}{dt}u(t) - \frac{d}{dt}[u(t)e^{-t/\tau}] \\ &= \delta(t) - u(t)\left(\frac{-e^{-t/\tau}}{\tau}\right) - e^{-t/\tau}\delta(t) \\ &= (1 - e^{-t/\tau})\delta(t) + \frac{e^{-t/\tau}}{\tau}u(t) \end{aligned}$$

Fig. 6.22. Response of an *RC* circuit to: (*a*) pulse input; (*b*) impulse input.



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But $\delta(t)$ is zero for all $t \neq 0$, and at $t=0$ the coefficient of $\delta(t)$ is zero, so the first term in the expression vanishes to give

$$v_2(t)_{\text{impulse}} = \frac{1}{\tau} e^{-t/\tau} u(t) \tag{6.85}$$

In this expression $u(t)$ is to be regarded as a multiplying factor indicating that the response is zero for $t<0$ and $(1/\tau)e^{-t/\tau}$ for $t \geq 0$.

Now let us compare the response of the circuit to the two inputs for times equal to or greater than the duration of the pulse, that is for $t \geq \Delta$. For the pulse input the value of the output voltage at $t=\Delta$ will, according to (6.83), be

$$v_2(\Delta)_{\text{pulse}} = \frac{1}{\Delta}(1 - e^{-\Delta/\tau})$$

If it is assumed that the duration of the pulse is small in relation to the time constant of the circuit, this expression may be approximated by

$$\begin{aligned} v_2(\Delta)_{\text{pulse}} &\simeq \frac{1}{\Delta}\left(1 - 1 + \frac{\Delta}{\tau} - \frac{1}{2}\left(\frac{\Delta}{\tau}\right)^2 + \ldots\right) \\ &\simeq \frac{1}{\Delta}\left(\frac{\Delta}{\tau} - \frac{1}{2}\left(\frac{\Delta}{\tau}\right)^2 + \ldots\right) \\ v_2(\Delta)_{\text{pulse}} &\simeq \frac{1}{\tau}\left(1 - \frac{\Delta}{2\tau} + \ \ldots\right) \end{aligned} \tag{6.86}$$

For the impulse input the output at $t=\Delta$ is, by (6.85),

$$v_2(\Delta)_{\text{impulse}} = \frac{1}{\tau} e^{-\Delta/\tau}$$

which may be approximated by

$$\begin{aligned} v_2(\Delta)_{\text{impulse}} &\simeq \frac{1}{\tau}\left(1 - \frac{\Delta}{\tau} - \frac{1}{2}\left(\frac{\Delta}{\tau}\right)^2 - \ldots\right) \\ &\simeq \frac{1}{\tau}\left(1 - \frac{\Delta}{\tau} + \ldots\right) \end{aligned} \tag{6.87}$$

Comparing (6.86) and (6.87), and neglecting second and higher order terms, we see that the difference ε between the impulse response and the pulse response (at $t=\Delta$) is

$$\varepsilon = \frac{1}{2}\frac{\Delta}{\tau} \tag{6.88}$$

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If, for example, the time constant of the circuit is a factor of ten greater than the pulse duration, then $\varepsilon = 0.05$. For $t > \Delta$ the difference between pulse and impulse responses will be less than that given by (6.88) since both response curves decay exponentially, with the same time constant, to zero and will therefore converge.

We may conclude that it is possible to predict the response of a simple RC circuit to a rectangular pulse of unit area (for times greater than the pulse duration) by determining the response to the unit impulse function. The shorter the pulse duration in relation to the circuit time constant the better the accuracy of prediction. It also follows that the response to a pulse of area A may be found from the response to an impulse of magnitude A. Provided the condition $\tau > \Delta$ is fulfilled, the ratio of pulse height to pulse width is immaterial. Indeed, the pulse may be of any shape since the magnitude of the equivalent impulse function depends only upon the area of the pulse. Consequently, we are able to determine the response of a circuit to a pulse of arbitrary shape (including pulses which cannot be expressed analytically) simply by finding the area enclosed by the pulse, and then determining the response of the circuit to the equivalent impulse function.

The foregoing argument has been developed on the basis of the simple RC circuit, but the same general conclusions are found to be true for any first order circuit. The conclusions are also valid for higher order circuits, but in such cases it is the shortest effective time constant of the particular circuit which must be used as the criterion.

6.8.3 Worked example

A photomultiplier tube, used in a scintillation counting system, produces at its output pulses of current of the form shown in fig. 6.23(a). The photomultiplier is connected to an amplifier whose input circuit can be modelled by a resistance of 1 MΩ in parallel with a capacitance of 30 pF. Estimate the form of the voltage response at the input of the amplifier subsequent to the arrival of a single pulse.

Solution: The photomultiplier can be regarded as having an infinite output resistance so that it behaves essentially as an ideal current source. The circuit model is therefore as shown in fig. 6.23(b) where $i(t)$ is of the form shown in fig. 6.23(a).

The time constant of the circuit is $RC = 10^6 \times 30 \times 10^{-12} = 30\,\mu$s whereas the pulse duration is approximately one microsecond; consequently the pulse may be replaced by an impulse function at the origin.

The differential equation relating $v(t)$ and $i(t)$ is

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$$v(t) = \frac{R(1/CD)}{R + (1/CD)} i(t) \qquad \text{or} \qquad \left(\mathrm{D} + \frac{1}{RC}\right) v(t) = \frac{1}{C} i(t)$$

Putting $i(t)$ equal to the unit step function of current $u(t)$,

$$\left(\mathrm{D} + \frac{1}{RC}\right) v_{\text{step}} = \frac{1}{C} u(t)$$

where v_{step} is the step response.

The steady state response to unit constant current will clearly be $v_{\text{ss}} = R$, and the natural response will be $v_n = A\mathrm{e}^{-t/RC}$, hence

$$v_{\text{step}} = v_n + v_{\text{ss}} = A\mathrm{e}^{-t/RC} + R$$

The initial condition is $v = 0$ at $t = 0^+$ giving $A = -R$, so;

$$v_{\text{step}} = R(1 - \mathrm{e}^{-t/RC})$$

The impulse response is obtained by differentiating the step response. Using a procedure similar to that leading to (6.85) we find

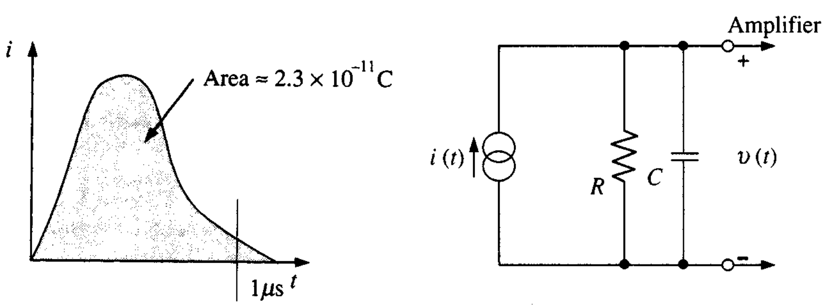
$$v_{\text{impulse}} = \frac{1}{C} \mathrm{e}^{-t/RC} u(t)$$

This is the response to *unit* impulse; the response to an impulse of magnitude Q will be $(Q/C)\mathrm{e}^{-t/RC}$. With $Q = 2.3 \times 10^{-11}$ coulomb, the form of the voltage response at the input of the amplifier is

$$v = 0.76\mathrm{e}^{-t/30 \times 10^{-6}}$$

Physically, we may interpret the result in the following way. During the pulse, lasting for about 1 μs, a charge of 2.3×10^{-11} C is delivered to the capacitance causing the voltage to rise to 0.76 V. The capacitance then discharges and the voltage decays with a time constant of 30 μs.

Fig. 6.23. Diagrams for worked example (section 6.8.3).



(*a*) Input pulse

(*b*) Circuit model

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6.8.4 Ramp and other singularity functions

The *ramp function* is the integral of the step function and is defined by:

$$\rho(t)=\int_{-\infty}^{t} u(t)\,\mathrm{d}t=\int_{0}^{t} \mathrm{d}t \tag{6.89}$$

or

$$\left.\begin{aligned}\rho(t)&=t \qquad t\geq 0\\ &=0 \qquad t<0\end{aligned}\right\} \tag{6.90}$$

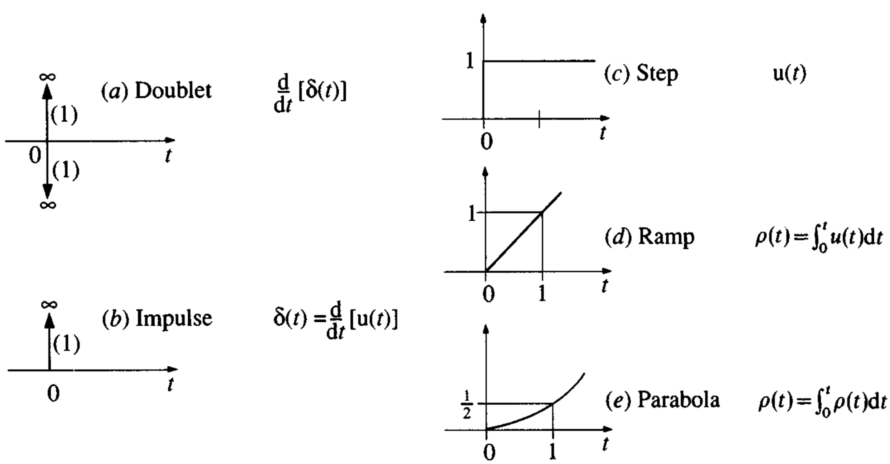
The integral of a step function of amplitude E corresponds to a ramp function of slope E, that is,

$$\int_{-\infty}^{t} Eu(t)\,\mathrm{d}t=E\rho(t) \tag{6.91}$$

Other singularity functions, useful in more advanced network analysis, are obtained by further differentiation or integration of the basic impulse, step, and ramp functions. For example, differentiation of the unit impulse function produces the *unit doublet* consisting of positive and negative going unit impulses at the origin. Integration of the ramp function produces the *unit parabola.*

The relationships among all of the functions mentioned in this section are shown in fig. 6.24. In this book only the impulse, step and ramp functions will be considered further.

Fig. 6.24. Relationships among the unit singularity functions.



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6.8.5 Delayed functions

The singularity functions considered so far have a point of discontinuity at time zero; however, for some purposes it is useful to extend the concept to embrace functions having a discontinuity at some other, positive, value of time. This may be accomplished simply by changing the argument of the function. Thus, if the unit step function is defined by

$$\left.\begin{aligned} u(t-a) &= 0 \qquad (t-a)<0 \\ &= 1 \qquad (t-a)\geq 0 \end{aligned}\right\} \tag{6.92}$$

Fig. 6.25. Delayed singularity functions.

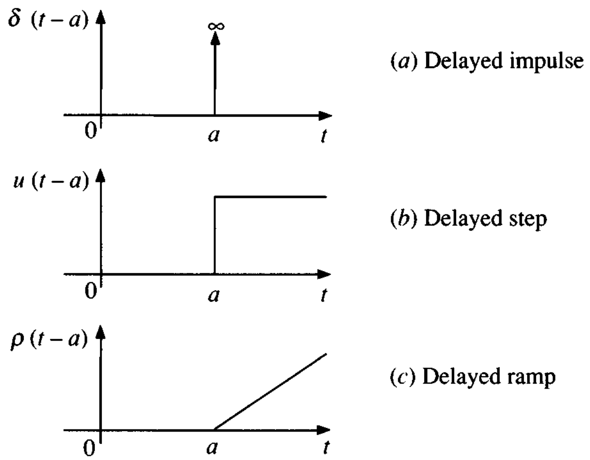
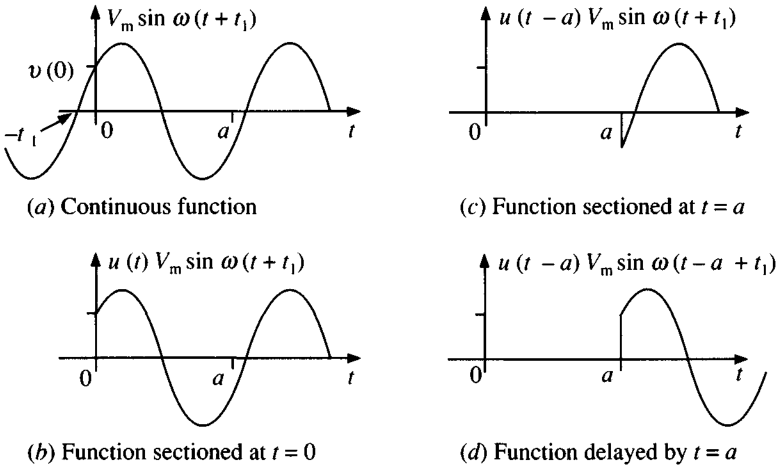


Fig. 6.26. Use of delayed step function to provide time sectioning and time delay of a signal waveform.



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the step will be delayed until $t=a$. The impulse and ramp functions may be treated similarly, as shown in fig. 6.25.

The delayed unit step function allows one to specify analytically the time at which a function commences; a process which is sometimes called *sectioning* of the function. For example, the sinusoid $v=V_m \sin\omega(t+t_1)$ shown in fig. 6.26(a) is continuous for all negative and positive time. Multiplication by $u(t)$ sections the function at the origin (fig. 6.26(b)) while multiplication by $u(t-a)$ sections the function at $t=a$ (fig. 6.86(c)).

Changing the argument of a function $f(t)$ to $f(t-a)$ and multiplying by $u(t-a)$ shifts the original function bodily along the time axis so that it commences at $t=a$ with the same value v_0 that it had at the origin. The result of this operation on the sinusoid is illustrated in fig. 6.26(d).

6.9 The Laplace transform

The method of analysis now to be described employs the Laplace transform by means of which functions of time are transformed into functions of a new variable s in such a way that what was initially a differential equation becomes an algebraic equation. For the functions of time normally encountered in linear circuit applications, these transformations are unique so that to each function of t there corresponds a function of s and, conversely, to each function of s there corresponds a function of t. Therefore, from the algebraic equation(s) we obtain a function of s that may, by the inverse Laplace transform, be converted to a function of t. This new function is the solution of the original differential equation.

The Laplace transform has applications to situations such as we have considered in this chapter where a known driving function is applied at $t=0$ to a circuit and it is desired to find the response of the circuit for all $t>0$, having given the conditions in the circuit at $t=0$ (the initial conditions). The method has several features in common with the D-operator approach to circuit analysis, in particular the method by which the circuit equations are set up and manipulated is essentially the same, however, the greater mathematical generality of the Laplace transform allows it to be used for a wider variety and range of problems. An important advantage of the method is that initial conditions are included automatically in the transformed circuit equations; a disadvantage is that it can involve a formidable amount of algebraic manipulation, and it sometimes tends to obscure the underlying physics of the problem under consideration.

The theory of the Laplace transform may be developed in relation to the Fourier series and Fourier integral which are used, respectively, in the representation of non-sinusoidal periodic functions and of pulses and other functions of finite duration. In our approach we shall simply define the Laplace transform; calculate the transforms for some functions of time that are commonly encountered, and show how the results may be applied to the solution of some specific problems. We shall not go into the question of the conditions that a function must satisfy in order that a transform should exist; we assume (with justification) that all the functions we use meet these criteria.

6.9.1 Definition of the Laplace transform

The Laplace transform $F(s)$ of a function $f(t)$ is defined by*

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{6.93}$$

The symbol $\mathscr{L}\{f(t)\}$ is to be read: 'the Laplace transform of $f(t)$'. The variable s, which has dimensions of angular frequency, may be real or complex and is usually expressed by

$$s = \sigma + j\omega \tag{6.94}$$

σ must be positive and sufficiently large to ensure that the integral converges. For the functions of t that we shall be concerned with, this condition is satisfied.

The inverse Laplace transform is defined by

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds \tag{6.95}$$

where the symbol $\mathscr{L}^{-1}$ indicates the process of finding the inverse of the function $F(s)$.

In the application of the Laplace transform to practical circuit problems, the integrals (6.93) and (6.95) are rarely used directly. The integral (6.93) has been evaluated for a large number of functions and one refers to tables of Laplace transform pairs to effect the appropriate transforms, both forward and inverse. We shall not, therefore, concern ourselves further with (6.95). A short table of transform pairs is given in Appendix D; we indicate below how some of the more useful of the entries in this table are derived.

6.9.2 Laplace transforms for some functions of time

Note that pair numbers given below refer to the table of Laplace transform pairs in Appendix D.

(1) $f(t) = t^n$

* The lower limit in the integral (6.93) is zero, and the definition used here is called the *one-sided* Laplace transform, which is applicable to functions that are zero for $t<0$. The *two-sided* Laplace transform, which we do not deal with in this book, is defined by a similar integral but with lower limit $-\infty$.

Consider first $f(t)=t$ then using (6.93)

$$F(s)=\int_0^\infty te^{-st}\,dt$$

Integrate by parts: let $t=u$ and $e^{-st}\,dt=dv$, so $du=dt$ and $v=-\frac{1}{s}e^{-st}$, then

$$\int u\,dv=uv-\int v\,du=\left[-\frac{t}{s}e^{-st}\right]_0^\infty+\int_0^\infty\frac{1}{s}e^{-st}\,dt=0+\frac{1}{s^2}$$

So,

$$\mathscr{L}\{t\}=\frac{1}{s^2}$$

Repeated application of integration by parts gives the general result:

$$\mathscr{L}\{t^n\}=\frac{n!}{s^{(n+1)}} \tag{6.96}$$

[Pair No. 1]

(2) $f(t)=e^{at}$

$$F(s)=\int_0^\infty e^{at}e^{-st}\,dt=\int_0^\infty e^{-(s-a)t}\,dt=\frac{1}{s-a}$$

For the integral to converge, $s>a$. In the applications considered here, a is negative, so this inequality obtains. Then

$$\mathscr{L}(e^{at})=\frac{1}{s-a}$$

and

$$\mathscr{L}(e^{-at})=\frac{1}{s+a} \tag{6.97}$$

[Pair No. 2]

(3) $f(t)=\sin\omega t$.

This is calculated conveniently by using the identity

$$\sin\omega t=\frac{1}{2j}(e^{j\omega t}-e^{-j\omega t})$$

together with the result (6.97)

$$F(s)=\frac{1}{2j}\int_0^\infty[e^{-(s-j\omega)t}\,dt-e^{-(s+j\omega)t}dt]$$

$$=\frac{1}{2j}\left[\frac{1}{s-j\omega}-\frac{1}{s+j\omega}\right]$$

So,

$$\mathscr{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} \tag{6.98}$$

[Pair No. 6]

(4) $f(t) = \cos\omega t$

Again express the function as the sum of exponentials and use the result (6.97)

$$\mathscr{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2} \tag{6.99}$$

[Pair No. 8]

(5) $f(t) = c$ (constant)

$$F(s) = \int_0^\infty c e^{-st}\,\mathrm{d}t = c\left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{c}{s}$$

So,

$$\mathscr{L}\{c\} = \frac{c}{s} \tag{6.100}$$

[Pair No. 15]

(6) $f(t) = \mathrm{u(t)}$ (unit step)

Since $u(t) = 1$ for $t \geq 0$ we may use the result (6.100)

$$\mathscr{L}\{u(t)\} = \frac{1}{s} \tag{6.101}$$

[Pair No. 16]

(7) Transforms of $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$; $\quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{f(t)}$; $\quad \frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)$

So far we have simply shown how, by a mathematical manipulation, it is possible to obtain a function of s that corresponds to some given function of t. The use of the transform in circuit problems requires that we also obtain transforms of derivatives of functions of t. To find these we must know the values of $f(t)$ and its derivatives at $t = 0^+$. Let

$$f(0^+) = f_0; \qquad \frac{\mathrm{d}}{\mathrm{d}t}f(0^+) = f_1; \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}f(0^+) = f_2 \ldots \text{etc.}$$

Again using (6.93)

$$\mathscr{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t$$

Integrate by parts: $\mathrm{e}^{-st} = u$ and $\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t = \mathrm{d}v$; hence,

$$\mathrm{d}u = -s\mathrm{e}^{-st}\,\mathrm{d}t \text{ and } v = f.$$

$$\mathscr{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} = [f\mathrm{e}^{-st}]_0^\infty + s\int_0^\infty f e^{-st}\,\mathrm{d}t$$

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The first term on the right is $(-f_0)$ and the second term is just s times the Laplace transform of $f(t)$, therefore,

$$\mathscr{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\}=sF(s)-f_0 \qquad (6.102)$$

[Pair No. 24]

So, to write the transform of the derivative of a function we write s times the transform of the function and subtract the value of the function at $t=0^+$.

The voltage–current relationship for inductance provides an important example of the use of this operational transform pair. We have

$$v=L\frac{\mathrm{d}i}{\mathrm{d}t}$$

which, upon transforming both sides, becomes

$$V(s)=L(sI(s)-i_0) \qquad (6.103)$$

where $i_0(\equiv i(0^+))$ is the initial value of the current in the inductance.

The transform of the second derivative of a function is also found by integrating by parts. With $\mathrm{e}^{-st}=u$ and $\dfrac{\mathrm{d}^2f}{\mathrm{d}t^2}\mathrm{d}t=\mathrm{d}v$, then

$$\mathscr{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right\}=\left[\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-st}\right]_0^\infty+s\int_0^\infty\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t$$

When limits are substituted, the first term becomes $\dfrac{-\mathrm{d}f(0^+)}{\mathrm{d}t}=-f_1$. The second term is s times the transform of $\dfrac{\mathrm{d}f}{\mathrm{d}t}$, which we have just found. So,

$$\mathscr{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right\}=s^2F(s)-sf_0-f_1 \qquad (6.104)$$

[Pair No. 25]

Generalizing the above results we obtain:

$$\mathscr{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right\}=s^nF(s)-s^{n-1}f_0-s^{n-2}f_1-\ldots-f_{n-1} \qquad (6.105)$$

[Pair No. 26]

(8) Transforms of $\int f(t)\,\mathrm{d}t$ and $\int_0^t f(t)\,\mathrm{d}t$

$$\mathscr{L}\left\{\int f(t)\,\mathrm{d}t\right\}=\int_0^\infty\left[\int f(t)\,\mathrm{d}t\right]\mathrm{e}^{-st}\,\mathrm{d}t$$

Integrating by parts with $\int f\,\mathrm{d}t=u$ and $\mathrm{e}^{-st}\,\mathrm{d}t=\mathrm{d}v$ we obtain

$$\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u=\left[\left\{\int f\mathrm{d}t\right\}\frac{\mathrm{e}^{-st}}{-s}\right]_0^\infty-\int_0^\infty f\frac{\mathrm{e}^{-st}}{-s}\mathrm{d}t$$

The second term is simply $(1/s)F(s)$, and when limits are substituted in the first term we have

$$-\frac{1}{s}\left\{\left[\int f\,\mathrm{d}t\right]\mathrm{e}^{-\infty}-\left[\int f\,\mathrm{d}t\right]_{t=0}\right\}=\frac{1}{s}\left[\int f\,\mathrm{d}t\right]_{t=0}$$

But this is just $(1/s)$ times the value of the integral at the moment of the switching operation and therefore represents the initial condition. The transform is often written

$$\mathscr{L}\left\{\int f(t)\,\mathrm{d}t\right\}=\frac{1}{s}F(s)+\frac{1}{s}f^{-1}(0^+) \tag{6.106}$$

[Pair No. 27]

where $f^{-1}(0^+)$ represents the value of the integral at $t=0^+$.

The above expression gives the transform of the indefinite integral; the transform of the definite integral is obtained as follows:

$$\int_0^t f\,\mathrm{d}t=\int f\,\mathrm{d}t-\left[\int f\,\mathrm{d}t\right]_{t=0}=\int f\,\mathrm{d}t-f^{-1}(0^+)$$

Transforming term by term we have

$$\mathscr{L}\left\{\int_0^t f\,\mathrm{d}t\right\}=\mathscr{L}\left\{\int f\,\mathrm{d}t\right\}-\mathscr{L}\{f^{-1}(0^+)\}$$

The first term is the transform of the indefinite integral, which is given by (6.106). The second term represents the transform of a constant which is obtained from (6.100). Hence,

$$\left\{\int_0^t f\,\mathrm{d}t\right\}=\frac{1}{s}F(s)+\frac{1}{s}f^{-1}(0^+)-\frac{1}{s}f^{-1}(0^+)$$

So the transform of the definite integral is given by:

$$\mathscr{L}\left\{\int_0^t f(t)\,\mathrm{d}t\right\}=\frac{1}{s}F(s) \tag{6.107}$$

[Pair No. 28]

Using this expression we may find the transform of the voltage-current relationship for capacitance with an initial voltage v_0 (Table 1.1 equation (1.31)),

$$v=\frac{1}{C}\int_0^t i\,\mathrm{d}t+v_0$$

Transforming term by term we obtain

$$V(s)=\frac{1}{C}\,\frac{I(s)}{s}+\frac{v_0}{s} \tag{6.108}$$

The same result may be obtained directly from (6.106) by noting that the initial value of the integral, $f^{-1}(0^+)$, is just the charge on the capacitance $(q_0 = Cv_0)$.

As an example of the use of the transform relationships derived above, let us again consider the RL circuit of fig. 6.1. With the switch closed at $t=0$ the circuit equation for fig. 6.1(a) is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = Vu(t)$$

where $u(t)$ indicates that the constant voltage V is switched into the circuit at $t=0$. Using (6.102) we find that the first term transforms to $L(sI(s) - i_0) = LsI(s)$ since i_0, the initial current, is zero. The second term transforms to $RI(s)$; and the term on the right, using (6.101), transforms to V/s. The complete transform equation is then

$$LsI(s) + RI(s) = \frac{V}{s}$$

or

$$I(s) = \frac{V}{s}\,\frac{1}{Ls+R} = \frac{V}{L}\left[\frac{1}{s(s+R/L)}\right] \tag{6.109}$$

Now referring to our table of transform pairs (Appendix D), we see that transform pair number 3 allows us to find the inverse of the expression in brackets directly, that is,

$$i(t) = \frac{V}{L}\cdot\frac{L}{R}(1 - \mathrm{e}^{-Rt/L}) = \frac{V}{R}(1 - \mathrm{e}^{-Rt/L})$$

This result is identical to (6.3).

In the case of the sinusoidal driving voltage (fig. 6.1(b)), the circuit equation is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = V_{\mathrm{m}}\sin\omega t u(t)$$

which upon transformation becomes

$$LsI(s) + RI(s) = \frac{V_{\mathrm{m}}\omega}{s^2 + \omega^2}$$

Here we have used (6.98) to transform the sinusoidal driving voltage. Proceeding as before:

$$I(s) = \frac{V_{\mathrm{m}}\omega}{L}\left[\frac{1}{(s+R/L)(s^2+\omega^2)}\right] \tag{6.110}$$

Now we refer to our table of transform pairs but we discover that in this case the appropriate form of the function in brackets does not appear. More extensive tables are available (see for example reference 9), and such tables would include the function we require. However, it will be instructive to consider how a table containing a relatively small number of transform pairs may be extended to allow the inversion of functions such as (6.110); this is the subject of the following section.

6.9.3 Partial fractions

Because we are dealing with linear circuit elements we may use the following property of the Laplace transform in our calculations.

$$\mathscr{L}\{C_1 f_1(t)+C_2 f_2(t)\}=C_1\mathscr{L}\{f_1(t)\}+C_2\mathscr{L}\{f_2(t)\} \tag{6.111}$$

This equation means that if we have a function $F(s)$ for which the corresponding $f(t)$ is required, we may express $F(s)$ as the sum of several terms, find the inverse transform of each separately, and add the resulting functions of time to get the complete solution of the original differential equation. In order to express $F(s)$ as the sum of several terms we can often make use of the method of partial fractions.

The functions of s that we are concerned with in this text are in general *rational* functions, that is, they can be expressed as the ratio of two polynomials:

$$F(s)=\frac{N(s)}{D(s)} \tag{6.112}$$

We assume that $F(s)$ is a proper fraction (numerator of a lower order in s than denominator).*

Now the fundamental theorem of algebra states that any polynomial in s with real coefficients may be expressed as the product of factors of one or both of the following types:

(a) linear factors of the form, $as+b$

(b) irreducible quadratic factors of the form cx^2+dx+e, which does not have real, linear factors.

The coefficients a, b, c, d, e are real. If, therefore, $F(s)$ is the ratio of two polynomials, one may factorise the denominator. By the method of partial

* An improper fraction may be reduced by division to a form consisting of a polynomial plus a proper fraction. For example,

$$\frac{X^4+3X^2+2}{X^2-2X}=X^2+2X+7+\frac{14X+2}{X^2-2X}$$

However, for all the problems that will concern us, $F(s)$ is a proper fraction.

fractions the original expression for $F(s)$ may then be replaced by the sum of a series of fractions whose denominators are found from the factors of the denominator of $F(s)$. The appropriate method of determining the numerators of the partial fractions depends upon the factors that appear in the denominator of $F(s)$. We consider three different cases.

Case A. If the denominator of $F(s)$ contains only linear factors none of which is repeated, then,

$$F(s)=\frac{N(s)}{(s-a)(s-b)(s-c)\ldots(s-m)}=\frac{A}{(s-a)}+\frac{B}{(s-b)}+\ldots+\frac{M}{(s-m)} \tag{6.113}$$

To find the coefficient A, multiply through by $(s-a)$ and let $s=a$. Then

$$(s-a)F(s)=\frac{N(s)(s-a)}{(s-a)(s-b)\ldots(s-m)}=A+\frac{B(s-a)}{(s-b)}+\ldots+\frac{M(s-a)}{(s-m)}$$

When $s=a$, all terms on the right are zero except A. So,

$$\begin{aligned} A&=[(s-a)F(s)]_{s=a}\\ B&=[(s-b)F(s)]_{s=b} \end{aligned} \tag{6.114}$$

and so on.

Example. Let

$$F(s)=\frac{7s-2}{s^3-s^2-2s}=\frac{7s-2}{s(s+1)(s-2)}=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s-2)}$$

$$A=[(s)F(s)]_{s=0}=\frac{7s-2}{(s+1)(s-2)}=1$$

$$B=[(s+1)F(s)]_{s=-1}=-3 \qquad C=[(s-2)F(s)]_{s=2}=2$$

and

$$F(s)=\frac{1}{s}-\frac{3}{(s+1)}+\frac{2}{(s-2)}$$

The reader may care to derive the corresponding function of t, that is the inverse transform, using the table of transform pairs in Appendix D. (Answer: $1-3e^{-t}+2e^{2t}$)

Case B. If a linear factor $(s-b)$ is repeated p times in the denominator, then the partial fraction expansion must include p terms of the form:

$$\frac{A_1}{(s-b)}+\frac{A_2}{(s-b)^2}+\ldots+\frac{A_p}{(s-b)^p} \tag{6.115}$$

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The coefficient A_p is then found from

$$A_p = [(s-b)^p F(s)]_{s=b} \tag{6.116}$$

The other coefficients are found by repeated differentiation

$$\begin{aligned} A_{p-1} &= \frac{d}{ds}[(s-b)^p F(s)]_{s=b} \\ A_{p-2} &= \frac{1}{2!}\frac{d^2}{ds^2}[(s-b)^p F(s)]_{s=b} \\ A_{p-3} &= \frac{1}{3!}\frac{d^3}{ds^3}[(s-b)^p F(s)]_{s=b} \end{aligned} \tag{6.117}$$

and so on.

Example.

$$F(s) = \frac{s^2+4s-15}{s^3-3s^2+4} = \frac{s^2+4s-15}{(s+1)(s-2)^2} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$$A = [(s+1)F(s)]_{s=-1} = \left[\frac{s^2+4s-15}{(s-2)^2}\right]_{s=-1} = -2$$

$$C = [(s-2)^2 F(s)]_{s=2} = \left[\frac{s^2+4s-15}{(s+1)}\right]_{s=2} = -1$$

$$B = \frac{d}{ds}[(s-2)^2 F(s)]_{s=2} = \frac{d}{ds}\left[\frac{s^2+4s-15}{(s+1)}\right]_{s=2} = 3$$

Therefore,

$$F(s) = \frac{-2}{(s+1)} + \frac{3}{(s-2)} + \frac{-1}{(s-2)^2}$$

Again the reader may care to find the inverse of this function (Answer: $-2e^{-t} + 3e^{2t} - te^{2t}$)

Case C. Suppose there is an irreducible quadratic term of the form $s^2 + as + b^2$ in the denominator of $F(s)$. Because a and b are real numbers, this term gives two complex values of s that are complex conjugates of one another. That is

$$s^2 + as + b^2 = [s + (\alpha + j\omega)][s + (\alpha - j\omega)] \tag{6.118}$$

where α and ω are real and $\alpha = a/2$, $\omega^2 = b^2 - (a/2)^2$.

The partial fraction expansion then contains the following two terms:

$$\frac{A_1}{s+(\alpha+j\omega)}+\frac{A_2}{s+(\alpha-j\omega)} \tag{6.119}$$

where A_1 and A_2 are, in general, complex. Then

$$\begin{aligned} A_1 &= [\{s+(\alpha+j\omega)\}F(s)]_{s=-(\alpha+j\omega)} \\ A_2 &= [\{s+(\alpha-j\omega)\}F(s)]_{s=-(\alpha-j\omega)} \end{aligned} \tag{6.120}$$

Because $F(s)$ is a rational function in s with real coefficients, A_1 and A_2 must be complex conjugates, that is, $A_2 = A_1{}^*$. In polar form:

$$A_1 = A_1 e^{j\phi_1}$$

and

$$A_2 = A_2 e^{j\phi_2} = A_1 e^{-j\phi_1} \tag{6.121}$$

Now the inverse transform of the sum of the two terms in (6.119) is

$$\begin{aligned} f(t) &= A_1 e^{-(\alpha+j\omega)t} + A_2 e^{-(\alpha-j\omega)t} \\ &= A_1 e^{j\phi_1} e^{-(\alpha+j\omega)t} + A_1 e^{-j\phi_1} e^{-(\alpha-j\omega)t} \\ &= A_1 e^{-\alpha t}[e^{j(\omega t-\phi_1)} + e^{-j(\omega t-\phi_1)}] \end{aligned}$$

or

$$f(t) = 2A_1 e^{-\alpha t}\cos(\omega t-\phi_1) \tag{6.122}$$

Example.

$$F(s)=\frac{s^2+3s+7}{(s+1)(s^2+2s+5)}=\frac{A}{s+1}+\frac{B_1}{s+(1+j2)}+\frac{B_2}{s+(1-j2)}$$

Then

$$\begin{aligned} A &= [(s+1)F(s)]_{s=-1} = \tfrac{5}{4} = 1.25 \\ B_1 &= [\{s+(1+j2)\}F(s)]_{s=-(1+j2)} \\ &= \frac{(-1-j2)^2+3(-1-j2)+7}{(-1-j2+1)(-1-j2+1-j2)} = -\frac{1}{8}+j\frac{1}{4} \end{aligned}$$

In polar form, $B_1 = 0.28\underline{/116^\circ}$.

The inverse transform of the last two terms is from (6.122)

$$2B_1 e^{-\alpha t}\cos(\omega t-\phi_1) = 2\times 0.28e^{-t}\cos(2t-116^\circ)$$

and the complete function of t is

$$f(t) = 1.25e^{-t} + 0.56e^{-t}\cos(2t-116^\circ)$$

We may use the foregoing theory to derive two useful transform pairs.

(1) Let $F(s)=\dfrac{1}{s^2+as+b^2}=\dfrac{1}{(s+\alpha)^2+\omega^2}$

where $\alpha=a/2$ and $\omega^2=b^2-(a/2)^2$.

By (6.118) and (6.119) we may write $F(s)$ as

$$F(s)=\frac{A_1}{s+(\alpha+\mathrm{j}\omega)}+\frac{A_2}{s+(\alpha-\mathrm{j}\omega)}$$

then,

$$A_1=[\{s+(\alpha+\mathrm{j}\omega)\}F(s)]_{s=-(\alpha+\mathrm{j}\omega)}$$

$$=\frac{1}{-\mathrm{j}2\omega}=\frac{1}{2\omega}\underline{/90^\circ}$$

and, using (6.122), we obtain

$$f(t)=2A_1\mathrm{e}^{-\alpha t}\cos(\omega t-\phi_1)=\frac{1}{\omega}\mathrm{e}^{-\alpha t}\cos(\omega t-90^\circ)$$

or

$$f(t)=\frac{1}{\omega}\mathrm{e}^{-\alpha t}\sin\omega t$$

We then have

$$\mathscr{L}\{\mathrm{e}^{-\alpha t}\sin\omega t\}=\frac{\omega}{(s+\alpha)^2+\omega^2} \qquad (6.123)$$

[Pair No. 10]

(2) Let $F(s)=\dfrac{s}{s^2+as+b^2}=\dfrac{s}{(s+\alpha)^2+\omega^2}$

By an argument similar to that given above

$$A_1=[\{s+(\alpha+\mathrm{j}\omega)\}F(s)]_{s=-(\alpha+\mathrm{j}\omega)}$$

$$=\frac{\alpha+\mathrm{j}\omega}{\mathrm{j}2\omega}=\frac{\sqrt{(\alpha^2+\omega^2)}}{2\omega}\underline{/\theta-90^\circ}$$

where $\theta=\tan^{-1}(\omega/\alpha)$.

So, by (6.122),

$$f(t)=\frac{\sqrt{(\alpha^2+\omega^2)}}{\omega}\mathrm{e}^{-\alpha t}\cos(\omega t-\theta+90^\circ)$$

$$=\frac{\sqrt{(\alpha^2+\omega^2)}}{\omega}\mathrm{e}^{-\alpha t}\sin(\theta-\omega t)$$

$$=\frac{\sqrt{(\alpha^2+\omega^2)}}{\omega}\mathrm{e}^{-\alpha t}(\sin\theta\cos\omega t-\sin\omega t\cos\theta)$$

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But $\sin\theta = \omega/\sqrt{(\alpha^2+\omega^2)}$ and $\cos\theta = \alpha/\sqrt{(\alpha^2+\omega^2)}$ hence

$$f(t) = e^{-\alpha t}\left(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right)$$

Hence

$$\mathscr{L}\left\{e^{-\alpha t}\left(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right)\right\} = \frac{s}{(s+\alpha)^2+\omega^2} \tag{6.124}$$

[Pair No. 12]

Let us now return to the problem of finding the inverse of (6.110), namely,

$$I(s) = \frac{V_m\omega}{L}\left[\frac{1}{(s+R/L)(s^2+\omega^2)}\right] \tag{6.110}$$

The term in brackets may be expanded using the procedures detailed under *Case C* above. With $\alpha=0$ in (6.119), and letting $R/L=a$, we have

$$F(s) = \frac{1}{(s+a)(s^2+\omega^2)} = \frac{A}{s+a} + \frac{\boldsymbol{B}_1}{s+\mathrm{j}\omega} + \frac{\boldsymbol{B}_2}{s-\mathrm{j}\omega}$$

By (6.114)

$$A = [(s+a)F(s)]_{s=-a} = \frac{1}{a^2+\omega^2}$$

and the inverse of the first term is $e^{-at}/(a^2+\omega^2)$.
By (6.120)

$$\boldsymbol{B}_1 = [(s+\mathrm{j}\omega)F(s)]_{s=-\mathrm{j}\omega} = \frac{1}{(-\mathrm{j}\omega+a)(-2\mathrm{j}\omega)} = \frac{1}{-2\omega(\omega+\mathrm{j}a)}$$

$$= -\frac{1}{2\omega\sqrt{(\omega^2+a^2)}}\underline{/-\phi_1} \qquad \text{where } \phi_1 = \tan^{-1}\frac{a}{\omega}$$

Therefore, the inverse of the last two terms is, from (6.122),

$$-\frac{1}{\omega\sqrt{(\omega^2+a^2)}}\cos(\omega t+\phi_1)$$

and the complete expression for the inverse of (6.110) becomes

$$i(t) = \frac{V_m\omega}{L}\left[\frac{e^{-at}}{\omega^2+a^2} - \frac{1}{\omega\sqrt{(\omega^2+a^2)}}\cos(\omega t+\phi_1)\right]$$

Putting $a=R/L$

$$i(t) = \frac{V_m\omega L}{(\omega L)^2+R^2}e^{-Rt/L} - \frac{V_m}{\sqrt{[(\omega L)^2+R^2]}}\cos(\omega t+\phi_1) \tag{6.125}$$

Now referring to (6.1), the phase angle for the circuit of fig. 6.1(*b*) is given by $\tan\theta = \omega L/R$, hence, $\sin\theta = \omega L/\sqrt{[(\omega L)^2+R^2]}$. Also, $\tan\phi_1 = R/\omega L$, hence,

$\phi_1 = 90 - \theta$. So, putting $\omega L/\sqrt{[(\omega L)^2 + R^2]} = \sin\theta$ in the first term of (6.125), and $\phi_1 = 90 - \theta$ in the second term of (6.125) gives

$$i(t) = \frac{V_m e^{-Rt/L}}{\sqrt{[(\omega L)^2 + R^2]}} \sin\theta + \frac{V_m}{\sqrt{[(\omega L)^2 + R^2]}} \sin(\omega t - \theta)$$

which is identical to (6.38) with $\lambda = 0$.

If one compares the above method of solving the circuit of fig. 6.1(*b*) with that given in section 6.6.1, it will be seen that it is algebraically more complicated. In general, the Laplace transform method is not to be recommended for solving first order circuits with constant voltage or sinusoidal driving sources. However, as will become apparent in the following sections, the method has advantages when dealing with circuits of second or higher order and having finite initial energy states. The Laplace transform method can also be advantageous for first order circuits containing driving sources other than constant voltage or sinusoidal.

6.9.4 Network analysis by Laplace transform

In applying the Laplace transform method to circuit problems two approaches are possible. The first is to set up the complete circuit integro-differential equations and then transform these into algebraic equations in s. With this approach difficulties can arise when taking into account initial energy states of the circuit, particularly if the differential equations contain second or higher order derivatives. The second approach, which we adopt here, is to transform the voltage–current relationships for each circuit element before setting up the circuit equations. Using this approach it is often helpful to reconstruct the original time-domain circuit in the s-domain. This new circuit will contain complete information concerning the initial energy states of the original circuit.

We have already found in section 6.9.2 (equation (6.103)) the transform corresponding to the voltage–current relationships for an inductance carrying initial current i_0:

$$v(t) = \frac{L\mathrm{d}i(t)}{\mathrm{d}t} \Rightarrow V(s) = sLI(s) - Li_0 \tag{6.126}$$

The corresponding current–voltage relationship is

$$i(t) = \frac{1}{L}\int_0^t v(t)\,\mathrm{d}t + i_0 \Rightarrow I(s) = \frac{V(s)}{sL} + \frac{i_0}{s} \tag{6.127}$$

The circuit interpretation of these relationships is shown in fig. 6.27. In (6.126) we see that the voltage $V(s)$ is the sum of two terms: (1) a voltage drop, $sL \times I(s)$, where sL is interpreted as a reactance (dimensions of ohms);

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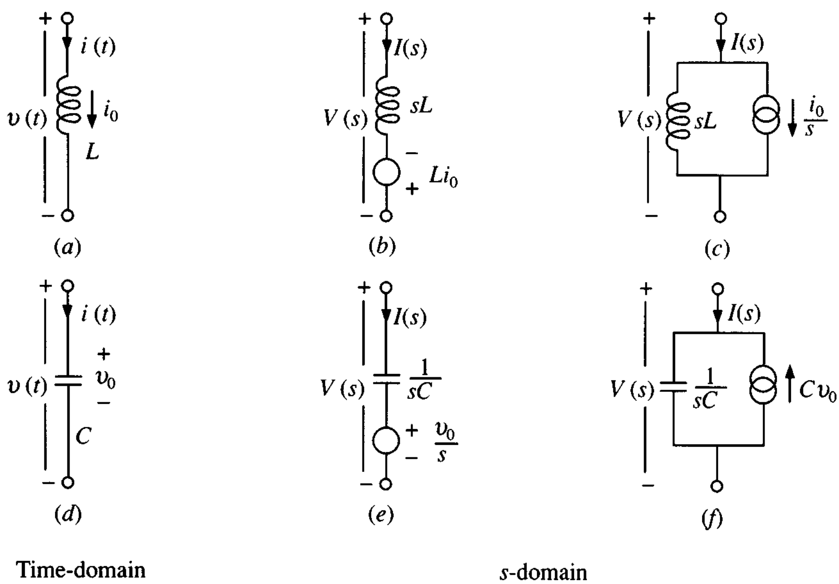
(2) a constant voltage Li_0. The s-domain circuit consists, therefore, of an inductance L in series with a constant voltage source, as shown in fig. 6.27(b). Likewise in (6.127) the current $I(s)$ is the sum of two terms: $V(s)/sL$ and i_0/s. The s-domain circuit consists, therefore, of an inductance L in parallel with a constant current source (fig. 6.27(c)).

It will be appreciated that the relationships (6.126) and (6.127) in the s-domain are mathematically identical; one can be derived from the other by simple algebraic manipulation. From the circuit point of view this manipulation corresponds to the Thévenin–Norton transformation (discussed in section 2.9.1). Fig. 6.27(b) is a Thévenin circuit (inductance in series with an ideal voltage source), while fig. 6.27(c) is a Norton circuit consisting of the same inductance in parallel with a current source the magnitude of which is given by $Li_0/sL = i_0/s$. It is of interest to note that the source Li_0 in the s-domain circuit of fig. 6.27(b) corresponds to an impulse in the time-domain circuit since the inverse transform of a constant is an impulse (transform pair No. 18). Likewise, the source i_0/s in fig. 6.27(c) corresponds to a step function in the time-domain circuit (transform pair No. 16).

The transform corresponding to the voltage–current relationship for capacitance, charged to an initial voltage v_0, was also derived in section 6.9.2 (equation 6.108):

Fig. 6.27. Time- and s-domain circuits for inductance and capacitance. i_0 and v_0 are initial values of current and voltage.



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$$v(t)=\frac{1}{C}\int i(t)\,dt+v_0 \Rightarrow V(s)=\frac{I(s)}{sC}+\frac{v_0}{s} \tag{6.128}$$

The corresponding current–voltage relationship is:

$$i(t)=C\frac{dv(t)}{dt} \Rightarrow I(s)=sCV(s)-Cv_0 \tag{6.129}$$

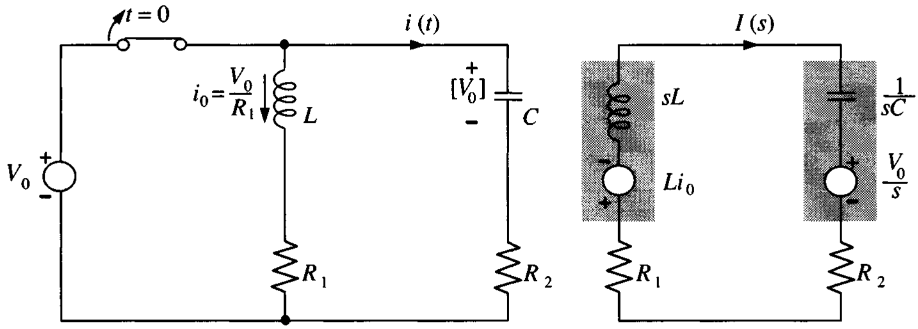
Figures 6.27(d), (e) and (f) show the circuit interpretation of these relationships; again it will be evident that the circuits of figs. 6.27(e) and (f) are related through the Thévenin–Norton transformation.

The relationships shown in fig. 6.27 enable one to transform any circuit into the s-domain. Once the s-domain circuit is established any of the formal procedures and techniques of steady-state circuit analysis may be applied. For the inexperienced reader it is recommended that inductive and capacitive elements with initial current and voltages be transformed using the series forms given in figs. 6.27(b) and (e). The Thévenin–Norton transformation can always be applied subsequently according to the demands of a particular problem.

As an example of how the relationships shown in fig. 6.27 are applied to derive the transform of a circuit, consider the time-domain circuit of fig. 6.28(a). The switch has been closed for an appreciable time so that C is charged to the source voltage V_0, and a constant current $i_0(=V_0/R_1)$ flows through L. At $t=0$ the switch is opened. We wish to find an expression for the current $i(t)$ in the circuit for $t>0$.

To derive the s-domain circuit of fig. 6.28(b) we consider each of the storage elements in turn. The inductance with its initial current i_0 is, according to fig. 6.27(b), transformed to an inductance connected in series

Fig. 6.28. Example of circuit transformation using the relationships shown in figs. 6.2.7 (a), (b) and (d), (e).



(a) time–domain (b) s–domain

with an ideal voltage source. The capacitance with its initial voltage V_0 also transforms to a series combination of capacitance and ideal voltage source. We choose the series (Thévenin) circuit transformations in each case because this leads to the simplest possible single-mesh circuit in the s-domain. Note that care must be exercised to ensure that the correct polarities are assigned to the voltage sources in the s-domain. For the inductance, the polarity of its associated source must be such as to drive current in the *same* direction as that of the initial current in the time-domain circuit. For the capacitance the polarity of the s-domain source must be identical to that of the initial voltage in the time-domain. These observations apply irrespective of the directions of the assigned currents $i(t)$ or $I(s)$.

Applying Kirchhoff's voltage law to the s-domain circuit we have

$$\left(sL+\frac{1}{sC}+R_1+R_2\right)I(s)=-\left(\frac{V_0}{s}+Li_0\right)$$

or

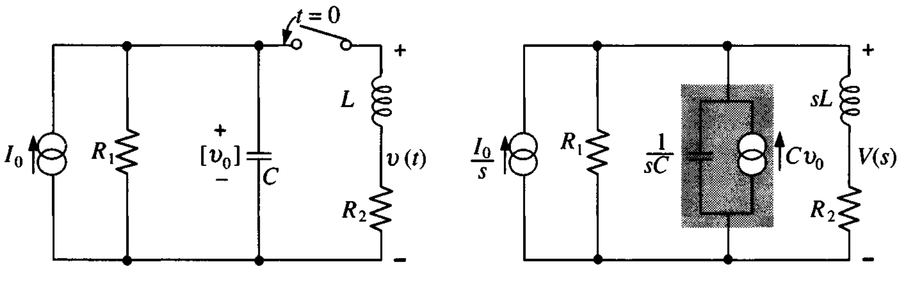
$$I(s)=\frac{-(V_0/s+Li_0)}{sL+1/sC+R_1+R_2}$$

Inversion of this expression yields the required function of current in the time domain.

A further example of the application of the Laplace transform method is shown in fig. 6.29. In fig. 6.29(a) C is charged to an initial voltage $v_0(=I_0R_1)$. The switch is closed at $t=0$; we wish to find the voltage $v(t)$ for $t>0$.

In this case it is slightly more convenient to use the parallel transformation of fig. 6.27(f) since this leads directly to an s-domain circuit with one

Fig. 6.29. Example of circuit transformation using (for the capacitance) the relationship shown in figs. 6.27 (d), (f).



(a) time–domain (b) s–domain

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independent node for which a nodal analysis, to find $V(s)$, is clearly appropriate. (The choice of the series transformation of fig. 6.27(e) would have led, albeit indirectly, to precisely the same nodal analysis.) Note that the ideal current generator of magnitude I_0 in the t-domain circuit transforms to I_0/s in the s-domain (transform pair no. 15).

Applying nodal analysis to the s-domain circuit we obtain:

$$\left[\frac{1}{R_1}+sC+\frac{1}{sL+R_2}\right]V(s)-\frac{I_0}{s}-Cv_0=0$$

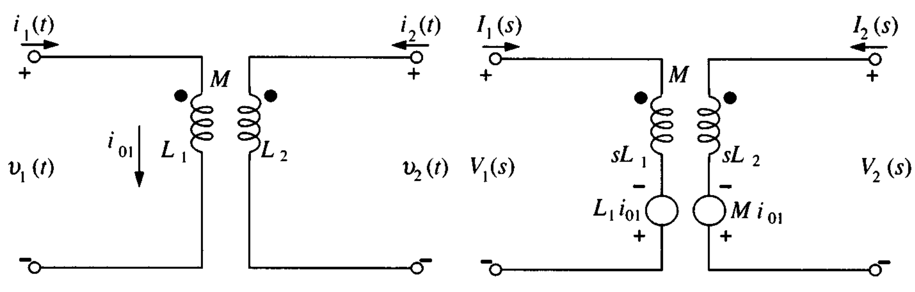
or

$$V(s)=\frac{I_0/s+Cv_0}{1/R_1+sC+1/(sL+R_2)}$$

Inversion of this function yields $v(t)$.

The transform relationships for inductance expressed by (6.126), and illustrated in figs. 6.27(a) and (b), may be readily extended to the case of mutual inductance. We have seen that an initial current i_0 in an inductance gives rise to a constant voltage source Li_0 in the s-domain. Referring to the circuit of fig. 6.30(a), in which there are two coils of inductances L_1 and L_2 coupled by mutual inductance M, if the first coil carries initial current i_{01}, then in the s-domain circuit this will give rise to a voltage source of magnitude L_1i_{01} in series with L_1. In addition, a source of magnitude Mi_{01} will arise in series with L_2. This follows from the theory of mutual inductance presented in section 1.10. The polarity of this source will depend upon the way in which the two coils are wound with respect to one another. This information is provided by the dot convention (see sections 1.10 and 3.12). Notice that in fig. 6.30(b), the polarities of the two sources Li_{01} and Mi_{01} bear precisely the same relationship with the corresponding ends of

Fig. 6.30. Time- and s-domain circuits for mutual inductance.



(a) time-domain (b) s-domain

the two coils; that is, the negative side of the Li_{01} source is joined to the non-dotted end of L_1, and the negative side of the Mi_{01} is joined to the non-dotted end of L_2. (Of course, the same end result would be obtained if the positive sides of either or both sources were connected to the dotted ends of the coils.)

Finally, it follows that if there is an initial current i_{02} in L_2, this will produce in the s-domain circuit a source $L_2 i_{02}$ in series with L_2 and, additionally, a source Mi_{02} in series with L_1.

6.9.5 Worked example

The circuit of fig. 6.31(a) is in equilibrium with the switch open. At $t=0$ the switch is closed. Find the current $i_2(t)$ for $t>0$.

Solution: Since the circuit is in equilibrium for $t<0$, the current through L_1 at the instant of closing the switch must be $i_0 = V_0/R_1 = 5$ A for the given circuit values. The direction of i_0 is from left to right in the circuit diagram. In the s-domain circuit (fig. 6.31(b)) this current gives rise to the voltage source $L_1 i_0$ with its polarity such that it drives current from left to right (into the dotted end of L_1). In addition, i_0 gives rise to the source Mi_0, with polarity such that it also drives current into the dotted end of L_2.

The procedure for solving circuits containing mutual inductance is outlined in section 3.12. Currents $I_1(s)$ and $I_2(s)$ are assigned to the two meshes in the s-domain circuit. (In practical problems it is sufficient to denote currents by I_1, I_2 etc.)

Applying KVL to mesh (1) we obtain

$$(sL_1 + R_1)I_1 - R_1 I_2 + sMI_2 = \frac{V_0}{s} + L_1 i_0$$

and for mesh (2)

$$(R_1 + R_2 + sL_2)I_2 - R_1 I_1 + sMI_1 = Mi_0$$

Note that the terms due to mutual inductance, sMI_2 and sMI_1, are both positive because both assigned currents enter corresponding (dotted) ends of coils.

Substituting numerical values and rearranging the above two equations we obtain

$$(s+1)I_1 + (s-1)I_2 = \frac{5}{s} + 5$$

and

$$(s-1)I_1 + (s+2)I_2 = 5$$

Solution of these equations yields, after some algebraic manipulation,

$$I_2 = \frac{s+1}{s(s+0.2)}$$

By (6.113) we may write

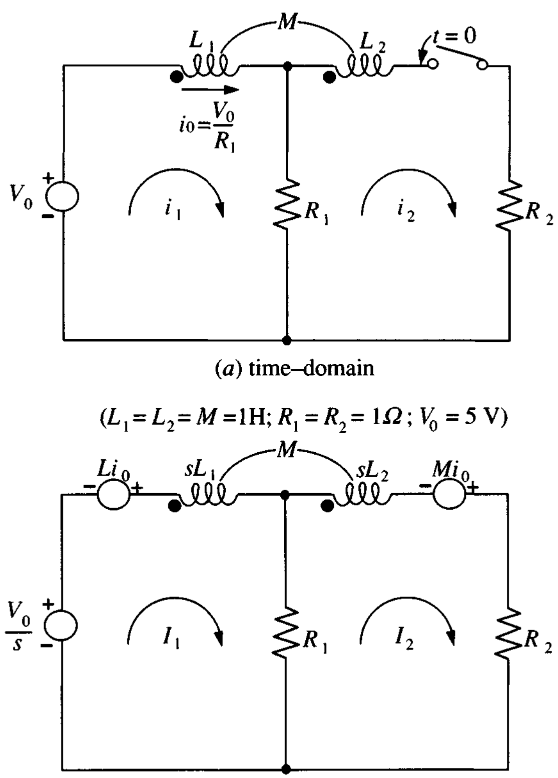
$$I_2 = \frac{A}{s} + \frac{B}{s+0.2}$$

and by (6.114)

$$A = [sF(s)]_{s=0} = \frac{1}{0.2} = 5$$

$$B = [(s+0.2)F(s)]_{s=-0.2} = \frac{-0.2+1}{-0.2} = -4$$

Fig. 6.31. Circuits for worked example (section 6.9.5).



(*b*) s–domain

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Therefore,

$$I_2(s)=\frac{5}{s}-\frac{4}{s+0.2}$$

The first term of this expression is inverted using transform pair No. 15, and the second term using transform pair No. 2. So,

$$i_2(t)=5-4e^{-0.2t}$$

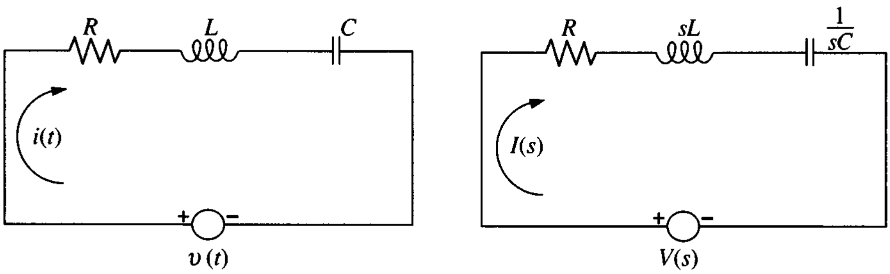
If the simultaneous equations above are solved for I_1, a similar procedure shows that the current in the first mesh is $i_1(t)=10-6e^{-0.2t}$.

It may be remarked that the current $i_2(t)$ at the instant $t=0^+$ is, according to the expression above, equal to 1 A, while the current $i_1(t)$ is 4 A. Before closing the switch, that is, at $t=0^-$, the currents were 0 and 5 A respectively; evidently the currents in the inductances are changing instantaneously, which might be taken to imply that the energies are doing likewise. The reason for this apparent anomaly is that, if energy is to be conserved, the total flux linkage (inductance × current) must be conserved at the instant of switching. Both currents contribute to the total flux linkage. Thus, at $t=0^-$: $L_1 i_0=(1)(5)=5$. At $t=0^+$: $L_1 i_1+Mi_2=(1)(4)+(1)(1)=5$. The product $(L\times i)$ is often referred to as the *electrokinetic momentum* (see reference 2).

6.9.6 Generalized impedance, network function and impulse response

Consider the general branch driven by a voltage source $v(t)$ as shown in fig. 6.32(a). The circuit is initially dead so that the circuit transforms into that shown in fig. 6.32(b). In the s-domain the circuit equation is

Fig. 6.32. Time- and s-domain circuits for the general *RLC* branch driven by an ideal voltage source.



(a) time–domain (b) s–domain

$$\left(R+sL+\frac{1}{sC}\right)I(s)=V(s)$$

or

$$\frac{V(s)}{I(s)}=R+sL+\frac{1}{sC}$$

The ratio $V(s)/I(s)$ may be interpreted as an impedance and we can write

$$Z(s)=\frac{V(s)}{I(s)}=R+sL+\frac{1}{sC} \tag{6.130}$$

Now in the detailed theory of the Laplace transform it is shown that s is complex with dimensions of angular frequency. (It has already been indicated in section 6.5.3 how the concept of complex frequency can arise in circuit theory.) The complex frequency is usually written

$$s=\sigma+\mathrm{j}\omega \tag{6.131}$$

If we let $\sigma=0$, then

$$Z(s)=Z(\mathrm{j}\omega)=R+\mathrm{j}\omega L+\frac{1}{\mathrm{j}\omega C} \tag{6.132}$$

which is recognized as the steady-state a.c. circuit impedance. We conclude that (6.132) is merely a special case of (6.130), and indeed, from this viewpoint the a.c. theory developed in chapter 3 can be regarded as a special case of the more general approach afforded by the Laplace transform.

In the s-domain the ratio of voltage to current at any terminal pair or port of a network is denoted by $Z(s)$ and is referred to as the *generalized impedance* (or sometimes the generalized driving point impedance). Likewise the ratio of current to voltage is called the *generalized admittance.** (The word 'generalized' is often omitted when the context is clear.)

For example, the generalized admittance of a combination of R, L and C connected in parallel may be written

$$Y(s)=\frac{1}{R}+\frac{1}{sL}+sC \tag{6.133}$$

It will be apparent, therefore, that the analytical techniques and terminology developed in chapter 3 for a.c. quantities may be translated directly in terms of these generalized concepts. In particular, the important

* The term *immittance* is often used when referring in a non-specific way to either impedance or admittance.

ideas concerning the transfer function of a network, introduced in section 3.9, may be extended to the s-domain, as indicated in fig. 6.33.

If $e(t)$ is a driving function or excitation in the time-domain network and $r(t)$ is the response to this excitation, and if $E(s)$ and $R(s)$ are the corresponding Laplace transforms, then we may define a *network function* $H(s)$ by

$$H(s)=\frac{R(s)}{E(s)}$$

or

$$R(s)=H(s)E(s) \tag{6.134}$$

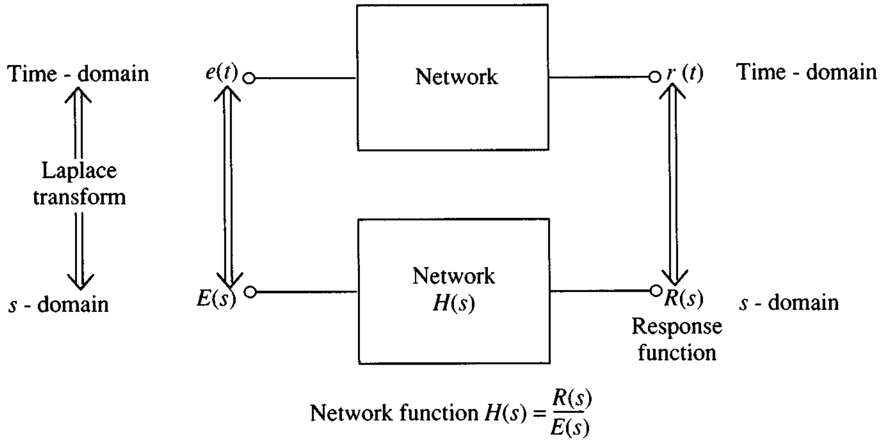
$E(s)$ and $R(s)$ are called the *excitation function* and *response function* respectively.

If $R(s)$ and $E(s)$, which may be voltage or current functions in s, refer to the same port, then $H(s)$ is a driving point immittance (impedance or admittance). If they refer to different ports, then $H(s)$ is termed a transfer function.

As an example serving to illustrate some of the points discussed above, let us find the transfer function for the circuit shown in fig. 6.34, and its response to a ramp function input. The circuit forms a two-arm divider with parallel elements in each arm; the admittance divider formulation is therefore the most appropriate way of finding the transfer function. The admittances of the two arms are:

$$Y_1(s)=\frac{1}{R_1}+sC_1=\frac{1+sC_1R_1}{R_1}$$

Fig. 6.33. Definition of the network function.



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and

$$Y_2(s)=\frac{1}{R_2}+sC_2=\frac{1+sC_2R_2}{R_2}$$

Therefore, the transfer function is

$$H(s)=\frac{V_2(s)}{V_1(s)}=\frac{Y_1(s)}{Y_1(s)+Y_2(s)}$$

$$=\frac{(1+sC_1R_1)/R_1}{(1+sC_1R_1)/R_1+(1+sC_2R_2)/R_2} \qquad (6.135)^*$$

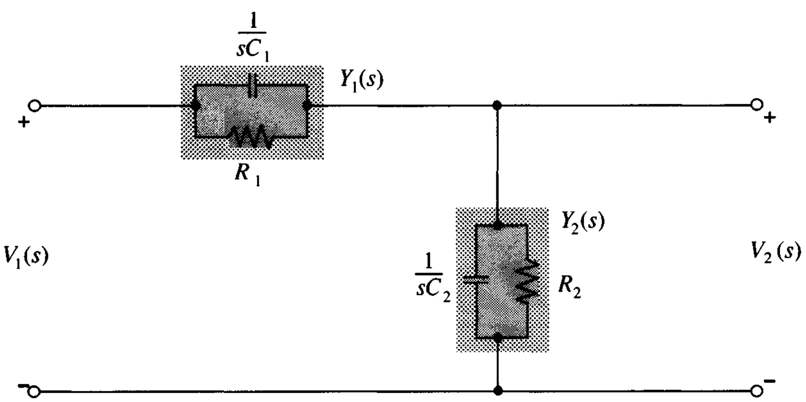
If the input $e(t)$ is a ramp function, then the excitation function $E(s)$ is (from transform pair No. 20) $1/s^2$. Thus, the response in the s-domain will be

$$R(s)=H(s)E(s)=\frac{(1+sC_1R_1)/R_1}{(1+sC_1R_1)/R_1+(1+sC_2R_2)/R_2}\cdot\frac{1}{s^2}$$

This example illustrates the way in which one can utilize the techniques of a.c. circuit analysis in the s-domain to obtain the response of a network not only to sinusoidal input waveforms but to any waveform whose Laplace transform can be found.

An important special case arises when the excitation $e(t)$ is the unit

Fig. 6.34. s-domain circuit for a voltage divider.



* If the time constants in the two arms are equal, that is, if $C_1R_1=C_2R_2$, then (6.135) reduces to $H(s)=R_2/(R_1+R_2)$, which is independent of frequency. Waveforms of arbitrary shape will, therefore, be transferred without distortion, (except for a scaling factor). For this reason the circuit with equal time constants is known as a frequency compensated divider. (This forms the basis of the well-known 'oscilloscope probe'.)

impulse function. In this case the response $r(t)$ is, by definition, equal to the impulse response $h(t)$ (see section 6.8.2). Now the transform of the unit impulse function is equal to unity, consequently the response in the s-domain is given by

$$R(s)=H(s)E(s)=H(s)\times 1=H(s)$$

This means that the response function in the s-domain for unit impulse excitation in the time domain, is simply the network function itself. It follows from this that the impulse response $h(t)$ must be the inverse transform of the network function $H(s)$. The time and s-domain relationships for this particular case are depicted in fig. 6.35.

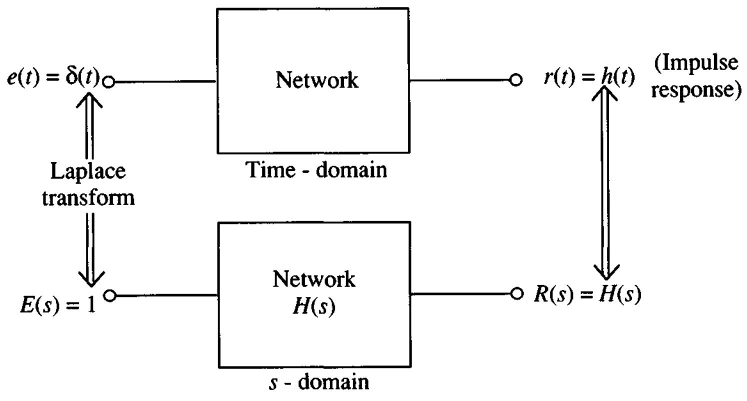
This relationship between impulse response and network function provides a relatively simple way of determining the impulse response of a network: first find the network function using the 'a.c. steady-state' approach described above; then find the inverse transform of the network function using tables of transform pairs. For example, let us find the impulse response of the simple RC circuit of fig. 6.36. For this circuit, the network (transfer) function is given by

$$H(s)=\frac{V_2(s)}{V_1(s)}=\frac{1/sC}{1/sC+R}=\frac{1}{\tau(s+1/\tau)}$$

where $\tau=CR$. Hence, the impulse response is

$$h(t)=\frac{1}{\tau}\mathrm{e}^{-t/\tau}$$

Fig. 6.35. Illustrating the relationships between the network function and impulse response.



The reader should compare this procedure with that used earlier to obtain (6.85).

6.9.7 Third and higher order networks

We have observed in preceding sections that a circuit containing a single energy storage element leads to a first order differential equation, while a circuit containing two independent storage elements leads to a second order differential equation. For example, the equation for the general series branch, containing two storage elements is

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri+\frac{1}{C}\int i\,\mathrm{d}t=v$$

which upon differentiation gives the second order equation

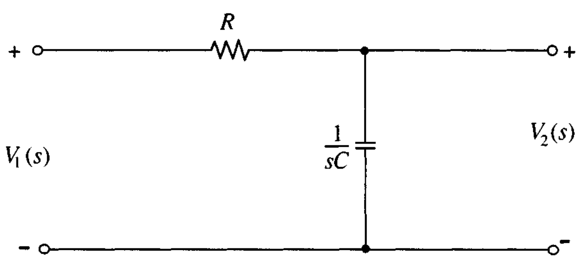
$$L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2}+R\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{i}{C}=\frac{\mathrm{d}v}{\mathrm{d}t}$$

We now consider in a more general way the relationship between the number and types of storage element in a network, and the form of the network equation and its solution. Generalizing the above result for the *RLC* series branch, the equation relating the response $r(t)$ at any port in a network to the excitation $e(t)$ at the same or a different port of that network may be written in differential form as:

$$a_n\frac{\mathrm{d}^n r}{\mathrm{d}t^n}+a_{n-1}\frac{\mathrm{d}^{n-1} r}{\mathrm{d}t^{n-1}}+\ldots+a_1\frac{\mathrm{d}r}{\mathrm{d}t}+a_0$$
$$=b_m\frac{\mathrm{d}^m e}{\mathrm{d}t^m}+b_{m-1}\frac{\mathrm{d}^{m-1} e}{\mathrm{d}t^{m-1}}+\ldots+b_1\frac{\mathrm{d}e}{\mathrm{d}t}+b_0 \qquad (6.136)$$

where the a_i and b_i are functions of the network elements only, and n is equal to the number of independent storage elements.

Fig. 6.36. Simple *RC* circuit in the *s*-domain.



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Setting the RHS of (6.136) to zero results in an equation that characterizes the natural behaviour of the network. Its solution gives the natural response which contains n terms of the form

$$r(t)=A_1e^{s_1t}+A_2e^{s_2t}+\ldots+A_ne^{s_nt} \tag{6.137}$$

in which $A_1 \ldots A_n$ are arbitrary constants (governed by the initial energy states of the network and by the form of the excitation function), and $s_1 \ldots s_n$ are the roots of the auxiliary equation:

$$s^n+\frac{a_{n-1}}{a_n}s^{n-1}+\ldots+\frac{a_1}{a_n}s+\frac{a_0}{a_n}=0 \tag{6.138}$$

a polynomial of degree n.

The corresponding network equation may be formulated in the s-domain by transforming both sides of (6.136)

$$\begin{aligned}&(a_ns^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0)R(s)\\&=(b_ms^m+b_{m-1}s^{m-1}+\ldots+b_1s_1+b_0)E(s)\end{aligned}$$

or

$$R(s)=\frac{b_ms^m+b_{m-1}s^{m-1}+\ldots+b_1s+b_0}{a_n\left(s^n+\dfrac{a_{n-1}}{a_n}s^{n-1}+\ldots+\dfrac{a_1}{a_n}s+\dfrac{a_0}{a_n}\right)}E(s) \tag{6.139}$$

To find the response $r(t)$ from this expression requires the inverse transform to be found, and this in turn necessitates finding the roots of the denominator polynomial, which is of degree n and identical to (6.138).

Thus, whether the network equation is formulated in the time-domain or the s-domain, its solution entails finding the roots of a polynomial of degree n where n is the number of independent storage elements in the network. (Exceptions to this general rule are provided by certain network configurations containing elements of identical value, in which case the network equation may be of lower order than n.)

Before attempting to solve a network problem it is good practice to count the number of independent storage elements and to note their type. The following rules can then offer a general (although not infallible) guide to the form of the network equation and its solution. The polynomial referred to below is that given by (6.138) or the denominator of (6.139).

Rule 1 If n is the number of independent storage elements in a network, then the following parameters are numerically equal to n:

(a) order of circuit differential equation

(b) degrees of polynomial

(c) number of roots

(d) number of exponential terms in the solution
(e) number of arbitrary constants.

Rule 2 If storage elements are all of one type, then the roots of the polynomial will be real and there will be no oscillatory terms in the natural response. If they are not of one type, the roots may be complex leading to oscillatory terms in the natural response.

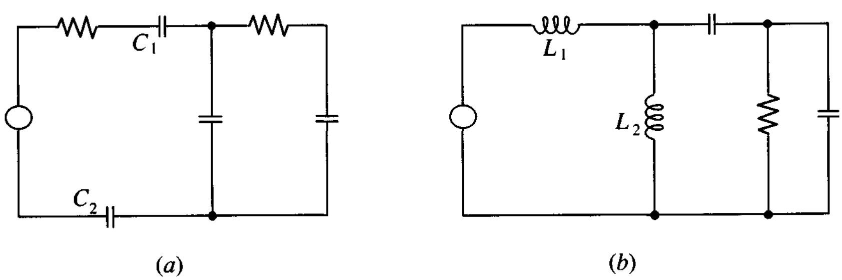
Rule 3 Coefficients of the polynomial must be real and positive in all cases.

It must be emphasized that the number of independent storage elements in a network is not necessarily the same as the number of separate, identifiable elements. Two inductances in the same branch, for instance, combine to form a single independent storage element. Both circuits in fig. 6.37 contain four separate storage elements yet lead to a third order equation; in each case elements can be combined by series or parallel addition. In fig. 6.37(*b*) a Thévenin–Norton transformation is required to allow the two leftmost inductances to be combined in parallel.

Finding the roots of third and higher degree polynomials can constitute a major proportion of the work involved in solving complex networks. An algebraic formula is available for finding the roots of a cubic, but it is distinctly more difficult to apply than the quadratic formula. For quartic and polynomials of higher degree, resort has to be made to algebraic methods aimed at reduction of the polynomial in question into cubic, quadratic or linear components (Lin's method). Iterative methods also exist for finding the real roots of a polynomial to any desired degree of accuracy (for example, Horner's method) but the amount of repetitive manipulation involved is considerable.

For these reasons it has become commonplace to use numerical methods to find the roots of third and higher order polynomials. Program C4 in Appendix C, allows one to compute the real and complex roots of higher order polynomials.

Fig. 6.37. Circuits containing four storage elements: (*a*) reduces to third order by combining C_1 and C_2 in series, (*b*) reduces to third order by combining L_1 and L_2 in parallel.



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Although the amount of work involved in evaluating the network polynomial is the same irrespective of whether the network equation is formulated in the time domain or the s-domain, the latter has advantages when dealing with third and higher order networks. This stems from the fact that initial energy states can be incorporated quite simply into the s-domain network equation. The time-domain solution, on the other hand, requires the evaluation of arbitrary constants after the complete solution of the network differential equation has been found. This can be a difficult operation since it involves the evaluation of higher derivatives in terms of the initial energy states of the network.

6.9.8 Worked example

(a) Find the transfer function $H(s) = V_2(s)/V_1(s)$ for the circuit shown in fig. 6.38.

(b) Show that the expression for the natural response of this circuit contains three exponential terms, and determine the time constants associated with these terms for the given values of R and C.

(c) If $v_1(t)$ is a steady sinusoidal function, show that the phase shift of $v_2(t)$ with respect to $v_1(t)$ is zero for a particular frequency, and determine this frequency for the given values of R and C.

Solution

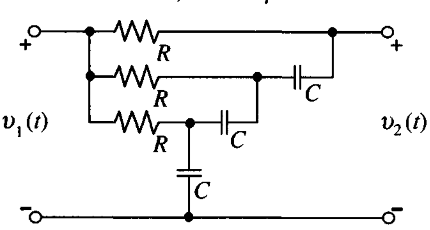
(a) Referring to the s-domain circuit (fig. 6.38(b)), the circuit contains three independent nodes O, P, Q. The voltage at O is the assigned voltage V_2; let the voltages at P and Q be V_P and V_Q respectively. Applying nodal analysis we have: at node O

$$\frac{V_2 - V_1}{R} + \frac{V_2 - V_P}{1/sC} = 0$$

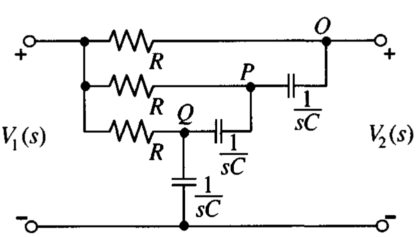
$$(sCR+1)V_2 - sCRV_P = V_1$$

Fig. 6.38. Circuits for worked example (section 6.9.8).

R = 1 kΩ; C = 0.1 μF



(a) time-domain



(b) s-domain

Let $sCR = a$ (The algebraic manipulation is thus greatly simplified.) Then

$$(a+1)V_2 - aV_p = V_1$$

Similarly, at nodes P and Q we obtain

$$-aV_2 + (2a+1)V_P - aV_Q = V_1$$

and

$$-aV_P + (2a+1)V_Q = V_1$$

Elimination of V_P and V_Q from the above three simultaneous equations yields:

$$V_2(a^3 + 6a^2 + 5a + 1) = V_1(6a^2 + 5a + 1)$$

The transfer function is therefore

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{6s^2(RC)^2 + 5sRC + 1}{s^3(RC)^3 + 6s^2(RC)^2 + 5sRC + 1}$$

(b) The natural response is obtained from the roots of the denominator polynomial in the transfer function. Normalizing, by letting the product $RC = 1$, the polynomial becomes

$$s^3 + 6s^2 + 5s + 1$$

Using program C4 in Appendix C, we find that this has roots at $s = -5.05$; $s = -0.643$; $s = -0.308$.

In the time-domain the natural response is therefore of the form:

$$A_1 e^{-5.05t} + A_2 e^{-0.643t} + A_3 e^{-0.308t}$$

or

$$A_1 e^{-t/0.198} + A_2 e^{-t/1.56} + A_3 e^{-t/3.25}$$

For the given component values, $RC = 10^{-4}$, hence, the required time constants are 19.8 μs; 156 μs; 325 μs.

(c) For a steady sinusoidal input, the transfer function is

$$H(j\omega) = \frac{6(j\omega RC)^2 + 5j\omega RC + 1}{(j\omega RC)^3 + 6(j\omega RC)^2 + 5j\omega RC + 1}$$

$$= \frac{1 - 6(\omega RC)^2 + j5\omega RC}{1 - 6(\omega RC)^2 + j[5\omega RC - (\omega RC)^3]}$$

Let $1 - 6(\omega RC)^2 = p$; $5\omega RC = q$ and $5\omega RC - (\omega RC)^3 = r$, then,

$$H(j\omega) = \frac{p + jq}{p + jr} = \frac{(p+jq)(p-jr)}{p^2 + r^2}$$

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$$=\frac{p^2+qr}{p^2+r^2}+j\frac{p(q-p)}{p^2+r^2}$$

If the phase shift is to be zero, then the imaginary term in this expression must vanish, that is $p(q-p)=0$. Putting $(q-p)=0$ leads to the trivial result $\omega=0$, therefore,

$$p=1-6(\omega RC)^2=0$$

which gives

$$\omega=\frac{1}{\sqrt{6}RC}$$

For the given values of R and C, $\omega=4082$ rad/s.

This circuit is one of a small number of RC circuits capable of producing a voltage step-up. For the zero phase-shift condition, the transfer function becomes

$$H(j\omega)=\frac{q}{r}=\frac{5\omega CR}{5\omega CR-(\omega RC)^3}=\frac{5/\sqrt{6}}{5/\sqrt{6}-(1/\sqrt{6})^3}=\frac{30}{29}$$

This property of the circuit has been utilized in one type of oscillator.

6.9.9 Further Laplace transform theorems

In this section we consider some useful theorems that allow us to extend the table of transform pairs and increase the facility with which transforms and inverse transforms may be found. The theorems are introduced without proof; interested readers will find proofs in references 1, 2 and 9.

Theorem 1: shift in the s-domain
If $F(s)=\mathscr{L}[f(t)]$,

then

$$F(s+\alpha)=\mathscr{L}[e^{-\alpha t}f(t)] \tag{6.140}$$

Example: find $\mathscr{L}[e^{-\alpha t}\cos\omega t]$.
Since

$$\mathscr{L}[\cos\omega t]=\frac{s}{s^2+\omega^2} \qquad \text{[Pair No. 8]}$$

then

$$\mathscr{L}[e^{-\alpha t}\cos\omega t]=\frac{s+\alpha}{(s+\alpha)^2+\omega^2} \qquad \text{[Pair No. 11]}$$

Example: given $F(s)=\dfrac{1}{(s+2)^2}$, find $f(t)$.

Since

$$\mathscr{L}^{-1}\left[\frac{1}{s^2}\right]=t \qquad \text{[Pair No. 1]}$$

then

$$\mathscr{L}^{-1}\left[\frac{1}{(s+2)^2}\right]=f(t)=t\mathrm{e}^{-2t}$$

Theorem 2: shift in the time-domain (also known as '*real translation*')
If $\mathscr{L}[f(t)]=F(s)$

then

$$\mathscr{L}[f(t-a)]=\mathrm{e}^{-at}F(s) \tag{6.141}$$

Example: find the Laplace transform of the delayed impulse function $\delta(t-a)$.

Since

$$\mathscr{L}[\delta(t)]=1 \qquad \text{[Pair No. 18]}$$

then

$$\mathscr{L}[\delta(t-a)]=\mathrm{e}^{-as} \qquad \text{[Pair No. 19]}$$

Theorem 3: differentiation with respect to s
If $\mathscr{L}[f(t)]=F(s)$

then

$$\mathscr{L}[tf(t)]=-\frac{\mathrm{d}}{\mathrm{d}s}(F(s)) \tag{6.142}$$

Example: find the Laplace transform of $t\sin\omega t$. Since

$$\mathscr{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2} \qquad \text{[Pair No. 6]}$$

$$\mathscr{L}[t\sin\omega t]=-\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\omega}{s^2+\omega^2}\right]=\frac{2s\omega}{(s^2+\omega^2)^2}$$

Theorems 4 and 5: initial and final value theorems
If $\mathscr{L}[f(t)]=F(s)$

then

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$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{initial value theorem} \tag{6.143}$$

and

$$\lim_{t\to \infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{final value theorem} \tag{6.144}$$

These theorems are applicable only if $f(t)$ and its first derivative are Laplace transformable. In addition, for the final value theorem, complex factors in the denominator polynomial of $sF(s)$ must have positive real parts. The theorems are often useful for checking whether a function of current or voltage derived in the s-domain gives physically sensible results.

Example. Consider a step function of magnitude V applied to the RC circuit of Fig. 6.36. The transfer function is $1/(sRC+1)$ so that

$$V_2(s) = \frac{1}{(sRC+1)} \cdot \frac{V}{s} \text{ and } sV_2(s) = \frac{V}{sRC+1}$$

Then, by the initial value theorem,

$$\lim_{t\to 0} v_2(t) = \lim_{s\to\infty} \left[\frac{V}{sRC+1}\right] = 0$$

and by the final value theorem

$$\lim_{t\to \infty} v_2(t) = \lim_{s\to 0} \left[\frac{V}{sRC+1}\right] = V$$

These results are what might be expected from physical considerations: since the voltage on the capacitance cannot change instantaneously, its voltage must be zero at $t=0^+$ (assuming zero initial voltage). It is also apparent that the capacitance will charge up to a final value V.

6.10 Pole-zero methods

Let us again consider the expression (6.139) relating the response and excitation functions of a network. Since the ratio $R(s)/E(s)$ is equal to the network function $H(s)$, (6.139) may be rewritten as

$$\frac{R(s)}{E(s)} = H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

$$= \frac{b_m}{a_n} \cdot \frac{\left(s^m + \frac{b_{m-1}}{b_m}s^{m-1} + \ldots + \frac{b_1}{b_m}s + \frac{b_0}{b_m}\right)}{\left(s^n + \frac{a_{n-1}}{a_n}s^{n-1} + \ldots + \frac{a_1}{a_n}s + \frac{a_0}{a_n}\right)} \tag{6.145}$$

Recalling that the a_i and b_i in this expression are functions only of the network Ls, Cs and Rs, we see that $H(s)$ is the ratio of two polynomials with real coefficients, that is, it is a rational function. Therefore, according to the fundamental theorem of algebra, the numerator and denominator polynomials may be factorized to give:

$$H(s)=H_0\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \tag{6.146}$$

where $H_0=\dfrac{b_m}{a_n}$ is a scaling factor, and the z_i and p_i are roots respectively of the numerator and denominator polynomials.

Now if we examine the behaviour of $H(s)$ as the complex frequency s is varied, we see that at the particular values of $s=p_1$, $s=p_2$ etc., each of the factors in the denominator polynomial of $H(s)$ in turn becomes zero and the function becomes infinite. We say that *poles* of the function exist at $s=p_1, p_2 \ldots$ or that $p_1, p_2 \ldots$ are poles of $H(s)$. Likewise, at the particular values of $s=z_1$, $s=z_2$ etc., $H(s)$ becomes zero and we refer to $z_1, z_2 \ldots$ as *zeros* of the function $H(s)$. Put another way: the poles are the roots of the denominator polynomial, the zeros are the roots of the numerator polynomial. If we trace the derivation of (6.146) from (6.139), we see that the poles of $H(s)$ determine the form of the natural response. For example, consider the function

$$H(s)=\frac{2s^3-4s^2+4s}{s^4+6s^3+13s^2+24s+36}$$

which when factorized becomes

$$H(s)=\frac{2s(s-1+\mathrm{j}1)(s-1-\mathrm{j}1)}{(s+3)^2(s+\mathrm{j}2)(s-\mathrm{j}2)}$$

The first factor in the numerator, s, may be written $(s-0)$ from which it will be apparent that a zero exists at $s=0$. Other zeros occur when*

$$s-1+\mathrm{j}1=0 \qquad \text{i.e., when } s=1-\mathrm{j}1$$

and

$$s-1-\mathrm{j}1=0 \qquad \text{i.e., when } s=1+\mathrm{j}1$$

Poles of the function occur when

* Strictly, a zero also occurs at $s=\infty$. This arises because as $s\rightarrow\infty$ the function $H(s)\rightarrow 1/s\rightarrow 0$. Poles and zeros at infinity are of importance only in the more advanced theory of the method.

$$(s+3)^2=0 \qquad \text{i.e. when } s=-3 \text{ (twice)}$$
$$s+\mathrm{j}2=0 \qquad \text{i.e. when } s=-\mathrm{j}2$$
$$s-\mathrm{j}2=0 \qquad \text{i.e. when } s=+\mathrm{j}2$$

The locations of the poles and zeros may be mapped in the complex frequency plane, or s-plane, as shown in fig. 6.39. We shall see that such a map, which is known as a *pole-zero diagram*, provides an extremely informative way of displaying the essential features of a network function, and of illustrating the characteristics of the network from which the function is derived. Note that the scaling factor ($\times 2$ in this case) does not affect the positions of the poles and zeros, and is often omitted from the pole-zero diagram.

In order to illustrate the use of the pole-zero diagram, we again consider the general series branch shown in fig. 6.32. The circuit equation is

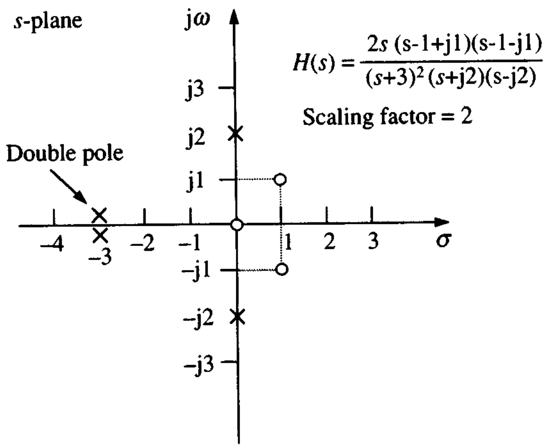
$$\left(R+sL+\frac{1}{sC}\right)I(s)=V(s)$$

which may be written

$$I(s)=\left[\frac{1}{R+sL+\dfrac{1}{sC}}\right]V(s)$$

In the above expression $V(s)$ is the excitation function, $I(s)$ is the response function, and the term in brackets is the network function – in this case an admittance function

Fig. 6.39. Pole-zero diagram for the function $H(s)$; poles are indicated by crosses, zeros by circles.



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$$Y(s)=\frac{1}{R+sL+\frac{1}{sC}}=\frac{1}{L}\cdot\frac{s}{\left(s^2+\frac{sR}{L}+\frac{1}{LC}\right)} \tag{6.147}$$

The scaling factor for this function is $1/L$, and there is a single zero at $s=0$, that is, at the origin in the pole-zero diagram. To find the poles, it is necessary to factorize the denominator of (6.147). Let s_1, s_2 be the roots of $(s^2+sR/L+1/LC)$ then

$$Y(s)=\frac{s}{L(s-s_1)(s-s_2)}$$

where

$$s_1, s_2=-\frac{R}{2L}\pm\sqrt{\left[\left(\frac{R}{2L}\right)^2-\frac{1}{LC}\right]}$$

It will be convenient at this point to use the notation introduced in section 6.5.3. Let $R/2L=\alpha$, $1/LC=\omega_0{}^2$ then

$$s_1, s_2=-\alpha\pm\sqrt{(\alpha^2-\omega_0{}^2)} \tag{6.148}$$

Now our study in section 6.5.3 of the behaviour of the *RLC* circuit revealed that three different types of natural response could occur: overdamped, underdamped and critically damped. Let us consider the admittance function $Y(s)$ in terms of these responses.

If $\alpha^2>\omega_0{}^2$ in (6.148) then we have the overdamped case and the roots s_1, s_2 are negative, real and unequal. Again using the notation of section 6.5.3, let $s_1=-m$ and $s_2=-n$, then for the overdamped case the admittance function becomes:

$$Y(s)=\frac{1}{L}\,\frac{s}{(s+m)(s+n)}$$

Poles occur at $-m$ and $-n$ as shown in fig. 6.40(*a*). It follows from (6.148) that these poles are symmetrically disposed about the point $-\alpha$ on the σ-axis.

Next, consider critical damping; this occurs when $\alpha^2=\omega_0{}^2$ in (6.148). Then $s_1=s_2=-\alpha$ and the admittance function becomes:

$$Y(s)=\frac{1}{L}\,\frac{s}{(s+\alpha)^2}$$

In this case a *repeated*, *double* or *second-order* pole is said to occur at $-\alpha$ (fig. 6.40(*b*)).

Finally, consider underdamped or oscillatory response. In this case $\omega_0^2 > \alpha^2$ and

$$s_1, s_2 = -\alpha \pm j\omega_n$$

where ω_n is the damped natural frequency given by

$$\omega_n = \sqrt{(\omega_0^2 - \alpha^2)} \tag{6.149}$$

Then

$$Y(s) = \frac{1}{L}\frac{s}{(s+\alpha+j\omega_n)(s+\alpha-j\omega_n)} \tag{6.150}$$

and the poles are located as shown in fig. 6.40(*c*).

It should be noted that complex poles always occur as conjugate pairs, consequently, the pole-zero diagram is mirror symmetric about the σ-axis.

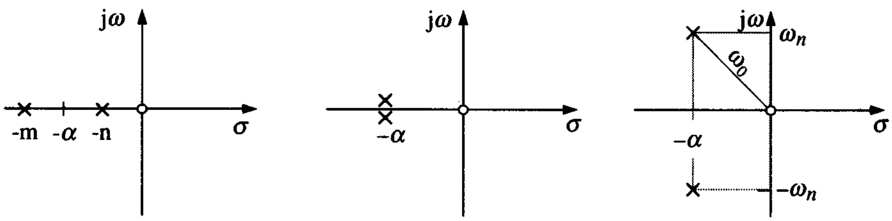
If the damping in the branch is varied by changing the value of R only (keeping L and C constant), then $\omega_0[=1/\sqrt{(LC)}]$ is constant and, from (6.149),

$$\omega_n^2 + \alpha^2 = \omega_0^2 = \text{const.}$$

Thus, the locus of the poles in the s-plane, as R is varied, is a semicircle of radius $1/\sqrt{(LC)}$.

The sequence of events as R is varied, so that the circuit changes from the overdamped case through critical damping to the underdamped case, is illustrated in the pole-zero diagram of fig. 6.41. Starting with a high value of R the poles are located at $(1, 1')$. As R is reduced, the poles move along the σ-axis converging towards point (2) located at $\sigma = -1/\sqrt{(LC)}$. On further reduction of R, the two poles diverge, moving along the semicircle of radius $1/\sqrt{(LC)}$ to points $(3, 3')$. As R becomes vanishingly small the poles lie on the jω-axis at points $(4, 4')$. This is not, of course, a practicable possibility

Fig. 6.40. Pole-zero diagrams for the admittance function of the general series branch.

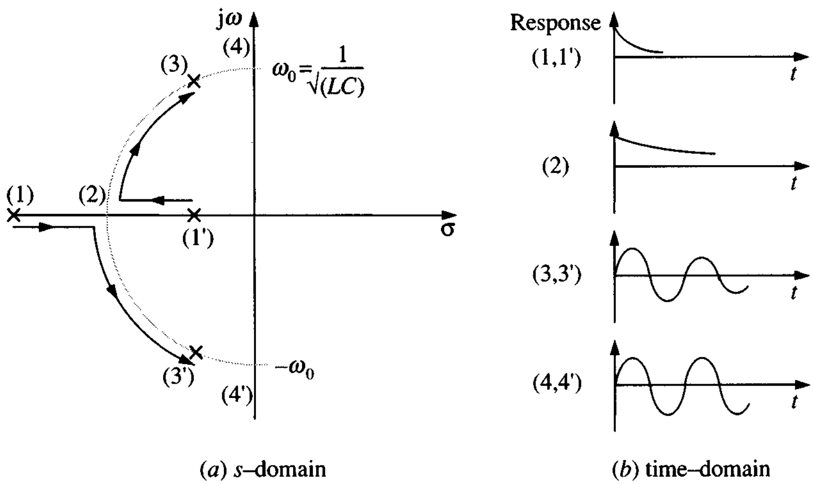


(*a*) Overdamped (*b*) Critically damped (*c*) Underdamped for a passive circuit (except of the superconducting variety) and the poles will always lie somewhere in the left-hand half plane of the pole-zero diagram. Poles in the right-hand half plane imply exponentially increasing functions in the time-domain, which are possible only with circuits containing active devices.* (From this point of view the pole-zero diagram plays an important role in the theory of control systems and the conditions that must obtain for their stability.)

It is sometimes helpful in the interpretation of the pole-zero diagram to think of the modulus of the function under consideration as a surface above the s-plane. In the present instance the surface of $|Y(s)|$, corresponding to the overdamped and underdamped cases, would look somewhat as depicted in fig. 6.42. While this visualization can be informative, it must be appreciated that it is the *map* of the poles and zeros in the s-plane, showing the way in which they move with change of particular network parameters, that provides the all-important information.

An alternative way of representing a network function in the s-plane is obtained by expressing the factors of the function in polar form. In the expression (6.146) for $H(s)$, a factor such as $(s - z_1)$ in the numerator, which is the difference between two complex quantities, can be written as

Fig. 6.41. Illustrating the effect of variation of R on the natural response of the general RLC series branch (L and C constant).



* The detailed theory for passive circuits shows that zeros may exist in the right-hand half plane, but only in the case of transfer functions. Neither poles nor zeros may exist in this region in the case of driving point functions.

$$(s-z_1)=N_1\underline{/\psi_1}$$

where $N_1=|s-z_1|$ and $\psi_1=\arg(s-z_1)$.

In the pole-zero diagram (fig. 6.43), $(s-z_1)$ is a line segment of length N_1 directed from the point z_1 to the point s, making an angle ψ_1 with the horizontal. (This follows from the normal laws of vector addition.)

A factor $(s-p_1)$ in the denominator of (6.146) may likewise be expressed by

$$(s-p_1)=D_1\underline{/\theta_1}$$

with a similar interpretation in the pole-zero diagram.

Expression (6.146) may then be written

$$H(s)=H_0\frac{(s-z_1)(s-z_2)\ldots}{(s-p_1)(s-p_2)\ldots}$$

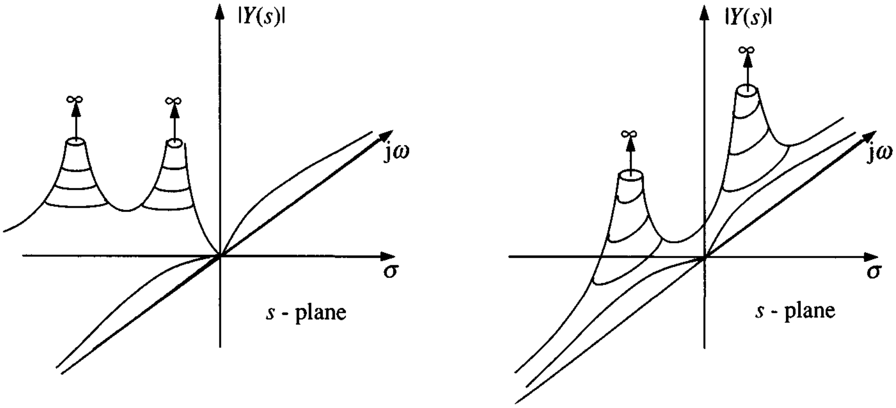
$$=H_0\frac{(N_1N_2\ldots)}{(D_1D_2\ldots)}\underline{/(\psi_1+\psi_2\ldots)-(\theta_1+\theta_2\ldots)} \qquad (6.151)$$

As an example consider the transfer function

$$H(s)=\frac{(s-1+\mathrm{j}1)(s-1-\mathrm{j}1)}{(s+3)(s+2+\mathrm{j}4)(s+2-\mathrm{j}4)}$$

which has the pole-zero diagram shown in fig. 6.44(*a*). Suppose s takes the particular value $(0+\mathrm{j}3)$, that is, the point s lies on the $\mathrm{j}\omega$ axis as shown in fig. 6.44(*b*). In polar form the function is

Fig. 6.42. Representation of the modulus of the network function $Y(s)$ in the s-plane. (*a*) Corresponding to fig. 6.40(*a*); (*b*) corresponding to fig. 6.40(*c*).



(*a*) Overdamped (*a*) Underdamped

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$$H(s)=\frac{N_1N_2}{D_1D_2D_3}\underline{/(\psi_1+\psi_2)-(\theta_1+\theta_2+\theta_3)}$$

Converting each of the factors of $H(s)$ to polar form we obtain:

$$\begin{aligned}
(s-1+\mathrm{j}1)&=-1+\mathrm{j}4=N_1\underline{/\psi_1}=4.12\underline{/104^\circ}\\
(s-1-\mathrm{j}1)&=-1+\mathrm{j}2=N_2\underline{/\psi_2}=2.24\underline{/111.6^\circ}\\
(s+3)&=3+\mathrm{j}3=D_1\underline{/\theta_1}=4.24\underline{/45^\circ}\\
(s+2+\mathrm{j}4)&=2+\mathrm{j}7=D_2\underline{/\theta_2}=7.28\underline{/74^\circ}\\
(s+2-\mathrm{j}4)&=2-\mathrm{j}1=D_3\underline{/\theta_3}=2.24\underline{/-26.7^\circ}
\end{aligned}$$

(Note that θ_3 is negative.)

Hence,

$$H(s)=\frac{4.12\times 2.24}{4.24\times 7.28\times 2.24}\underline{/(104+116.6)-(45+74-26.7)}$$

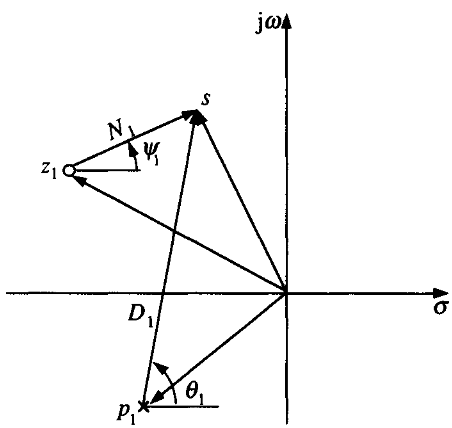
$$=0.133\underline{/128.3^\circ}$$

The above complex arithmetic may also be accomplished graphically using the construction shown in fig. 6.44(*b*).

This approach is particularly useful if one wishes to examine the steady-state behaviour of a network function as a function of frequency ω. In this case s lies on the jω axis, as in the above example, and the vectors from the various poles and zeros change in length and angle as the frequency is varied.

From (6.146), with $s=\mathrm{j}\omega$, the steady-state behaviour of the network function is given by

Fig. 6.43. Polar representation of factors in the network function.



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$$H(\mathrm{j}\omega)=H_0\frac{(\mathrm{j}\omega-z_1)(\mathrm{j}\omega-z_2)\ldots}{(\mathrm{j}\omega-p_1)(\mathrm{j}\omega-p_2)\ldots}$$
$$=M(\omega)\underline{/\phi(\omega)} \qquad (6.152)$$

where

$$M(\omega)=|H(\mathrm{j}\omega)|=\frac{(N_1N_2\ldots)}{(D_1D_2\ldots)}$$

and

$$\phi(\omega)=\arg H(\mathrm{j}\omega)=(\psi_1+\psi_2\ldots)-(\theta_1+\theta_2\ldots)$$

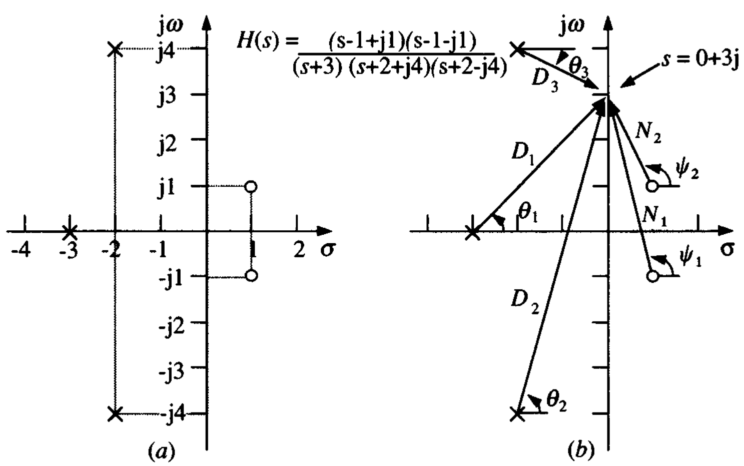
$M(\omega)$ is called the *amplitude response* function and $\phi(\omega)$ the *phase response* function. Evaluation of $M(\omega)$ and $\phi(\omega)$ at a few spot frequencies can quickly give one an idea of how the network behaves as a function of frequency. By shifting, adding or deleting poles and zeros in the diagram, the effect of changes to the circuit can be assessed; the circuit designer is thus able to produce the circuit characteristics required.

To illustrate these ideas we examine the admittance function of the general series branch relating to the underdamped case (equation (6.150) and fig. 6.40(*c*)).

$$Y(s)=\frac{1}{L}\,\frac{s}{(s+\alpha+\mathrm{j}\omega_n)(s+\alpha-\mathrm{j}\omega_n)}$$

In polar form with $s=\mathrm{j}\omega$ this becomes

Fig. 6.44. (*a*) Pole-zero diagram for the function $H(s)$. (*b*) Vectors drawn to the point $s=0+\mathrm{j}3$.



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$$Y(\mathrm{j}\omega)=\frac{1}{L}\frac{N_1}{D_1D_2}\underline{/\psi_1-(\theta_1+\theta_2)}$$

So,

$$M(\omega)=|Y(\mathrm{j}\omega)|=\frac{1}{L}\frac{N_1}{D_1D_2}$$

and

$$\phi(\omega)=\arg Y(\mathrm{j}\omega)=\psi_1-(\theta_1+\theta_2)$$

The vectors for the admittance function are shown for three different frequencies in fig. 6.45.

Consider first variation of the amplitude response function $M(\omega)$. For $\omega=0$: $N_1=0$ (because the zero is at the origin), therefore $M(\omega)=0$.

For $\omega=\omega_n$: $N_1=\omega_n$, $D_1=[\alpha^2+(2\omega_n)^2]^{\frac{1}{2}}$, $D_2=\alpha$ therefore $M(\omega)=\omega_n/L\alpha(\alpha^2+4\omega_n^2)^{\frac{1}{2}}$.

For $\omega\to\infty$: N_1, D_1, $D_2\to\infty$, but $N_1/D_1D_2\to 0$, therefore $M(\omega)\to 0$.

We conclude that with increasing frequency $M(\omega)$ rises from zero, to a maximum value then falls asymptotically to zero. (The shape of the $M(\omega)$ curve is seen in fig. 6.42 as the line profile of $|Y(s)|$ along the jω-axis.)

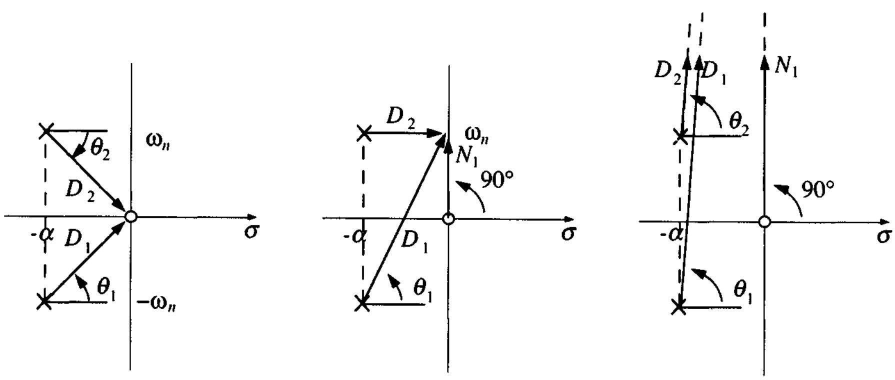
Now consider variation of the phase response function $\phi(\omega)$. For all ω, $\psi_1=90°$ because the zero is at the origin.

For $\omega=0:\theta_1=-\theta_2$, therefore $\phi(\omega)=90°$

For $\omega=\omega_n:\theta_2=0$, therefore $\phi(\omega)=90°-\theta_1$

For $\omega\to\infty:\theta_1=\theta_2\to 90°$, therefore $\phi(\omega)\to -90°$

Fig. 6.45. Amplitude–phase diagrams for the admittance function of the general series branch at three different frequencies.



(*a*) $\omega=0$ (*b*) $\omega=\omega_n$ (*c*) $\omega\to\infty$

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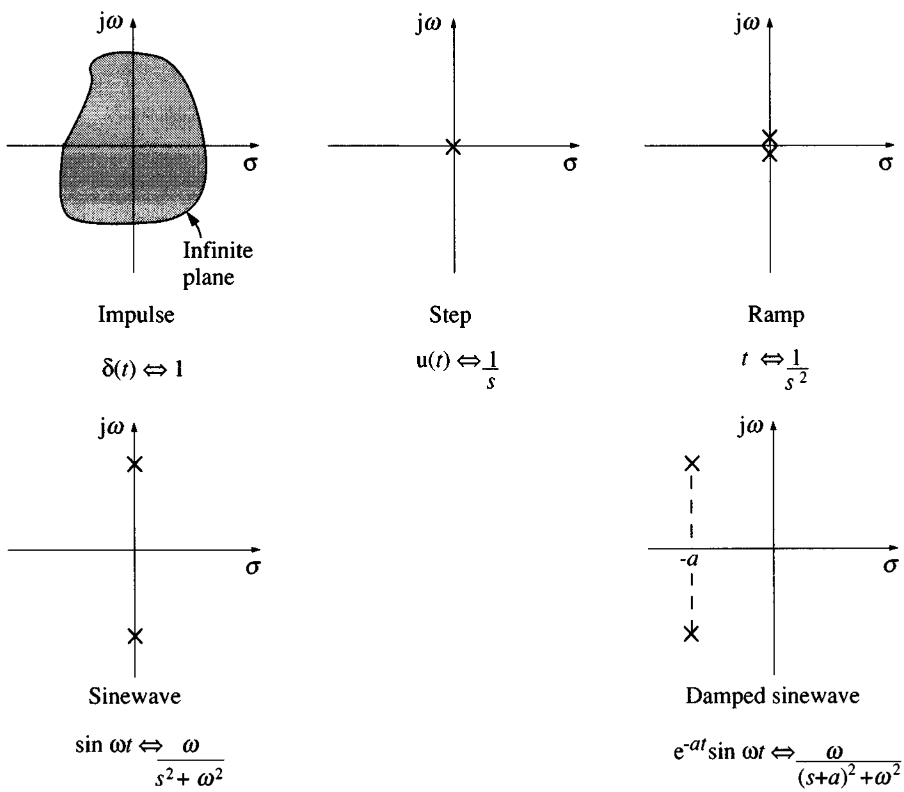
From the pole zero diagrams shown in fig. 6.45 we can also immediately see the effect of varying one of the circuit parameters. Reducing $\alpha(=R/2L)$, say, will shift the poles nearer to the jω-axis and increase $M(\omega)$ for values of ω near to ω_n, (that is, near to resonance) because D_2 varies rapidly with α. On the other hand, reducing α will have little effect at very small or very large ω, because vectors D_1 and D_2 then vary slowly with α.

In our study so far of pole-zero methods we have considered only network functions. Excitation functions may also be represented in the s-plane. Pole-zero diagrams for five common excitation functions are shown in fig. 6.46. The impulse has unit value everywhere in the s-plane; the step has a single pole at the origin, while the ramp has a double pole. The sine wave is represented by complex conjugate poles on the jω-axis; multiplying the sine wave by a damping term e^{-at}, shifts the poles into the left-hand half-plane. The latter two pole zero diagrams and their related transforms illustrate the ideas underlying the concept of a complex frequency. In the

Fig. 6.46. Pole-zero diagrams for five common excitation functions.



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steady-state theory developed in section 3.3 a sine wave of constant amplitude was represented by the complex exponential:

$$V_{\mathrm{m}}\sin(\omega t+\theta)=\mathrm{Im}\,V_{\mathrm{m}}\mathrm{e}^{\mathrm{j}(\omega t+\theta)}=\mathrm{Im}\,V_{\mathrm{m}}\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega t}$$

The quantity $V_{\mathrm{m}}\mathrm{e}^{\mathrm{j}\theta}=\mathbf{V}_{\mathrm{m}}$ was termed a phasor.

For a damped sinusoid we have

$$\mathrm{e}^{-at}V_{\mathrm{m}}\sin(\omega t+\theta)=\mathrm{Im}\,V_{\mathrm{m}}\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{-at}\mathrm{e}^{\mathrm{j}\omega t}=\mathrm{Im}\,\mathbf{V}_{\mathrm{m}}\mathrm{e}^{(-a+\mathrm{j}\omega)t}$$

The frequency variable is now $(-a+\mathrm{j}\omega)$ instead of $\mathrm{j}\omega$. We define this new variable by $s=-a+\mathrm{j}\omega$.

In general $s=\sigma+\mathrm{j}\omega$ where σ is negative for an exponentially decreasing sinusoid and positive for an exponentially increasing sinusoid.

To complete our study of pole-zero methods let us derive diagrams for the response functions obtained by applying some of the excitation functions illustrated in fig. 6.46 to the general branch. In each case we find the response $I(s)$ assuming that the branch is overdamped, that is

$$I(s)=\frac{1}{L}\,\frac{s}{(s+m)(s+n)}\,V(s)$$

We also find the form of the response in the time domain.

Step function $V_{\mathrm{m}}u(t)$

$$I(s)=\frac{1}{L}\,\frac{s}{(s+m)(s+n)}\cdot\frac{V_{\mathrm{m}}}{s}=\frac{V_{\mathrm{m}}}{L}\cdot\frac{1}{(s+m)(s+n)}$$

In this case the pole associated with the excitation function cancels the zero at the origin (fig. 6.47(a)). The response function is of the form

$$I(s)=\frac{A_1}{s+m}+\frac{A_2}{s+n}$$

$$i(t)=A_1\mathrm{e}^{-mt}+A_2\mathrm{e}^{-nt}$$

Ramp function $V_{\mathrm{m}}tu(t)$

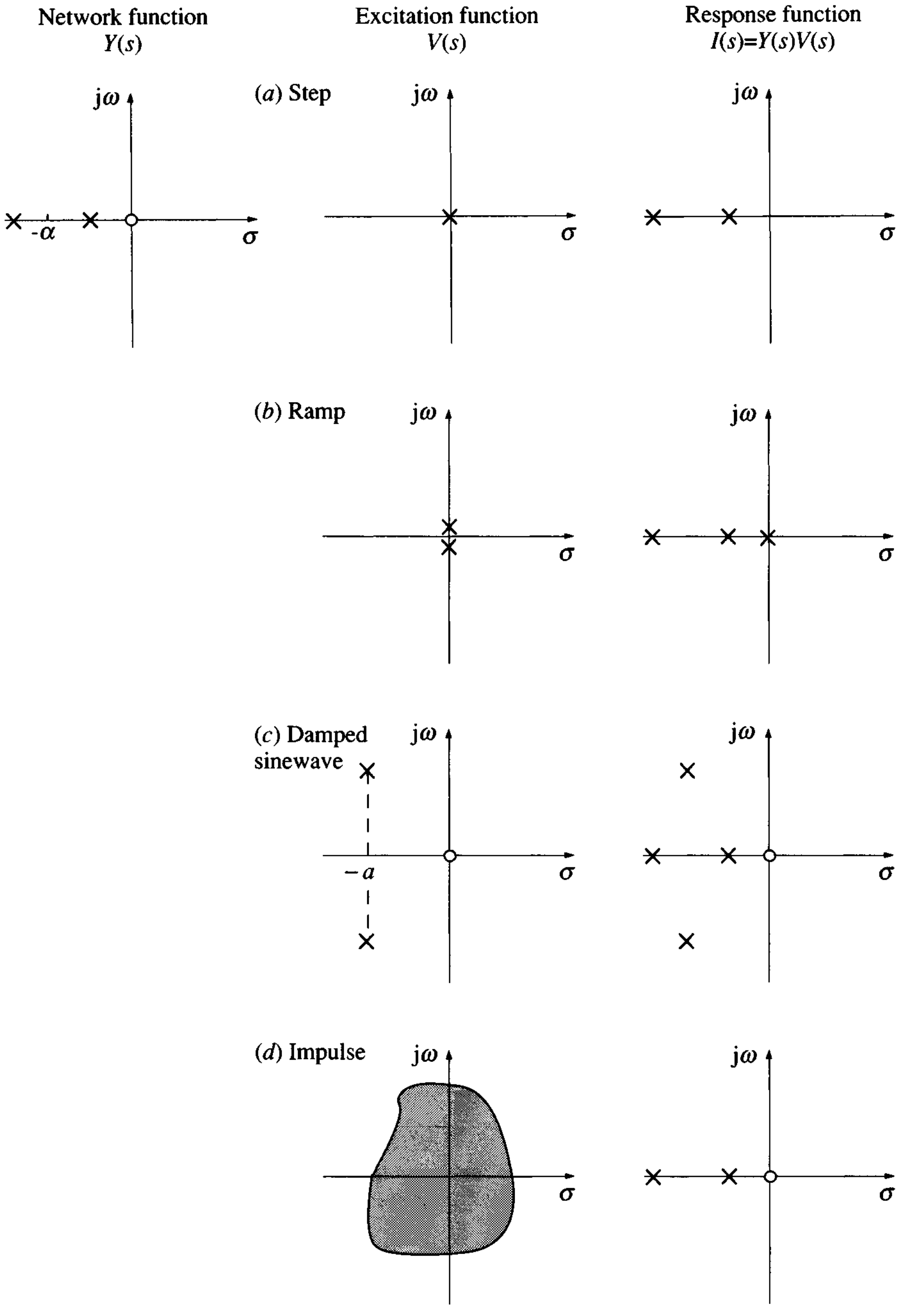
$$I(s)=\frac{1}{L}\,\frac{s}{(s+m)(s+n)}\cdot\frac{V_{\mathrm{m}}}{s^2}=\frac{V_{\mathrm{m}}}{L}\cdot\frac{1}{s(s+m)(s+n)}$$

Here the excitation function creates a pole at the origin (fig. 6.47(b))

$$I(s)=\frac{A_1}{(s+m)}+\frac{A_2}{(s+n)}+\frac{A_3}{s}$$

$$i(t)=A_1\mathrm{e}^{-mt}+A_2\mathrm{e}^{-nt}+A_3$$

Fig. 6.47. Pole-zero diagrams for the admittance function of the general series branch showing the response $I(s)$ for various excitation functions $V(s)$.



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We see that the pole at the origin in the s-plane corresponds to a d.c. component in the time domain.

Damped sine wave $V_m e^{-at}\sin\omega t$

$$I(s)=\frac{1}{L}\frac{s}{(s+m)(s+n)}\cdot\frac{V_m\omega}{(s+a)^2+\omega^2}$$

The excitation function creates additional complex poles (fig. 6.46(c)), and the response is of the form

$$I(s)=\frac{A_1}{(s+m)}+\frac{A_2}{s+n}+\frac{A_3}{(s+a+j\omega)}+\frac{A_4}{(s+a-j\omega)}$$

In the time domain

$$i(t)=A_1e^{-mt}+A_2e^{-nt}+Be^{-at}\sin(\omega t+\theta)$$

The last term in this expression is, of course, the forced response.

Impulse function $\delta(t)$

Since the transform of the impulse function is unity, multiplying the network function by the excitation function in this case leaves the network function unchanged, that is

$$I(s)=\frac{1}{L}\frac{s}{(s+m)(s+n)}=Y(s)$$

The pole-zero diagram of the response function is identical to that of the network function (fig. 6.47(d)) and response function is of the form

$$I(s)=\frac{A_1}{(s+m)}+\frac{A_2}{(s+n)}$$

In the time domain

$$i(t)=A_1e^{-mt}+A_2e^{-nt}$$

which is the impulse response of the series branch.

6.11 Worked example

The circuit of a fourth-order low-pass Butterworth filter is shown in fig. 6.48. Derive the transfer function $H(s)=V_2(s)/V_1(s)$ for this filter and show that its poles are equi-spaced on a semicircle of unit radius in the left-hand half plane of the pole-zero diagram. Determine graphically the amplitude response function for frequencies $\omega=0$; $\omega=0.5$; $\omega=1$, and $\omega=1.5$ rad/s.

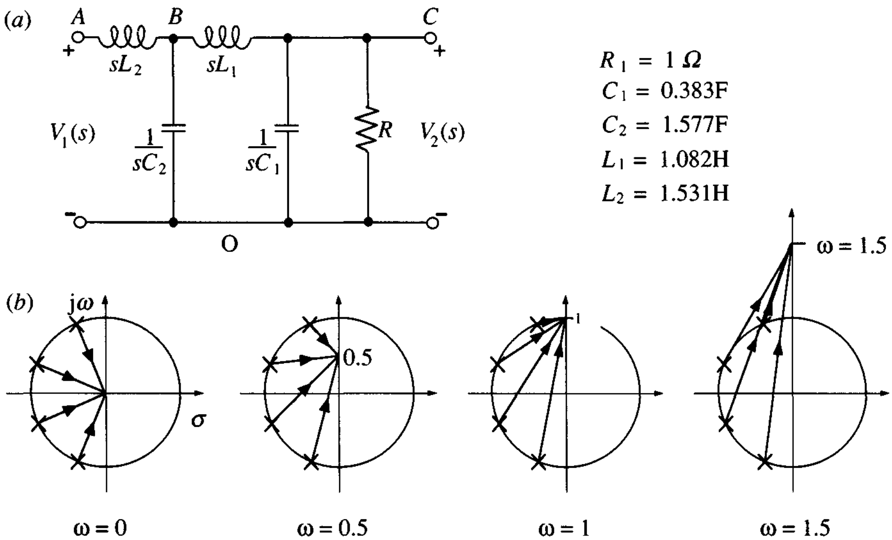
Solution. To obtain the transfer function the ladder method described in section 2.15.3 is used. Assume $V_2 = 1$ volt, then, with the nodes lettered as shown in the diagram:

$$I_{BC} = sC_1 + \frac{1}{R}$$

$$\begin{aligned}
V_{BO} &= I_{BC}sL_1 + 1 \\
I_{BO} &= sC_2 V_{BO} \\
I_{AB} &= I_{BO} + I_{BC} = sC_2 V_{BO} + I_{BC} \\
V_{AO} &= I_{AB}sL_2 + V_{BO} = (sC_2 V_{BO} + I_{BC})sL_2 + V_{BO} \\
&= s^2 C_2 L_2 V_{BO} + I_{BC}sL_2 + V_{BO} \\
&= V_{BO}(s^2 C_2 L_2 + 1) + sL_2 I_{BC} \\
&= (I_{BC}sL_1 + 1)(s^2 C_2 L_2 + 1) + sL_2 I_{BC} \\
&= \left(sC_1 + \frac{1}{R}\right)(s^3 L_1 L_2 C_2 + sL_1 + sL_2) + s^2 C_2 L_2 + 1 \\
&= s^4 C_1 C_2 L_1 L_2 + s^3 C_2 L_1 L_2 / R + s^2 (C_1 L_1 + C_1 L_2 + C_2 L_2) \\
&\quad + s(L_1 + L_2)/R + 1
\end{aligned}$$

Now, in the above calculation, V_2 is assumed to be 1 volt therefore, the ratio

Fig. 6.48. Diagrams for worked example (section 6.11). (*a*) Circuit for Butterworth filter (element values are normalized to an impedance level of 1 Ω and a frequency of $\omega = 1$ rad/s). (*b*) Pole-zero diagrams with amplitude vectors drawn for four selected frequencies.



V_1/V_2 must be equal to V_{AO}. The transfer function is therefore $H(s) = V_2(s)/V_1(s) = 1/V_{AO}$. Putting in numerical values we obtain

$$H(s) = \frac{1}{s^4 + 2.613s^3 + 3.414s^2 + 2.613s + 1}$$

Using program C4 in Appendix C it is found that the roots (poles) of the denominator polynomial in the above expression are:

$$s_1, s_2 = -0.38268 \pm j0.92388$$
$$s_3, s_4 = -0.92388 \pm j0.38268$$

In polar form these become:

$$1.0\underline{/112.5};\ 1.0\underline{/-112.5};\ 1.0\underline{/157.5};\ 1.0\underline{/-157.5}$$

and in order of increasing positive angle we have

$$1.0\underline{/112.5};\ 1.0\underline{/157.5};\ 1.0\underline{/202.5};\ 1.0\underline{/247.5}$$

Thus, the poles lie on a circle of unit radius in the left-hand half plane with an angular spacing between them of 45°.

The amplitude response function is given by (equation (6.152))

$$M(\omega) = \frac{1}{D_1 D_2 D_3 D_4}$$

where $D_1 \ldots D_4$ are the lengths of the vectors extending from each pole to a particular frequency on the jω-axis. Fig. 6.48(b) shows vectors drawn for the four selected frequencies. By direct measurement (from a diagram with a scale of unit frequency = 100 mm) we find:

$$\text{at } \omega = 0 \quad M(0) = \frac{1}{(1.0)(1.0)(1.0)(1.0)} = 1$$

$$\text{at } \omega = 0.5 \quad M(0.5) = \frac{1}{(0.58)(0.94)(1.28)(1.47)} = 0.98$$

$$\text{at } \omega = 1 \quad M(1) = \frac{1}{(0.39)(1.11)(1.67)(1.97)} = 0.70$$

$$\text{at } \omega = 1.5 \quad M(1.5) = \frac{1}{(0.69)(1.45)(2.11)(2.46)} = 0.19$$

It is seen that the amplitude response falls sharply at frequencies above $\omega = 1$, the normalized cut-off frequency. At cut-off frequency the amplitude response is, theoretically, $1/\sqrt{2} = 0.707$, that is, -3 dB referred to the response at $\omega = 0$.

6.12 Pulse and repeated driving functions

The techniques that have been described so far for determining the transient and steady-state conditions in a circuit allow us to deal with only a limited range of driving functions such as the step and sinusoidal functions. However, these techniques may be readily extended to cover single and repeated pulse waveforms of various shapes. Pulse wavetrains of simple shape, such as a repeated series of rectangular pulses, applied to a first order circuit, can be dealt with adequately by elementary methods. For more complicated waveshapes and higher order circuits, the formal methods of the Laplace transform are often to be preferred.

6.12.1 Pulse response of first order circuits

Consider a rectangular pulse of amplitude V and duration a applied to the RC circuit shown in fig. 6.49(a). For the duration of the pulse, $0 \leq t \leq a$, the circuit response is the same as that for an applied step function, which is from the theory presented in section 6.6.2,

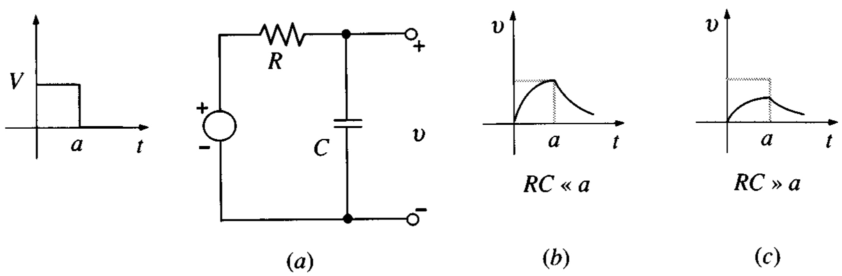
$$v(t) = V(1 - e^{-t/RC}) \qquad 0 \leq t \leq a \tag{6.153}$$

If the time constant RC is very short compared to the duration of the pulse, v will rise to a value substantially equal to the pulse amplitude V (the steady state value) and the output waveform will appear as in fig. 6.49(b). If the time constant is large compared with a, then v will not reach its maximum possible value before the end of the pulse (fig. 6.49(c)).

At the instant $t = a$ the input voltage drops to zero and the capacitance discharges through the resistance via the ideal source supplying the input pulse. The output voltage then decays from some initial value $v(a) = V(1 - e^{-a/RC})$ according to:

$$v(t) = v(a)e^{-(t-a)/RC} \qquad t \geq a \tag{6.154}$$

Fig. 6.49. Response of an RC circuit to an applied rectangular pulse of amplitude V and duration a for two different time constants RC.



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Expressions (6.153) and (6.154) can be used to find the response of the RC circuit to a train of rectangular pulses of the type shown in fig. 6.50(a). In this pulse train each pulse is of duration a, the interval between pulses is of duration b. (The ratio a/b is called the *mark–space ratio*.) We see that the pulse train is periodic with period $T=a+b$.

Fig. 6.50(b) shows the response for the condition $RC \ll a$: in this case the output voltage rises to its full (steady state) value V during each pulse and decays substantially to zero between pulses. For $RC \gg a$, we can encounter the condition shown in fig. 6.50(c) in which the output voltage has time to rise to only a small fraction of its maximum possible value during a pulse, and fails to decay to zero during the intervals between pulses. We now show that, under these circumstances, the output can build up so that it eventually reaches some steady-state value which is less than the pulse amplitude V.

Referring to fig. 6.50(c), let $v(t_1)=v_1$, $v(t_2)=v_2$ etc., then at the end of the first pulse we obtain, by (6.153),

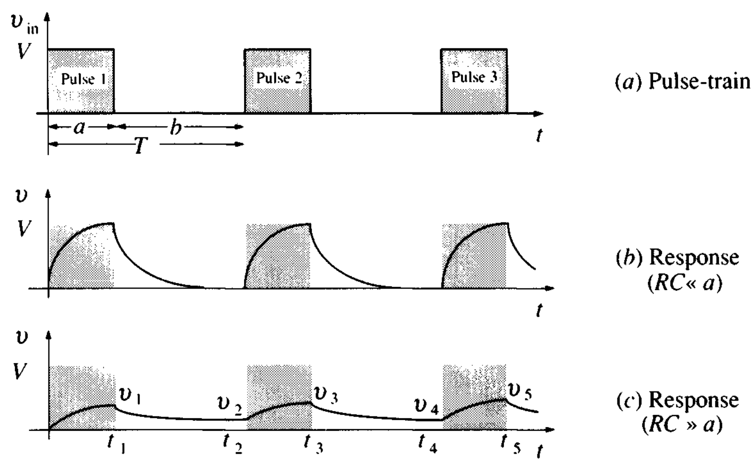
$$v_1 = V(1-e^{-t_1/RC}) \tag{6.155}$$

and at the end of the first interval, by (6.154),

$$v_2 = v_1 e^{-(t_2-t_1)/RC} \tag{6.156}$$

During the second pulse, the output will rise from an initial value v_2 to some value v_3, which may be determined in the usual way by considering the

Fig. 6.50. Response of an RC circuit to a train of pulses of mark-space ratio a/b and period $T=a+b$.



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response as a sum of transient and steady-state terms. The steady-state term is obviously V, so that the output during the second pulse will be given by

$$v = Ae^{-(t-t_2)/RC} + V \qquad t_2 \le t \le t_3$$

where A is a constant which is determined from the initial condition: $v = v_2$ at $t = t_2$. This gives $A = v_2 - V$, hence,

$$\begin{aligned} v &= (v_2 - V)e^{-(t-t_2)/RC} + V \\ &= V[1 - e^{-(t-t_2)/RC}] + v_2 e^{-(t-t_2)/RC} \qquad t_2 \le t \le t_3 \end{aligned}$$

Putting $t = t_3$ in this expression gives

$$v_3 = V[1 - e^{-(t_3-t_2/RC}] + v_2 e^{-(t_3-t_2)/RC}$$

But from (6.156)

$$v_2 = v_1 e^{-(t_2-t_1)/RC}$$

therefore

$$\begin{aligned} v_3 &= V[1 - e^{-(t_3-t_2)/RC}] + v_1 e^{-(t_3-t_1)/RC} \\ &= V[1 - e^{-a/RC}] + v_1 e^{-T/RC} \end{aligned} \qquad (6.157)$$

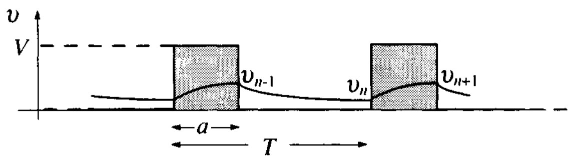
where a is the pulse duration and T is the period.

Combining (6.155) and (6.157) gives

$$v_3 = v_1 + v_1 e^{-T/RC}$$

which shows that v_3 exceeds v_1 by an amount $v_1 e^{-T/RC}$. By a similar process we can show that $v_5 > v_3$, $v_7 > v_5$ etc. Thus, the output voltage builds up until an equilibrium condition is reached at which the voltage exhibits a cyclic variation about some mean level. This condition is the steady-state response to the pulse train;* which is to be distinguished from the steady-state response (either zero or V) associated with each individual transition in the pulse train.

Fig. 6.51. Steady-state response of RC circuit to a pulse train: $v_{n-1} = v_{n+1}$.



* Some authors prefer to use the term 'quasi-steady state' for this type of equilibrium condition.

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We now determine the steady-state response of the RC circuit to an input pulse train under the condition $RC \gg a$ (fig. 6.50(c)). Referring to fig. 6.51, let v_n be the output voltage at the beginning of a particular pulse in the train, and let v_{n+1} and v_{n-1} be the voltages respectively at the end of this pulse and the end of the preceding pulse. Then, by analogy with 6.156 and 6.157,

$$v_n = v_{n-1}\mathrm{e}^{-(T-a)/RC}$$

and

$$v_{n+1} = V(1-\mathrm{e}^{-a/RC}) + v_{n-1}\mathrm{e}^{-T/RC}$$

But in the equilibrium (steady-state) condition $v_{n-1} = v_{n+1}$, therefore

$$v_{n-1} = \frac{V(1-\mathrm{e}^{-a/RC})}{(1-\mathrm{e}^{-T/RC})}$$

and

$$v_n = \frac{V(1-\mathrm{e}^{-a/RC})}{(1-\mathrm{e}^{-T/RC})} \cdot \mathrm{e}^{-(T-a)/RC} \tag{6.158}$$

The mean level is

$$\frac{v_{n-1}+v_n}{2} = \frac{V}{2}\frac{(1-\mathrm{e}^{-a/RC})}{(1-\mathrm{e}^{-T/RC})}[1+\mathrm{e}^{-(T-a)/RC}]$$

If, for example, the mark–space ratio is unity ($T = 2a$), then, the mean level is

$$\frac{V}{2}\frac{(1-\mathrm{e}^{-a/RC})}{(1-\mathrm{e}^{-2a/RC})}(1+\mathrm{e}^{-a/RC}) = \frac{V}{2}$$

So, for pulses with a mark–space ratio of unity, the output voltage of the RC circuit settles down to a mean level of just half the pulse amplitude. For other mark–space ratios, the steady-state output will be greater or less than $V/2$ depending on the particular value of a/b.

Another example for which elementary theory is well suited is provided by the circuit of fig. 6.52(a). Here an ideal current source drives a sawtooth waveform of current (fig. 6.52(b)) through an RL circuit. Such a circuit might represent, in idealized form, the deflection system of a cathode-ray tube display. L and R are the inductance and resistance of the deflection coils; the slow, positive-going ramp of current in the coils generates the linear sweep of the cathode-ray tube spot, while the fast, negative-going ramp corresponds to the flyback period. The form of the voltage generated across the deflection coils is of some interest to the circuit designer. In the following derivation of the source voltage waveforms, the current is assumed, for simplicity, to have unit amplitude.

The circuit equation is

$$v = L\frac{di}{dt} + Ri$$

In the interval $0 \leq t \leq t_1$, the current rises linearly from zero at a rate of k_1 amperes per second, hence,

$$v = L\frac{d}{dt}(k_1 t) + Rk_1 t = Lk_1 + Rk_1 t \qquad 0 \leq t \leq t_1$$

In this equation it must be remembered that the current is the driving function, and v is the unknown variable. Setting the RHS of the above equation to zero to find the transient response gives $v = 0$; from which we infer that the transient term is zero. The above equation, which is of straight-line form with intercept Lk_1 and slope Rk_1, represents, therefore, the total response to the driving current.

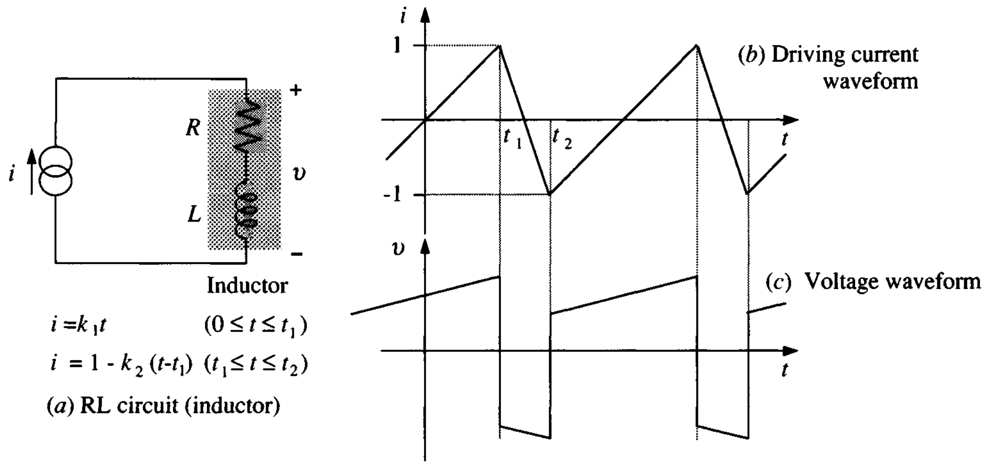
At the instant $t = t_1$, the current starts to fall and it is then expressed by $i = 1 - k_2(t - t_1)$, where k_2 is the new slope of the function. In the interval $t_1 \leq t \leq t_2$, we have, therefore,

$$v = L\frac{d}{dt}[1 - k_2(t - t_1)] + R[1 - k_2(t - t_1)]$$

or

$$v = -Lk_2 + R[1 - k_2(t - t_1)] \qquad t_1 \leq t \leq t_2$$

Fig. 6.52. Voltage response of an inductor driven by a sawtooth current waveform.



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Again this is an equation of straight-line form. At $t=t_2$ the current commences to rise with slope k_1 and the cycle repeats. The complete cyclic voltage waveform is shown in fig. 6.52(*c*). The reader might find it instructive to consider how this waveform would be modified if either L or R were reduced to zero.

6.12.2 Delayed singularity functions: transforms of recurrent waveforms

In this section we establish methods of describing repeated pulses or recurrent waveforms by means of delayed singularity functions; this in turn will allow us to obtain the Laplace transforms of such waveforms. For the sake of simplicity it will be convenient to assume waveforms of unit amplitude; the results obtained apply equally to waveforms of amplitude V provided that all derived waveforms are multiplied by a factor V.

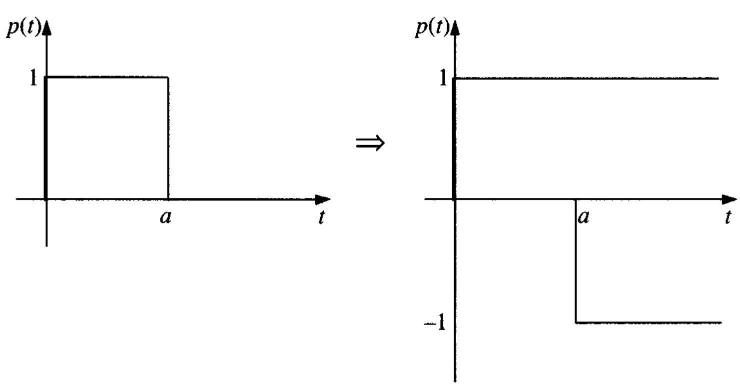
It will be recalled from our discussion in section 6.8.5 that the delayed unit step $u(t-a)$ provides a convenient way of representing the start of a function at a time $t=a$. This device can be used also to describe pulses and repeated functions. For example, the pulse shown in fig. 6.53 can be resolved into two step functions: a positive step starting at $t=0$ and a negative step starting at $t=a$. The expression for the pulse function is then written:

$$p(t)=u(t)-u(t-a)$$

The transform of the first term is $1/s$ while that for the second term is, according to the shift theorem, e^{-as}/s (transform pair No. 22). Thus, in the s-domain:

$$P(s)=\frac{1}{s}-\frac{e^{-as}}{s}=\frac{1}{s}(1-e^{-as}) \tag{6.159}$$

Fig. 6.53. Resolution of a pulse into two step functions.



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A pulse train of the type depicted in fig. 6.54(*a*) may be similarly resolved into a series of step functions:

$$f(t)=u(t)-u(t-a)+u(t-2a)-u(t-3a)+\dots \tag{6.160}$$

with transform

$$F(s)=\frac{1}{s}-\frac{e^{-as}}{s}+\frac{e^{-2as}}{s}-\frac{e^{-3as}}{s}+\dots$$
$$=\frac{1}{s}(1-e^{-as}+e^{-2as}-e^{-3as}+\dots)$$

The geometric progression within brackets may be expressed in closed form using the following identity:

$$\frac{1}{1+x}=1-x+x^2-x^3+\dots \qquad x<1 \tag{6.161}$$

Thus, the transform of the pulse train of fig. 6.54(*a*) is

$$F(s)=\frac{1}{s(1+e^{-as})} \tag{6.162}$$

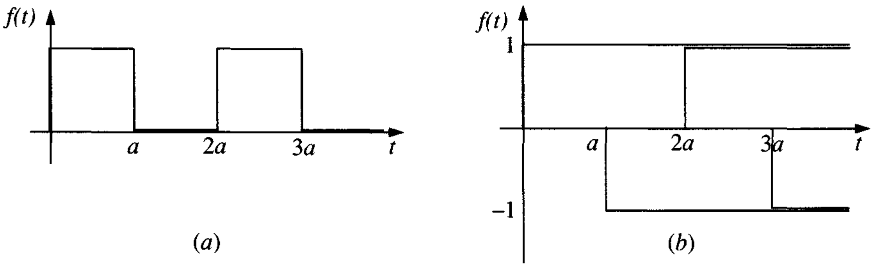
This technique for finding transforms of repeated functions may be expressed in more general terms as follows. Let $p(t)$ be a pulse of finite duration, as shown in fig. 6.55(*a*), and let $P(s)$ be its transform. Then, for the repeated function $f(t)$ of fig. 6.55(*b*) with period T, we have

$$f(t)=p(t)u(t)+p(t-T)u(t-T)+p(t-2T)u(t-2T)+\dots$$

and its transform is

$$F(s)=P(s)+P(s)e^{-Ts}+P(s)e^{-2Ts}+\dots$$
$$=P(s)(1+e^{-Ts}+e^{-2Ts}+\dots)$$

Fig. 6.54. Resolution of a pulse train (square wave) into an infinite series of step functions.



Now we use the identity

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots \qquad x<1 \tag{6.163}$$

to obtain

$$F(s)=\frac{P(s)}{1-e^{-Ts}} \tag{6.164}$$

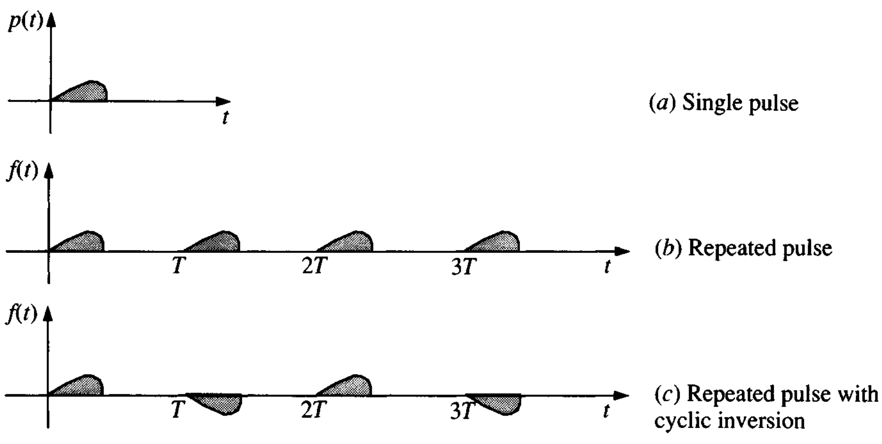
In a similar fashion it may be shown that the transform of the repeated function shown in fig. 6.55(*c*) is given by

$$F(s)=\frac{P(s)}{1+e^{-Ts}} \tag{6.165}$$

The expressions (6.164) and (6.165), which are sometimes referred to as the *periodicity theorem*, enable one to find the transform of a sequence of repeated pulses or a recurrent waveform given the transform of the individual pulse or waveform of which it is composed. For instance, the transform of the single rectangular pulse of duration a shown in fig. 6.53 is $\dfrac{1}{s}(1-e^{-as})$ from (6.159). If this pulse is repeated with period T, then, by the periodicity theorem (6.164), the transform of the rectangular pulse train will be

$$F(s)=\frac{P(s)}{1-e^{-Ts}}=\frac{1}{s}\,\frac{(1-e^{-as})}{(1-e^{-Ts})} \tag{6.166}$$

Fig. 6.55. A single pulse function $p(t)$ and its repeated versions $f(t)$.



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If the pulse train takes the form of a square wave, that is, if $T=2a$ as in fig. 6.54, then

$$F(s)=\frac{1}{s}\frac{(1-e^{-as})}{(1-e^{-2as})}=\frac{1}{s(1+e^{-as})} \tag{6.167}$$

which agrees with our previous result (6.162).

As a further example, consider the triangular pulse shown in fig. 6.56. We may resolve this function into three components: a positive ramp starting at $t=0$, a negative ramp starting at $t=1$ and a negative step also starting at $t=1$. For $t \geq 1$ the sum of the three components is zero, thus the function is cut off at $t=1$. Now the first component is the unit ramp $\rho(t)$, and the second component is the (negative) unit delayed ramp $\rho(t-a)$ (see section 6.8.4). So, we may express the function as:

$$P(t)=\rho(t)-\rho(t-1)-u(t-1)$$

Referring to the table of transforms it is found that

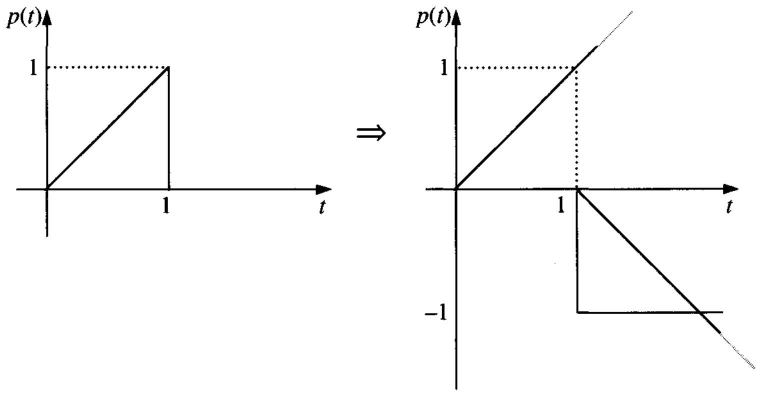
$$P(s)=\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}$$

and the transform of the repeated pulse (fig. 6.57(*a*)) is by (6.164)

$$F(s)=\frac{P(s)}{1-e^{-Ts}}=\frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}$$

By the use of (6.165) we may also immediately write the transform of the waveform shown in fig. 6.57(*b*) as

Fig. 6.56. Triangular pulse resolved into two ramp functions and one step function.



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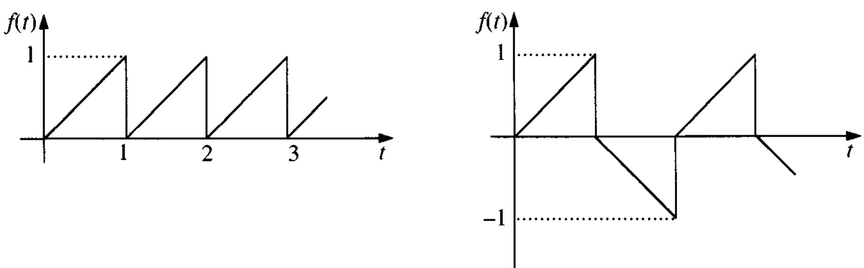
$$F(s)=\frac{P(s)}{1+e^{-Ts}}=\frac{1-e^{-s}-se^{-s}}{s^2(1+e^{-s})}$$

When applying the periodicity theorem, it is essential to ensure that the function $p(t)$ is correctly defined in terms of its resolved components. These must sum to zero for all t greater than the specified pulse length. This point is illustrated in our final example: the half-wave rectified sine wave shown in fig. 6.58(*a*).

Over the interval $0 \le t \le T/2$, $p(t)$ is specified by the function $u(t)\sin\omega t$ in which $u(t)$ indicates that $p(t)$ is initiated at $t=0$. However, as it stands, this function states that $p(t)$ is continuous for all $t>0$ whereas we require it to be zero for $t>T/2$. This is accomplished by the addition of an identical sinusoidal function, shifted so that it starts at $t=T/2$, which has the effect of cancelling the original function for $t>T/2$ (fig. 6.58(*b*)). Thus,

$$p(t)=u(t)\sin\omega t+u\left(t-\frac{T}{2}\right)\sin\omega\left(t-\frac{T}{2}\right)$$

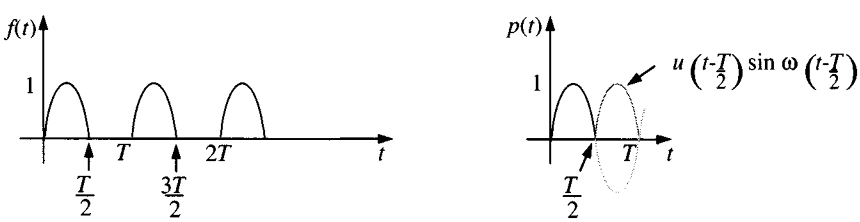
Fig. 6.57. Repeated versions of the triangular waveform of fig. 6.55.



(*a*) Repeated triangular waveform (sawtooth)

(*b*) Repeated triangular waveform with cyclic inversion

Fig. 6.58. Formation of a single half-sine-wave pulse by the combination of a continuous function with a shifted function.



(*a*) Half-wave rectified sine wave

(*b*) Single half-sine-wave pulse

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The transform of this function is

$$P(s)=\frac{\omega}{s^2+\omega^2}+\frac{\omega e^{-Ts/2}}{s^2+\omega^2}=\frac{\omega}{s^2+\omega^2}(1+e^{-Ts/2})$$

Now we can apply the periodicity theorem (6.164) to find the transform of the repeated version of this function. Note that the function is repeated with period T, consequently the exponent in the exponential of (6.164) is Ts, not $(T/2)s$. Therefore, the transform of the half-wave rectified sine wave of unit amplitude is:

$$F(s)=\frac{\omega}{(s^2+\omega^2)}\cdot\frac{(1+e^{-Ts/2})}{(1-e^{-Ts})}=\frac{\omega}{(s^2+\omega^2)(1-e^{-Ts/2})} \tag{6.168}$$

6.12.3 Response by the Laplace transform

In order to illustrate the transform method, and for comparison with a more elementary approach adopted in section 6.12.1, we return to a consideration of the *RC* circuit excited by a single pulse of amplitude V and duration a (fig. 6.59(a)). The transform of a rectangular pulse has already been derived (equation (6.159)) so that the circuit in the s-domain becomes as shown in fig. 6.59(b). The voltage across the capacitance is, therefore,

$$V(s)=\frac{1/sC}{R+1/sC}\cdot\frac{V}{s}(1-e^{-as})=V\left[\frac{1/RC}{s(s+1/RC)}\right](1-e^{-as}) \tag{6.169}$$

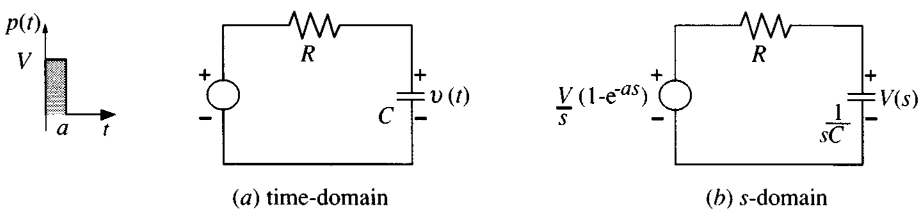
Because of the factor $(1-e^{-as})$, which is not a polynomial, it is not possible to expand the whole of this function in the usual way using partial fractions, but the term within square brackets can be expanded to give:

$$\frac{1/RC}{s(s+1/RC)}=\frac{1}{s}-\frac{1}{s+1/RC}$$

Thus,

$$V(s)=V\left[\frac{1}{s}-\frac{1}{s+1/RC}\right](1-e^{-as})$$

Fig. 6.59. *RC* circuit excited by a single rectangular pulse of duration a.



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which may be written

$$V(s)=V\left\{\left[\frac{1}{s}-\frac{1}{s+1/RC}\right]-\left[\frac{1}{s}-\frac{1}{s+1/RC}\right]e^{-as}\right\}$$

Upon inversion the first term of this expression yields $(1-e^{-t/RC})$, while the second term yields precisely the same function but delayed in time according to the shift factor e^{-as}. Hence, in the time domain we have:

$$v=V[(1-e^{-t/RC})u(t)-(1-e^{-(t-a)/RC})u(t-a)] \qquad (6.170)$$

Comparing this expression with our previous results we see that, for the duration of the pulse, only the first term of (6.170) is operative (because $u(t-a)=0$ for $t<a$)) and we obtain:

$$v=V(1-e^{-t/RC}) \qquad 0\leq t\leq a$$

which agrees with (6.153).

After the termination of the pulse, *both* terms of (6.170) are operative and

$$\begin{aligned} v&=V[(1-e^{-t/RC})-(1-e^{-(t-a)/RC})] \qquad t\geq a \\ &=V(e^{-(t-a)/RC}-e^{-t/RC}) \end{aligned}$$

which is in accord with (6.153) and (6.154).

Factors of the form $(1-e^{-as})$, like that appearing in (6.169), commonly occur in problems involving pulse driving functions. Such factors play no part in determining coefficients in the expansion of the s-domain circuit equation. Their effect is simply to establish two identical functions in the time domain: one starting at $t=0$, the other at $t=a$.

Continuing with our comparison of elementary and transform methods, we now use the latter to determine the response of the RC circuit to a train of rectangular pulses of unit amplitude (fig. 6.60(a)). The transform of the pulse train is given by (6.166), hence in the s-domain, we have the circuit of fig. 6.60(b). The circuit response is

$$V(s)=\frac{1/sC}{(R+1/sC)}\cdot\frac{1}{s}\,\frac{(1-e^{-as})}{(1-e^{-Ts})}=\left[\frac{1/RC}{s(s+1/RC)}\right]\frac{(1-e^{-as})}{(1-e^{-Ts})}$$

Expanding the term in square brackets gives

$$\begin{aligned} V(s)&=\left[\frac{1}{s}-\frac{1}{s+1/RC}\right]\frac{(1-e^{-as})}{(1-e^{-Ts})} \\ &=\left[\frac{1}{s}-\frac{1}{s+1/RC)}\right]\frac{1}{(1-e^{-Ts})}-\left[\frac{1}{s}-\frac{1}{s+1/RC}\right]\frac{e^{-as}}{(1-e^{-Ts})} \end{aligned}$$

Let $Q(s)=\left[\frac{1}{s}-\frac{1}{s+1/RC}\right]$,

then

$$V(s)=\frac{Q(s)}{1-e^{-Ts}}-\frac{Q(s)e^{-as}}{1-e^{-Ts}} \tag{6.171}$$

Now the inverse of $Q(s)$ is $(1-e^{-t/RC})$ therefore, by using the periodicity theorem (6.164) in reverse, we find that the inverse of the first term in (6.171) is

$$f_1(t)=[1-e^{-t/RC}]u(t)+[1-e^{-(t-T)/RC}]u(t-T) \\ +[1-e^{-(t-2T)/RC}]u(t-2T)+\ldots$$

To find the inverse of the second term in (6.171) we expand the factor $1/(1-e^{-Ts})$ using the identity (6.163):

$$Q(s)\frac{e^{-as}}{1-e^{-Ts}}=Q(s)e^{-as}(1+e^{-Ts}+e^{-2Ts}+e^{-3Ts}+\ldots)$$
$$=Q(s)[e^{-as}+e^{-(a+T)s}+e^{-(a+2T)s}+\ldots]$$

The series of shift factors within brackets leads to delayed functions in the time domain of the form

$$u(t-a)+u(t-a-T)+u(t-a-2T)+\ldots$$

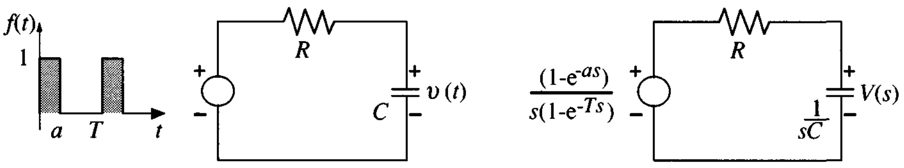
Thus, the inverse of the second term in (6.171) is

$$f_2(t)=[1-e^{-(t-a)/RC}]u(t-a)+[1-e^{-(t-a-T)/RC}]u(t-a-T) \\ +[1-e^{-(t-a-2T)/RC}]u(t-a-2T)+\ldots$$

The complete result is

$$v(t)=f_1(t)-f_2(t)$$

Fig. 6.60. *RC* circuit excited by a train of rectangular pulses of duration *a* and period *T*.



(*a*) time-domain (*b*) *s*-domain

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$$\begin{aligned}&=\{[1-e^{-t/RC}]u(t)+[1-e^{-(t-T)/RC}]u(t-T)\\&\quad+[1-e^{-(t-2T)/RC}]u(t-2T)+\ldots\}\\&\quad-\{[1-e^{-(t-a)/RC}]u(t-a)+[1-e^{-(t-a-T)/RC}]u(t-a-T)\\&\quad+[1-e^{-(t-a-2T)/RC}]u(t-a-2T)+\ldots\}\end{aligned}\qquad(6.172)$$

Let us use this expression to find the output of the circuit at the end of the second pulse, that is, at the instant $t=a+T$. In this case, all terms except the first two in each of the series contained in (6.172) vanish because of the operation of the delayed step functions. This leaves

$$\begin{aligned}v(t)&=\{[1-e^{-(a+T)/RC}]+[1-e^{-a/RC}]\}\\&\quad-\{[1-e^{-T/RC}][1-e^{0}]\}\end{aligned}$$

which upon re-arrangement becomes

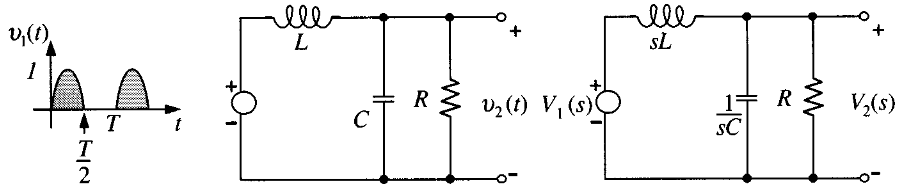
$$v(t)=(1-e^{-a/RC})+(1-e^{-a/RC})e^{-T/RC}$$

This result is consistent with the expressions (6.155) and (6.157) obtained previously in section 6.12.1. A comparison of the above method of solution with that of section 6.12.1 will reveal that, although the amount of algebraic manipulation required in the transform method is formidable, it offers a more systematic approach. This is an advantage that becomes more marked the greater the complexity of the network and the excitation. The following example will serve to illustrate this point.

Fig. 6.61 shows a circuit used in a.c.–d.c. power supplies. The rectifier in this type of power supply produces a series of half-sine-wave pulses, as shown in the figure, and the function of the circuit is to smooth out these pulsations so that the supply delivers a constant d.c. output voltage. The steady-state analysis of this type of circuit is best accomplished by the use of Fourier series methods (see chapter 7), but sometimes the circuit designer may wish to determine the behaviour of the circuit under the transient conditions prevailing when the supply is first switched on. The transform method provides such information.

Using the admittance divider principle, we have for the circuit transfer function

Fig. 6.61. *RLC* circuit driven by a half-wave rectified sine-wave.



(*a*) time-domain (*b*) *s*-domain

$$H(s)=\frac{V_2(s)}{V_1(s)}=\frac{1/sL}{1/sL+sC+1/R}=\frac{1}{LC(s^2+s/RC+1/LC)}$$

This type of circuit is always heavily damped so that the natural response will be non-oscillatory, and the two roots of the denominator polynomial in the above expression will be real. Let these roots be $-\alpha$ and $-\beta$ where $\alpha+\beta=1/RC$ and $\alpha\beta=1/LC$, then

$$H(s)=\frac{\alpha\beta}{(s+\alpha)(s+\beta)}$$

Now the transform of the half-wave rectified sine-wave has already been determined (equation 6.168), hence, the output voltage $V_2(s)$ is

$$V_2(s)=H(s)V_1(s)=\frac{\alpha\beta}{(s+\alpha)(s+\beta)}\cdot\frac{\omega}{(s^2+\omega^2)(1-e^{-Ts/2})}$$

$$=\alpha\beta\omega\left[\frac{1}{(s+\alpha)(s+\beta)(s^2+\omega^2)}\right]\frac{1}{(1-e^{-Ts/2})}$$

Partial fraction expansion of the term within square brackets gives

$$V_2(s)=\alpha\beta\omega\left[\frac{A_1}{s+\alpha}+\frac{A_2}{s+\beta}+\frac{A_3}{s+j\omega}+\frac{A_3^*}{s-j\omega}\right]\frac{1}{(1-e^{-Ts/2})}$$

where A_1, A_2, A_3, which are functions of α, β, ω, are found by the methods described in section 6.9.3. The inverse of the terms within square brackets is of the form

$$f(t)=A_1e^{-\alpha t}+A_2e^{-\beta t}+2A_3\cos(\omega t-\phi)$$

The function $1/(1-e^{-Ts/2})$ may be expanded using the identity (6.163)

$$\frac{1}{1-e^{-Ts/2}}=1+e^{-Ts/2}+e^{-Ts}+e^{-3Ts/2}+\ldots$$

which, as in our previous examples, is interpreted as a series of delayed functions in the time domain. Thus, the output voltage is finally given by

$$v_2(t)=\alpha\beta\omega\left[f(t)u(t)+f\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)+f(t-T)u(t-T)+\ldots\right]$$

6.13 Worked example

The TV video pulse train shown in fig. 6.62(a) is applied to the circuit of fig. 6.62(b). When the output voltage $v_2(t)$ of the RC circuit reaches a threshold voltage V_T, the field time base is triggered.

(a) Assuming that the circuit has attained steady-state conditions during the line pulse sequence, calculate the voltage $v_2(t)$ at the instant $t=t_A$.
(b) Calculate the voltage $v_2(t)$ at the end of the equalizing pulse sequence, i.e. at the instant $t=t_B$.
(c) If the field time base triggers at the instant $t=t_C=(t_B+48)\,\mu$s (middle of second field pulse period), estimate the threshold voltage V_T.

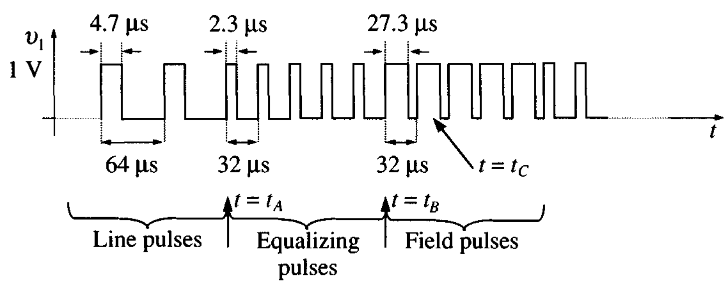
Solution

(a) Under steady-state conditions the output of the RC circuit at the end of a pulse period, for an input pulse train of duration a and period T, (fig. 6.51) is given by

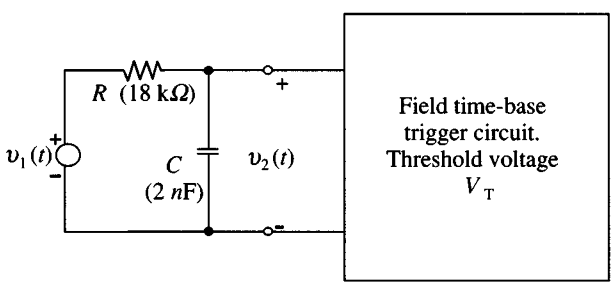
$$v_n=\frac{V(1-e^{-a/RC})}{(1-e^{-T/RC})}\cdot e^{-(T-a)/RC} \tag{6.158}$$

For the line pulse sequence: $a=4.7\,\mu$s, $T=64\,\mu$s; the circuit time constant $RC=(18\times 10^3)\times(2\times 10^{-9})=36\,\mu$s. Hence, by 6.158, with $V=1$ V, we have

Fig. 6.62. Diagrams for worked example (section 6.13).



(*a*) TV video waveform: pulse train for field synchronisation.



(*b*)

$$v_2 = \frac{(1 - e^{-4.7/36})}{(1 - e^{-64/36})} \cdot e^{-(64-4.7)/36} = 0.028\ \mathrm{V}$$

(b) The output voltage of an RC circuit subject to an input pulse train of unit magnitude is, during the transient period, expressed by equation 6.172. A simplification of the algebra and a resulting saving of labour is achieved if in this equation and those derived from it, units of time are normalized with respect to the circuit time constant RC. In this case 6.172 may be written:

$$\begin{aligned} v_2 = &\{[1 - e^{-t}]u(t) + [1 - e^{-(t-T)}]u(t-T) \\ &+ [1 - e^{-(t-2T)}]u(t-2T) + \ldots\} \\ &- \{[1 - e^{-(t-a)}]u(t-a) + [1 - e^{-(t-a-T)}]u(t-a-T) \\ &+ [1 - e^{-(t-a-2T)}]u(t-a-2T) + \ldots\} \end{aligned}$$

The equalizing pulse sequence in the waveform shown in fig. 6.62(a) contains five complete pulse periods, therefore, taking the first five terms in each of the series in the above expression, defining a new time origin at $t = t_A$, and putting $t = 5T$ we obtain

$$\begin{aligned} v_2 = &\{[1 - e^{-5T}] + [1 - e^{-4T}] + \ldots + [1 - e^{-T}]\} \\ &- \{[1 - e^{-(5T-a)}] + [1 - e^{-(4T-a)}] + \ldots + [1 - e^{-(T-a)}]\} \end{aligned}$$

Combining corresponding terms from each of the series in this expression gives

$$\begin{aligned} v_2 &= e^{-5T}(e^a - 1) + \ldots + e^{-T}(e^a - 1) \\ &= (e^a - 1)(e^{-5T} + e^{-4T} + e^{-3T} + e^{-2T} + e^{-T}) \end{aligned}$$

Substituting normalized values: $a = 2.3/36$, $T = 32/36$, yields

$$v_2 = 0.066(0.01 + 0.03 + 0.07 + 0.17 + 0.41) = 0.045\ \mathrm{V}$$

The above calculation assumes that the capacitance is uncharged, and therefore v_2 is zero, at the commencement of the equalizing pulse sequence, whereas it was found in section (a) above that a small but finite voltage (0.028 V) remains at the end of the line pulse sequence. This voltage will decay during the equalizing pulse sequence according to

$$v_2' = 0.028e^{-5T} = 3.29 \times 10^{-4}\ \mathrm{V}$$

By superposition, the net voltage at the end of the equalizing pulse sequence will be the sum

$$v_2 + v_2' = 0.045 + 0.0003$$

It is seen from the above calculation that the initial voltage results in only a very small contribution to the voltage on C at the end of the equalizing pulse sequence.*

(c) In the following calculation for estimating the threshold voltage V_T we neglect the small voltage on C at the beginning of the field pulse sequence. (Its effect could, as in part (b), be calculated by superposition.) Since the field time-base triggers in the middle of the second field pulse period, we have by equation 6.172 (expressed in normalized units of time)

$$V_T=[1-e^{-t}]+[1-e^{-(t-T)}]-[1-e^{-(t-a)}]$$

Substituting normalized values ($a=27.3/36$; $T=32/36$; $t=48/36$) gives

$$V_T=0.736+0.359-0.437=0.658\ \mathrm{V}$$

This example illustrates the application of the RC circuit as a sync-separator. The output of the circuit remains at a low level during line and equalizing pulse sequences but rises rapidly on inception of the longer field pulses. This action causes the field time base to trigger.

†6.14 Convolution

The methods discussed so far in this chapter for finding the response of a circuit to a given excitation are confined to circumstances in which the excitation is expressible as an analytical function whose Laplace transform can be determined. The convolution method described here is not restricted in this way, and it is applicable also to cases for which the excitation function can be expressed only by a numerical data sequence.

In the convolution method the excitation function in the time-domain is resolved into a sequence of impulses, the response of the circuit to each impulse in the sequence is found and, finally, the responses are superposed to give the overall response function. To establish the basis of the method, we first consider how a function may be expressed in terms of a sequence of impulses.

6.14.1 Representation of a function by an impulse train

Any function $f(t)$ may be represented by a train of step or impulse functions; here we consider only the latter case. Fig. 6.63(a) shows a

* The voltage on C at the end of the line pulse sequence is slightly different for odd and even fields of a standard TV video waveform because even fields end with a half line pulse period whereas odd fields end with a full period (the latter is shown in fig. 6.62(a)). The equalizing pulse sequence is included in the video waveform to allow the voltage on C to decay substantially to the same small value for both odd and even fields of the interlaced picture. Without equalizing pulses the field time base would trigger at slightly different times after the start of the field pulse sequence, resulting in line pairing.

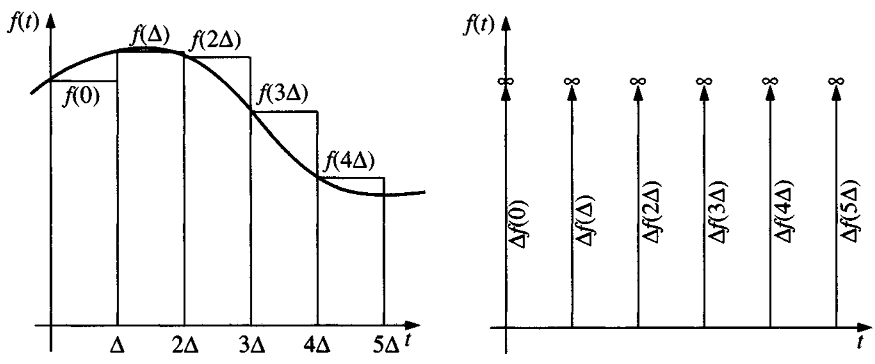
function sampled at intervals of time 0, Δ, 2Δ, 3Δ. ... During the first interval ($0<t<\Delta$) the function may be approximated by its value at the start of the interval, namely $f(0)$, as shown in the figure. During the second interval ($\Delta<t<2\Delta$) the function may be approximated by $f(\Delta)$; likewise during the third and succeeding intervals by $f(2\Delta)$, $f(3\Delta)$ and so on. The function is thus broken up into short pulses of duration Δ with areas $\Delta f(0)$, $\Delta f(\Delta)$, $\Delta f(2\Delta)$ etc. Recalling from our discussion in section 6.8.2 that a short pulse may be represented by an impulse of magnitude equal to its area, we may represent the complete function $f(t)$ by a train of impulses as shown in fig. 6.63(b). (In this figure the impulses are shown located at the start of their corresponding intervals whereas it might appear to be more logical to place them centrally. However, as we shall see later their precise location within the interval is immaterial.) The first impulse in the train will then be $\Delta f(0)\delta(t)$, the second impulse $\Delta f(\Delta)\delta(t-\Delta)$ and so on. Thus the function $f(t)$ may be expressed by the approximation:

$$f(t) \simeq \Delta f(0)\delta(t) + \Delta f(\Delta)\delta(t-\Delta) + \Delta f(2\Delta)\delta(t-2\Delta) + \ldots$$
$$+ \Delta f(n\Delta)\delta(t-n\Delta) + \ldots \ \Delta f(N\Delta)\delta(t-N\Delta)$$
$$\simeq \sum_{n=0}^{N} \Delta f(n\Delta)\delta(t-n\Delta) \tag{6.173}$$

where $N = t_N/\Delta$ and t_N is some time at which the function terminates.

It must be remembered that in this summation all terms are zero except the term in which the argument of the impulse function is zero, that is, when the sampling instant $n\Delta$ is equal to t.

Fig. 6.63. Representation of a continuous function $f(t)$ by an impulse train.



(a) $f(t)$ represented by pulses of duration Δ.

(b) $f(t)$ represented by an impulse train.

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Now $f(t)$ can be approximated to any desired degree of accuracy by letting Δ become as small as necessary and by increasing N correspondingly. In the limit, as $\Delta \rightarrow 0$ and $N \rightarrow \infty$, $n\Delta$ becomes a continuous time τ while Δ becomes the differential $d\tau$. The function is then represented by a continuum of impulses and can be expressed exactly by the integral

$$f(t)=\int_0^{t_N} f(\tau)\delta(t-\tau)d\tau \tag{6.174}$$

This integral may be interpreted in the following way. As the sampling time τ is varied over the range $0 \leq \tau \leq t_N$, the integrand, and therefore the integral, vanishes everywhere except when the argument of the impulse function is zero, that is, when $\tau = t$. At this instant the integral becomes

$$f(t)\int_0^{t_N} \delta(0)\delta\tau = f(t)$$

since, by definition, $\int\delta(0)=1$.

Because we are concerned only with sampling times τ extending up to the value t, the upper limit of the integral (6.174) may be replaced by t, thus

$$f(t)=\int_0^{t} f(\tau)\delta(t-\tau)d\tau \tag{6.175}$$

The property of the impulse function which enables it to be used to express a continuous function in the above manner is called the *sampling* or *sifting* property.

6.14.2 The convolution integral

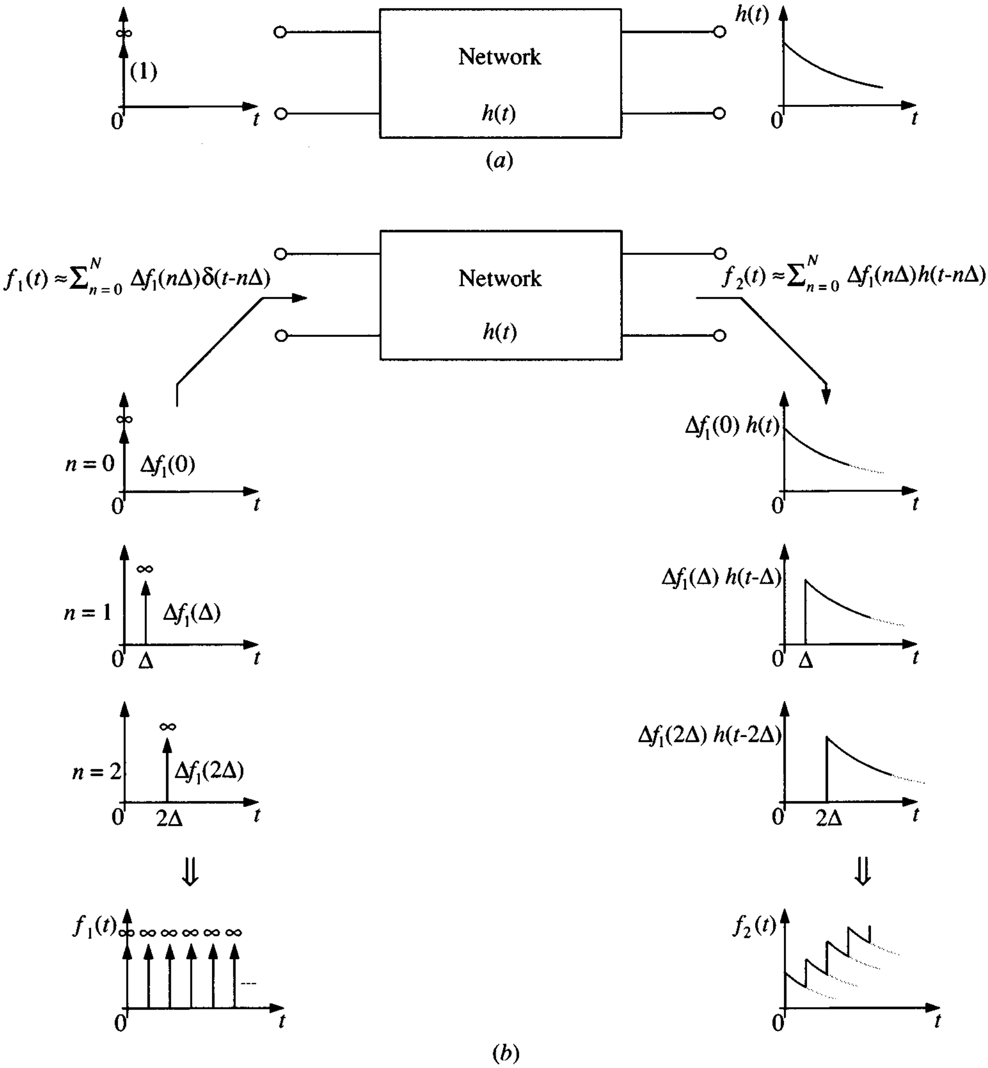
Consider a network having an impulse response $h(t)$ (fig. 6.64(a)). An impulse of unit magnitude applied to the input terminals of this network will produce an output $h(t)$ typically of the form shown in the figure.

Next, consider the same network with an excitation function $f_1(t)$,* described by the approximate expression (6.173), applied to its terminals (fig. 6.64(b)). The first impulse of the sequence will produce a response $\Delta f_1\Delta(0)h(t)$, the second impulse a response $\Delta f_1(\Delta)h(t-\Delta)$, and so on. We assume here that Δ is small compared with the effective time constant of the network (see section 6.8.2). Each succeeding impulse produces a response starting at the instant of the impulse and decaying in a fashion determined

* In previous sections the excitation and response functions in the time-domain have been denoted by $e(t)$ and $r(t)$ respectively. In this section we use $f_1(t)$ and $f_2(t)$, which conforms to the more usual notation found in texts on convolution theory.

by the particular impulse characteristics of the network. The output at any particular instant t will be the resultant of all the impulses and responses occurring prior to, and including, that particular instant. By linearity, the output function will be given by the superposition of all the individual responses, that is, by

Fig. 6.64. (a) Response of a network to unit impulse excitation. (b) Response to an input function $f_1(t)$ approximated by a sequence of impulses.



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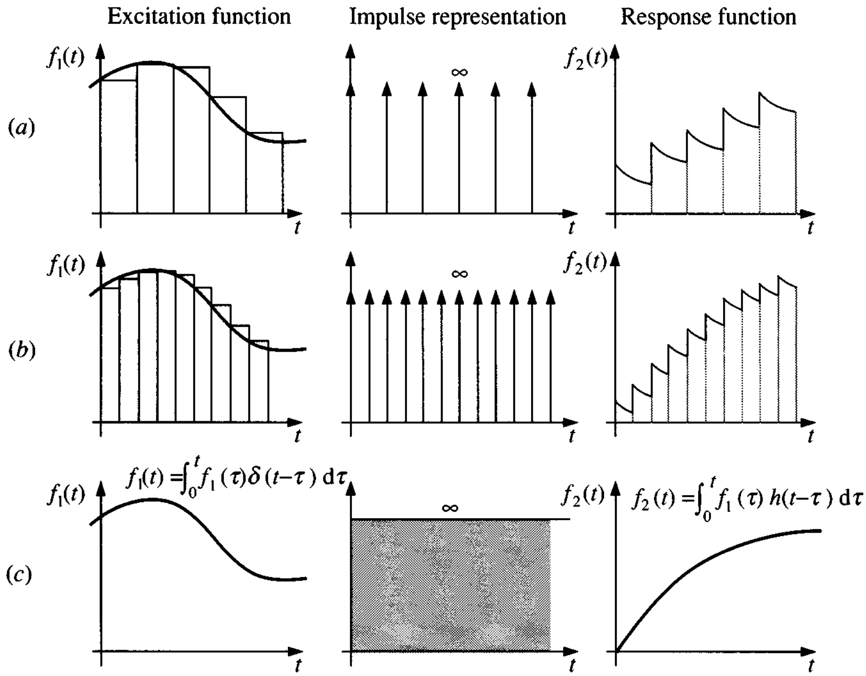
$$f_2(t) \simeq \sum_{n=0}^{N} \Delta f_1(n\Delta)h(t-n\Delta) \tag{6.176}$$

The result of superposing the first four responses is indicated in the lower diagram appertaining to the output sequence.

In fig. 6.64 we have chosen for the sake of clarity a large sampling interval Δ, which has resulted in a sawtooth-like output waveform. But it will be appreciated that by taking shorter and shorter intervals, a progressively more faithful representation of the true output waveform will be obtained. The effect of halving the interval Δ is illustrated in figs. 6.65(*a*) and (*b*) for both input and output waveforms. Notice that halving Δ also halves the magnitudes of the input impulses and therefore their corresponding output responses. The net effect is a substantial smoothing of the output waveform. Carrying this process to the limit, and letting $\Delta \to 0$, the approximation (6.176) becomes an exact integral, the *convolution integral*:

$$f_2(t) = \int_0^t f_1(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{6.177}$$

Fig. 6.65. The effect of sampling interval on response by convolution. (*a*), (*b*) The effect of halving the interval. (*c*) Sampling continuum.



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The limiting process used to derive this expression corresponds exactly to that leading to (6.175) and the symbols τ and t have the same meaning. Indeed, the convolution integral (6.177) follows directly from the integral (6.175) since for a network with an impulse response $h(t)$ the response to $\delta(t-\tau)$ is $h(t-\tau)$. These two integrals provide exact expressions for the excitation and response functions respectively (fig. 6.65(c)).

The convolution integral is often written in its most general form with the range of integration extending from $-\infty$ to $+\infty$. This then allows the range in any particular problem to be set in accordance with the constraints imposed by the conditions of that problem. In this book we are concerned almost entirely with functions that are zero for negative time, we shall, therefore, usually write the convolution integral, as may be convenient, either in the form (6.177) or as

$$f_2(t)=\int_0^{\infty} f_1(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{6.178}$$

Further insight into the convolution integral will be gained from consideration of the graphical interpretation of the integration processes shown in fig. 6.66. The curves in this figure are all plotted with τ as the independent variable while t is a fixed point on the abscissa. The reason for this is that in the evaluation of the convolution integral, t is regarded as a fixed parameter while τ is a variable of integration (a dummy variable) ranging from zero to some upper limit.

The integrand of the convolution integral contains two functions: $f_1(\tau)$ and $h(t-\tau)$. We obtain a graphical plot of the latter in three stages: first, the function $h(\tau)$ is plotted (fig. 6.66(a)), and then this is folded about the vertical axis to form its mirror image $h(-\tau)$ (fig. 6.66(b)). Finally, this function is shifted to the right by an amount t to form the function $h(t-\tau)$ (fig. 6.66(c)). Of course, the form of $h(t-\tau)$ in (c) could be deduced directly without going through the intermediate stages (a) and (b), but we have adopted this procedure to illustrate the folding and shifting process from which the name 'convolution' integral, was historically derived.*

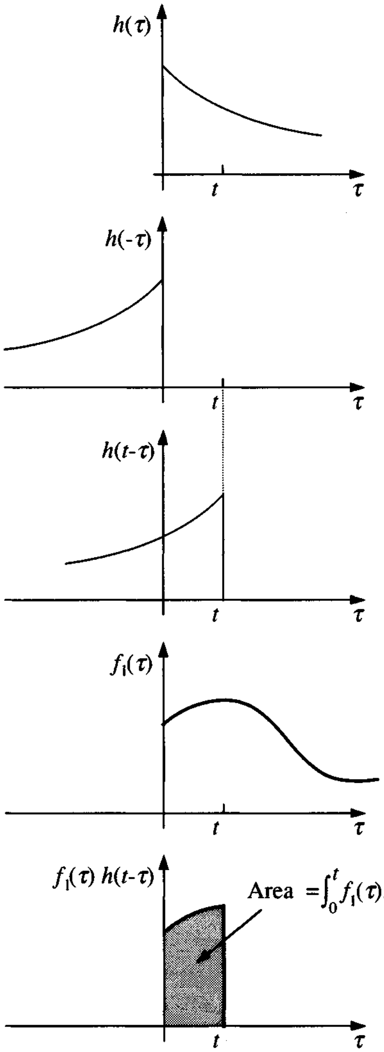
The next step in this graphical integration procedure is to form the product $f_1(\tau)h(t-\tau)$; this is shown in figs. 6.66(d) and (e). Finally, the area under this last curve, between the limits 0 and t, gives the value of the integral, which is equal to the response function $f_2(t)$ at the particular value of t considered. To find $f_2(t)$ at other values of t, the curve shown in fig. 6.66(c) must be shifted along the axis to the appropriate points. We may

* In the German language the name given to this important integral is the 'Faltung' (folding) integral.

then imagine the function $f_2(t)$ being generated by sliding or scanning the $h(t-\tau)$ function along the τ-axis of fig. 6.66(c) whilst performing the subsequent operations illustrated in the following figures (d) and (e). For this reason the $h(t-\tau)$ function is sometimes called the *scanning function.*

We now illustrate the convolution method using as an example the simple RC circuit excited by a pulse of unit amplitude and duration one second. This problem is one which has been dealt with at length in sections

Fig. 6.66. Graphical interpretation of convolution.



(a) Typical impulse response function

(b) Impulse response folded over

(c) Impulse response folded over and shifted

(d) Input function

Area $=\int_0^t f_1(\tau)h(t-\tau)\,d\tau = f_2(t)$

(e) Product of (c) and (d)

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6.12.1 and 6.12.3; the reader should compare the following treatment with the methods described in these sections.

The first step in applying the convolution method is to find the impulse response of the circuit. Usually this is most easily accomplished by first finding the transfer function. For the RC circuit,

$$H(s)=\frac{1/sC}{R+1/sC}=\frac{1}{RC}\,\frac{1}{(s+1/RC)}$$

Thus,

$$h(t)=\mathscr{L}^{-1}H(s)=\frac{1}{RC}\mathrm{e}^{-t/RC}u(t) \tag{6.179}$$

Assuming, for simplicity, that the time constant $RC=1$, then

$$h(t)=\mathrm{e}^{-t}u(t) \tag{6.180}$$

In the above expressions for $h(t)$ the step function $u(t)$ indicates that the response is zero for all negative time.

Now a rectangular pulse of unit amplitude may be described by (see section 6.12.2.)

$$v_1(t)=u(t)-u(t-a) \tag{6.181}$$

hence, using the convolution integral (6.178) we obtain

$$\begin{aligned}v_2(t)&=\int_0^\infty \{[u(\tau)-u(\tau-a)]\mathrm{e}^{-(t-\tau)}u(t-\tau)\}\,\mathrm{d}\tau\\ &=\int_0^\infty u(\tau)u(t-\tau)\mathrm{e}^{-(t-\tau)}\,\mathrm{d}\tau-\int_0^\infty u(\tau-a)u(t-\tau)\mathrm{e}^{-(t-\tau)}\,\mathrm{d}\tau\end{aligned}$$

We have chosen here to write the convolution integral with limits 0 and ∞; this has been done because the limits of integration now need to be determined in accordance with the particular parameters of the problem. In the following argument it will be of help to recall the definition of the unit step function, viz.,

$$\begin{aligned}u(t)&=1 \qquad t\geq 0\\ &=0 \qquad t<0\end{aligned}$$

Turning our attention to the integrand of the first integral above, $u(\tau)$ is zero for $\tau<0$ and unity for $\tau\geq 0$, so the lower limit of integration is zero. The function $u(t-\tau)$ is unity for $\tau\leq t$ and zero for $\tau>t$, so the upper limit of integration is t. Similar considerations apply to the second integral: in this case the function $u(\tau-a)$ is zero for $\tau<a$ and unity for $\tau\geq a$ hence

integration starts at the lower limit a. As before, the $u(t-\tau)$ defines the upper limit at t.

Using these new limits of integration the response becomes

$$v_2(t)=\int_0^t u(\tau)u(t-\tau)e^{-(t-\tau)}\,d\tau-\int_a^t u(\tau-a)u(t-\tau)e^{-(t-\tau)}\,d\tau$$

Now over the range of integration, all of the step functions are unity, so we obtain on integrating:

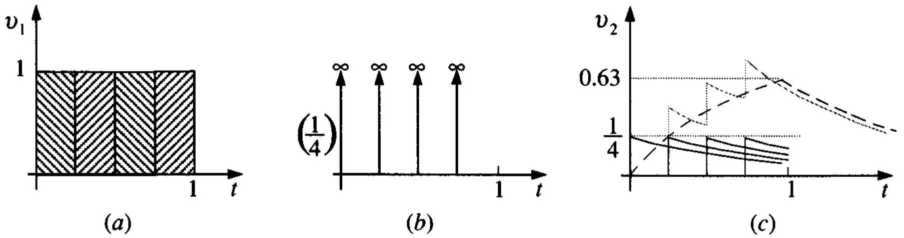
$$v_2(t)=[u(\tau)u(t-\tau)e^{-(t-\tau)}]_{\tau=0}^{t}-[u(\tau-a)u(t-\tau)e^{-(t-\tau)}]_{\tau=a}^{t}$$
$$v_2(t)=u(t)(1-e^{-t})-u(t-a)(1-e^{-(t-a)}) \tag{6.182}$$

(Some care is required when applying limits in the type of expressions encountered above: note particularly that, with $\tau=t$, $u(t-\tau)=u(t-t)=u(0)=1$.)

Equation (6.182) is in agreement with (6.170) obtained previously. Over $0\le t\le a$ only the first term is operative and the output rises exponentially. For $t\ge a$, both terms are operative and the output falls.

It will now be instructive to solve this same problem using a graphical method of convolution based on the approximate equation (6.176). We divide the pulse (assumed to be of 1 second duration) into n equal sub-intervals. Each sub-interval will be of duration $1/n$, and the area under the curve contained within each sub-interval will also be $1/n$. For the sake of clarity in the diagrams we choose a small number of sub-intervals, say $n=4$ (fig. 6.67(a)). The four pulses thus obtained are now replaced by four impulse functions of magnitude $\frac{1}{4}$ (fig. 6.67(b)). The response to each of these impulses will be $(\frac{1}{4})e^{-t}$ as shown in fig. 6.67(c). The total response, obtained by summing the individual responses, is shown by the dotted

Fig. 6.67. Response of the RC circuit excited by a single rectangular pulse: graphical convolution. (a) Pulse divided into four short pulses. (b) Each short pulse replaced by an impulse. (c) Response obtained by convolution (dotted curve) compared with theoretical curve (dashed curve).



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curve. This may be compared with the exact solution indicated by the dashed curve.

The above example is intended to help the reader new to the theory of convolution to gain familiarity with the concepts involved; for this particular problem, however, we should stress that one of the elementary methods of solution discussed in earlier sections would be more appropriate. Moreover, for problems not amenable to an analytical solution, it would be more usual to employ a numerical method for evaluating the convolution integral (see section 6.14.4) rather than the graphical procedure, used in this example.

The convolution integral is encountered frequently in many branches of engineering and physics. In general terms the convolution of two functions, say $\phi_1(t)$ and $\phi_2(t)$, is denoted symbolically by

$$\phi_1 * \phi_2 \equiv \int_{-\infty}^{\infty} \phi_1(\tau)\phi_2(t-\tau)\,\mathrm{d}\tau \tag{6.183}$$

where $*$ is read as '*convolved with*'. We can show very easily that ϕ^1 convolved with ϕ_2 is identical to ϕ_2 convolved with ϕ_1.

Let $z=t-\tau$ then $\mathrm{d}z=-\mathrm{d}\tau$ and (6.183) becomes

$$\phi_1 * \phi_2 = -\int_{\infty}^{-\infty} \phi_1(t-z)\phi_2(z)\,\mathrm{d}z$$

$$= \int_{-\infty}^{\infty} \phi_2(z)\phi_1(t-z)\,\mathrm{d}z = \phi_2 * \phi_1 \tag{6.184}$$

6.14.3 The convolution theorem

In preceding sections of this chapter we have developed a method of finding the response $f_2(t)$ of a circuit to a given excitation $f_1(t)$ using the Laplace transform. The method involves: (1) finding the transform of the excitation, $F_1(s)$; (2) finding the transfer function of the circuit, $H(s)$; and (3) finding the inverse transform of the product $F_1(s) \times H(s)$.

The convolution integral, on the other hand, allows one to find the response directly in the time domain. The two approaches to the problem, and the way in which they are related, are illustrated in fig. 6.68. It will be evident from this diagram that convolution in the time domain corresponds to multiplication in the transform domain; a relationship expressed by the *convolution theorem*:

$$f_2(t) = \mathscr{L}^{-1}[F_1(s)H(s)] = \int_0^{\infty} f_1(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{6.185}$$

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This theorem may be derived directly from the definition of the Laplace transform. Taking the transform of the RHS of (6.185), and calling this $F(s)$, we have

$$F(s)=\mathscr{L}\left[\int_0^\infty f_1(t)h(t-\tau)\,\mathrm{d}\tau\right]=\int_0^\infty\left[\int_0^\infty f_1(\tau)h(t-\tau)\,\mathrm{d}\tau\right]\mathrm{e}^{-st}\,\mathrm{d}t$$

Changing the order of integration:

$$F(s)=\int_0^\infty f_1(\tau)\left[\int_0^\infty h(t-\tau)\mathrm{e}^{-st}\,\mathrm{d}t\right]\mathrm{d}\tau$$

The integral within brackets is recognized as the Laplace transform of the delayed function $h(t-\tau)$: which is, from the shift theorem (6.141), $\mathrm{e}^{-s\tau}H(s)$. Therefore,

$$F(s)=\int_0^\infty f_1(\tau)\mathrm{e}^{-s\tau}H(s)\mathrm{d}\tau=H(s)F_1(s)$$

which proves (6.185).

The commutative property of convolution (6.184) also follows from the convolution theorem since

$$\mathscr{L}^{-1}[\Phi_1(s)\Phi_2(s)]=\phi_1(t)*\phi_2(t)$$

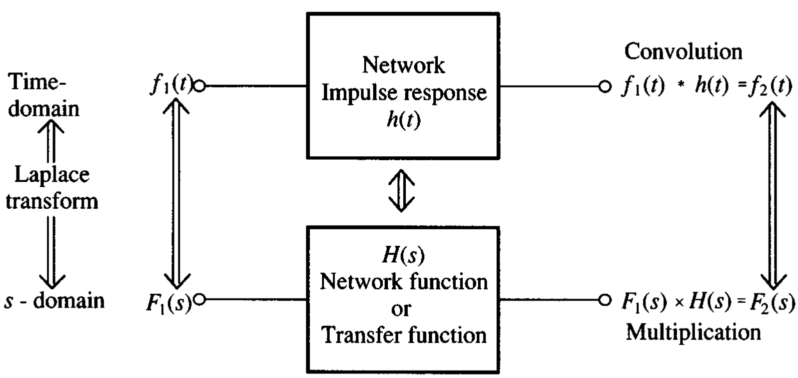
and

$$\mathscr{L}^{-1}[\Phi_2(s)\Phi_1(s)]=\phi_2(t)*\phi_1(t)$$

Clearly, these two expressions are identical.

For the majority of the analytical excitation functions encountered in circuit theory, convolution in the time domain involves a more difficult

Fig. 6.68. Illustrating the convolution theorem.



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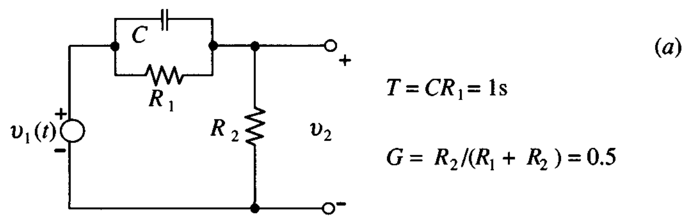
mathematical procedure than multiplication in the transform domain. However, when the excitation is arbitrary and cannot be expressed or modelled by an analytical function, or if the Laplace transforms of the functions involved cannot be readily found, then numerical evaluation of the convolution integral provides a powerful technique for obtaining the network response.

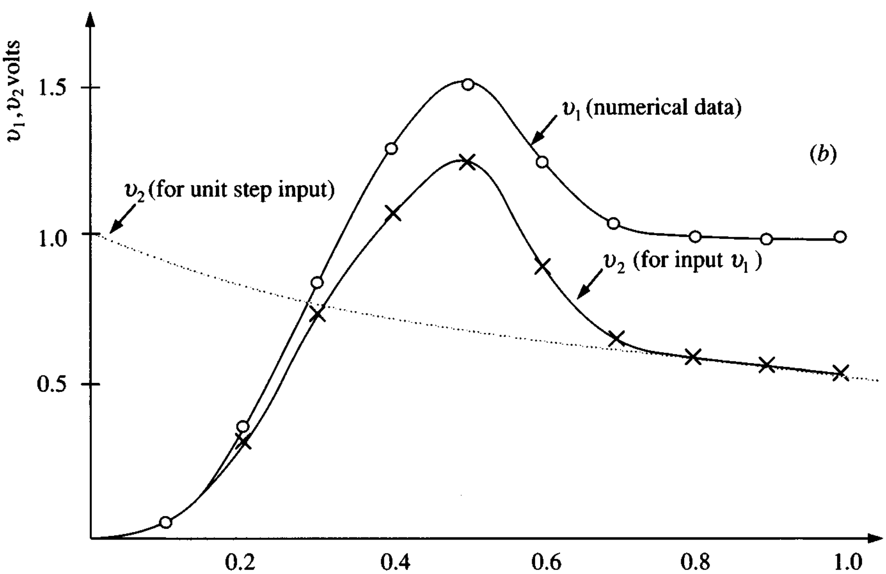
6.14.4 Worked example

The 'phase advance' circuit shown in fig. 6.69 is often used in electronic control system networks. Show that the voltage transfer function for this circuit is

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{s + 1/T}{s + 1/GT}$$

Fig. 6.69. Circuit and waveforms for worked example (section 6.14.4).





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where $T = CR_1$ and $G = R_2/(R_1 + R_2)$. Hence determine the impulse response $h(t)$ if $T = 1$ second and $G = 0.5$.

Find a general expression for the output $v_2(t)$ in terms of the input $v_1(t)$ and $h(t)$. Hence, determine the output if the input is: (*a*) the unit step function; (*b*) the waveform specified by the following numerical data:

t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
v_1 (volt)	0	0.07	0.38	0.83	1.3	1.5	1.23	1.04	1.0	1.0	1.0

Solution

By the method established in section 6.9.6 we obtain

$$H(s) = \frac{sC + 1/R_1}{sC + 1/R_1 + 1/R_2} = \frac{s + 1/CR_1}{s + (1/C)(R_1 + R_2)/R_1R_2} = \frac{s + 1/T}{s + 1/GT}$$

For $T = 1$ second and $G = 0.5$:

$$H(s) = \frac{s+1}{s+2}$$

Expressing $H(s)$ as a partial fraction (see section 6.9.3)

$$H(s) = 1 - \frac{1}{s+2}$$

hence, the impulse response is

$$h(t) = \mathscr{L}^{-1}[H(s)] = [\delta(t) - e^{-2t}]u(t)$$

Assuming that the input voltage is zero for $t < 0$, then the input is described by $v_1(t)u(t)$ and by convolution we obtain

$$\begin{aligned} v_2(t) &= \int_0^\infty v_1(\tau)h(t-\tau)\,d\tau \\ &= \int_0^\infty v_1(\tau)u(\tau)\{[\delta(t-\tau) - e^{-2(t-\tau)}]u(t-\tau)\}\,d\tau \\ &= \int_0^\infty v_1(\tau)u(\tau)\delta(t-\tau)u(t-\tau)d\tau \\ &\qquad - \int_0^\infty v_1(\tau)u(\tau)e^{-2(t-\tau)}u(t-\tau)\,d\tau \end{aligned}$$

In the above integrals the step functions define the range of integration as $0 \le \tau \le t$, and are unity over this range, therefore,

$$v_2(t) = \int_0^t v_1(\tau)\delta(t-\tau)\,d\tau - \int_0^t v_1(\tau)e^{-2(t-\tau)}\,d\tau$$

The first of these integrals is, from (6.175), simply equal to the input function itself, hence a general expression for the output voltage is

$$v_2(t) = v_1(t) - e^{-2t} \int_0^t v_1(\tau) e^{2\tau} d\tau \tag{6.186}$$

(a) *Step function input*

Putting $v_1(t) = u(t)$ in (6.186) gives

$$v_2(t) = u(t) - e^{-2t} \int_0^t u(\tau) e^{2\tau} d\tau$$

$$= u(t) - \frac{e^{-2t}}{2} u(t) [e^{2\tau}]_0^t$$

$$v_2(t) = \tfrac{1}{2}(1 + e^{-2t}) u(t)$$

This expression is shown plotted in fig. 6.69(*b*).

(b) *Numerical input data*

The input waveform, defined by the numerical data, is shown in fig. 6.69(*b*). Numerical integration is used to evaluate (6.186).

τ or t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$v_1(\tau)e^{2\tau}$	0.085	0.57	1.51	2.89	4.07	4.08	4.21	4.95	6.04	7.38
$I = \int_0^t v_1(\tau) e^{2\tau} d\tau$	0.004	0.04	0.14	0.36	0.71	1.12	1.53	1.99	2.54	3.21
$v_2 = v_1 - e^{-2t} I$	0.067	0.36	0.75	1.14	1.24	0.89	0.66	0.60	0.58	0.57

In the above table the integral I has been evaluated using the trapezoidal rule and a pocket calculator. The results are plotted in fig. 6.69(*b*). As to be expected, the output initially follows the input quite closely because of the capacitive coupling between input and output; with increasing time the output diverges from the input and decays towards its d.c. level of 0.5 V.

6.15 Summary

In this chapter a variety of methods have been developed for finding the response of a linear network to various forms of excitation. Broadly, these fall into three categories: time-domain techniques in which the network differential equations are set up and solved; Laplace transform techniques in which the network equations are formulated and solved in the transform domain; and convolution techniques in which the network is characterized by an impulse response and the network response is found by means of a convolution integral.

For any linear network the response $r(t)$ and excitation $e(t)$ are related by a differential equation of the form:

$$a_n \frac{d^n r}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{dr}{dt} + a_0$$
$$= b_m \frac{d^m e}{dt^m} + b_{m-1} \frac{d^{m-1} e}{dt^{m-1}} + \ldots + b_1 \frac{de}{dt} + b_0 \qquad (6.136)$$

The solution of this equation can be written:

$$\text{total response } r(t) = r_n(t) + r_{ss}(t)$$

where $r_n(t)$ is the transient or natural response, and $r_{ss}(t)$ is the steady-state or forced response.

The natural response contains terms of the form $A_i e^{-t/\tau_k}$ which die away with increasing time according to the network time constants τ_k. The constants A_i depend on both the initial energy states of the network and upon the form of the excitation; they can be evaluated only after the form of the complete solution of the network differential equation has been found. For high-order networks, the evaluation of these constants can be troublesome.

The forced response depends only on the excitation; for d.c. or steady sinusoidal excitation it is most easily found by using the standard techniques of d.c. and a.c. network analysis developed in chapters 2 and 3.

For many problems, it is often convenient to find the natural and forced responses separately, and then combine them to find the complete response. This approach is particularly advantageous for first and second order networks and where the excitation consists of step, sinusoidal or other simple functions.

The D-operator method facilitates the process of setting up the network differential equation and, for certain types of excitation, of finding its solution. In this method the network differential equation (6.136) is replaced by an equation of the form:

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) r(t)$$
$$= (b_m D^m + b_{m-1} D^{m-1} + \ldots + b_1 D + b_0) e(t)$$

in which D may be treated as an algebraic quantity.

The Laplace transform provides the most powerful and comprehensive means of analysing the transient and steady-state behaviour of linear networks. By taking the Laplace transform of the general network differential equation (6.136) we obtain

$$(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0)R(s)$$
$$= (b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0)E(s) \quad (6.139)$$

where s is the complex frequency, and $R(s)$ (the response function) and $E(s)$ (the excitation function) are the Laplace transforms of $r(t)$ and $e(t)$ respectively. The ratio of $R(s)$ and $E(s)$, defines a network function:

$$H(s) = \frac{R(s)}{E(s)} \quad (6.134)$$

In the practical application of the method, each element of the network is expressed as a generalized reactance or impedance function in s, which process allows the network function $H(s)$ to be formulated. The excitation function $E(s)$ is conveniently obtained from $e(t)$ using a table of Laplace transform pairs. The response function $R(s)$ is then formed from the product $H(s) \times E(s)$, and finally the inverse of $R(s)$ is found, again using a table of transform pairs, which yields the response $r(t)$ in the time domain.

An important advantage of the Laplace transform method is that information concerning the initial energy states of the network can be incorporated into the excitation function; thus, the necessity of evaluating the arbitrary constants, A_i, as required in the solution of the network differential equation, is obviated. Also, because the complex frequency s may be manipulated in the s-domain as an algebraic identity, the method retains, in this respect, the advantage of the D-operator method. The pole-zero diagram provides an important adjunct to the Laplace transform method. It affords the network designer with a ready means for the appreciation and understanding of network behaviour under both transient and steady-state conditions. No such pictorial device exists for the other methods of network analysis.

In general, the use of the Laplace transform method is advantageous when dealing with networks of order three or higher, and for excitations of complex form. For uncomplicated circuits and excitations of simple form, its use can involve unnecessary algebraic complexity.

The response of a network can be found by the Laplace transform method only if the excitation is expressible as an analytical function. The same applies to the use of the network differential equation. The convolution method, on the other hand, suffers from no such restriction; the network response may be found even in cases where the excitation is described in terms of a numerical data sequence. In the convolution method, $e(t)$ is expressed as a summation of delayed impulse functions; the network is characterized by an impulse function $h(t)$ (the inverse transform of $H(s)$), and the overall response is found by convolving $e(t)$ with $h(t)$ using the convolution integral. We have shown, by means of the convolution theorem, that this procedure corresponds to multiplication of $E(s)$ by $H(s)$ in the transform domain. The convolution method is particularly well suited to the numerical integration procedures available on most calculators and small computers.

6.16 Problems

1. A voltage having a step waveform of amplitude 50 V is applied to a circuit formed by a 1 MΩ resistor in series with a 1 μF capacitor. What is the time constant of the circuit? How long does the capacitor take to charge to: (a) 25 V; (b) 47.5 V?

2. In the circuit of fig. 6.70 the switch has been in position 1 for a long time. At $t=0$ it is thrown to position 2.

(a) What are i_c and v_c at $t=0^+$?

(b) Determine an expression for $i_c(t)$ for $t>0$.

(c) What are i_c and v_c at $t=\infty$?

(d) How much energy is stored in the capacitor at $t=\infty$?

(e) How much energy has been supplied by the battery during the charging process?

(f) Show that the energy of part (e) is twice that stored in the capacitance, regardless of the size of the resistance.

3. For the circuit of fig. 6.71, derive equations for i, i_1 and i_2 valid from the instant $t=0$ when switch S is closed. (C is uncharged initially.) S is reopened

Fig. 6.70. Circuit for problem 2.

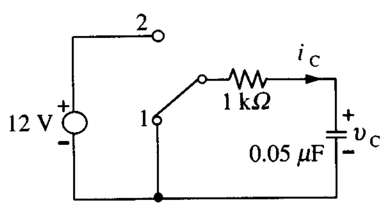
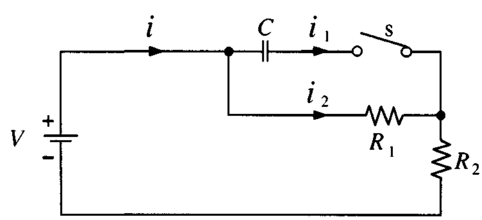


Fig. 6.71. Circuit for problem 3.



after one time constant has elapsed. If $V=1\,\text{V}$, $R_1=R_2=2\,\text{k}\Omega$ and $C=1\,\mu\text{F}$, what will be the values of i immediately before and immediately after opening S?

Sketch the variation of current i from $t=0$.

(Manchester University)

4. In the circuit shown in fig. 6.72 switches S_1 and S_2 are open.

(a) At $t=0$, S_2 is closed. Obtain expressions for the current in and the voltage across C for time $t=0.01$ s. C is initially uncharged.

(b) At $t=0.05$ s, S_2 is opened and at the same time S_1 is closed. Obtain an expression for the current in C subsequent to this operation and determine the voltage across C for $t=0.1$ s.

(c) With S_1 remaining closed, write down the circuit equations and the initial conditions for the sudden reclosing of S_2.

(Wales Science and Technology)

5. The operating coil of a relay working on a 20 V d.c. supply and activated by opening and closing switch S is represented by L, R in the circuit of fig. 6.73. The coil is shunted by a non-inductive resistor R_1 and the parameters of the circuit are as shown.

The relay closes when the current through its coils reaches 180 mA and opens when the current falls to 60 mA. Calculate the time lags for opening and closing respectively.

(Newcastle University)

Fig. 6.72. Circuit for problem 4.

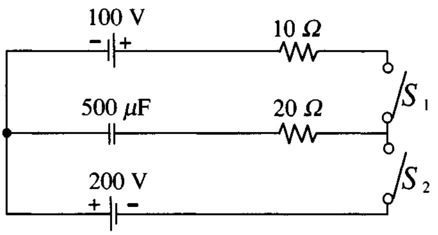
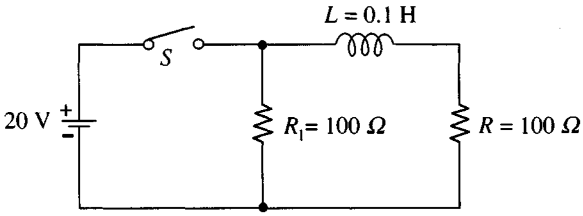


Fig. 6.73. Circuit for problem 5.



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6. In the circuit of fig. 6.74 the initial current in the inductance is 2 mA.
(a) With $e_s(t)=4\cos 10^3 t$, the switch is closed at $t=0$. Find an expression for $i_L(t)$ for $t>0$.
(b) If $e_s(t)=4\cos(10^3 t+\phi)$, determine the value of ϕ such that there will be no transient when the switch is closed at $t=0$.

7. The current through the deflecting coils of a cathode-ray tube is required to rise linearly with time from zero at the rate of 3 A/s. What must be the form of the applied voltage if the coils have an inductance of 0.1 H and a resistance of 5 Ω? What would be the form of the voltage if the resistance were zero?

8. The following problem relating to the circuit shown in fig. 6.75, is an exercise in determining initial and final conditions. The switch has been open for a long time.
(a) What are the values of i_1, i_2 and v_c? At $t=0$, the switch is closed.
(b) Calculate: i_1, i_2, v_c, $\mathrm{d}i_1/\mathrm{d}t$, $\mathrm{d}i_2/\mathrm{d}t$ and $\mathrm{d}v_c/\mathrm{d}t$ all at $t=0^+$.
(c) Calculate: i_1, i_2 and v_c at $t=\infty$.

9. The circuit between two terminals consists of two branches in parallel. One branch contains an inductance L and a resistance R_1 in series; the other a capacitance C and a resistance R_2 in series. By considering the

Fig. 6.74. Circuit for problem 6.

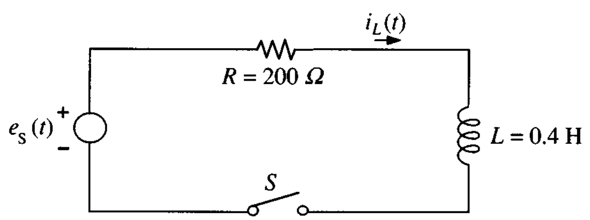
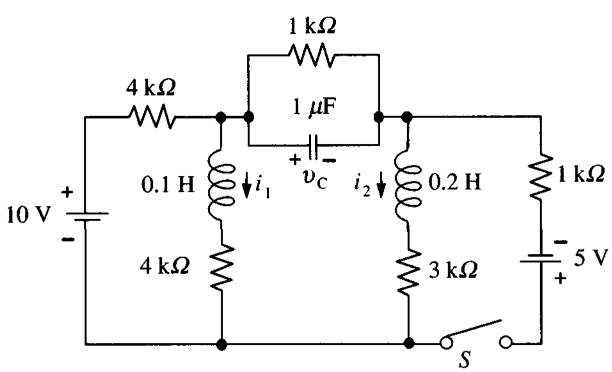


Fig. 6.75. Circuit for problem 8.



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currents which flow in the branches when a generator of e.m.f. E and zero internal impedance is suddenly connected to the terminals, find the relations between L, C, R_1 and R_2 such that the circuit behaves as a pure resistance.

10. In the circuit of fig. 6.76 determine the value of R in terms of L, C and r so that the potential difference between A and B will be non-oscillating when switch S is opened.
(Manchester University)

11. For the circuit of fig. 6.77; derive the differential equation, expressed in terms of the D-operator, relating $v_2(t)$ and $v_1(t)$. Assuming that the circuit is over-damped, write the form for the natural response for $v_2(t)$ and derive the time constants in the expression.

12. Two coils each having inductance L have mutual inductance M. One coil has a resistance R connected in parallel with it, and the other has a voltage E suddenly applied through a resistance R to its terminals. Find an expression for the subsequent voltage between the terminals of the first coil.

13. In the circuit of fig. 6.78 the initial conditions are: $i(0) = 2$ A, $v_c(0) = 1$ V. The switch is closed at $t = 0$.
(a) If $e_s(t) = 1$, find $v_c(t)$ for $t > 0$.
(b) If $e_s(t) = 2\cos 2t$, find $v_c\ (t)$ and $i(t)$ for $t > 0$.

Fig. 6.76. Circuit for problem 10.

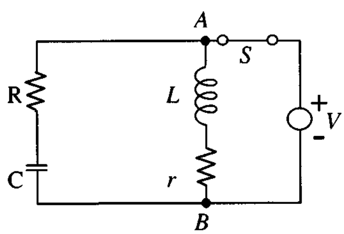
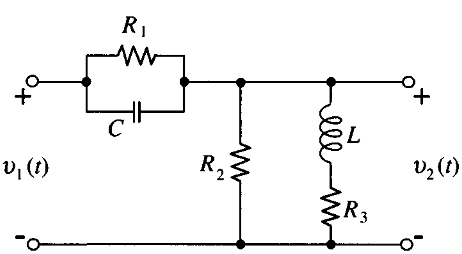


Fig. 6.77. Circuit for problem 11.



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14. When a 250 pF capacitor charged to 100 V is connected to an inductance coil, it is found that the discharge is oscillatory, and that the peak voltage falls to 10 V after 150 μs, corresponding to 150 cycles of oscillation.

It is also found that if the experiment is repeated with a pure resistor R_1 connected across the coil, the peak voltage falls from 100 V to 10 V after 90 μs, corresponding to 90 cycles of oscillation. Determine:

(a) the inductance and Q factor of the coil;

(b) the ohmic value of the resistance R_1.

The expression $v = V_0 \exp(-Rt/2L)\cos\omega t$ for the transient voltage v may be assumed.

(London University)

15. In the circuit of fig. 6.79 the switch is closed at $t=0$. Show that

$$i_1 = \frac{V}{R_1}\left[1 - e^{-\alpha t}\left\{\cosh\beta t - \frac{L_2R_1 - L_1R_2 \pm 2MR_1}{2(L_1L_2 - M^2)\beta}\sinh\beta t\right\}\right]$$

where

$$\alpha = \frac{L_1R_2 + L_2R_1}{2(L_1L_2 - M^2)}; \qquad \beta^2 = \alpha^2 - \frac{R_1R_2}{L_1L_2 - M^2}$$

(Manchester University)

Fig. 6.78. Circuit for problem 13.

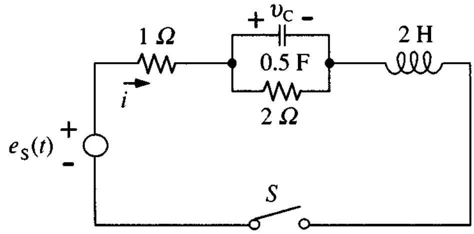
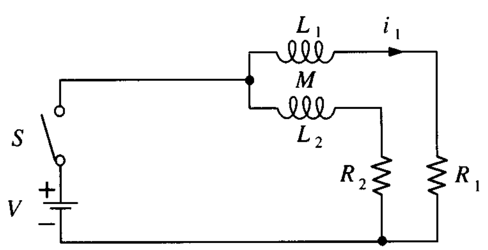


Fig. 6.79. Circuit for problem 15.



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16. In the circuit of fig. 6.80 the switch is initially open and the capacitor is charged so that $v_c(0) = 10$ V. At $t = 0$, the switch is closed. Use the Laplace transform method to find $i(t)$ and $v_c(t)$ for $t > 0$.

17. Discuss the benefits of using the Laplace transform method in network analysis.

In the network of fig. 6.81, $v_1(t)$ and $v_2(t)$ are the voltages on C_1 and C_2 respectively, and $i_L(t)$ is the current through L.

(a) If the components have the values indicated, show that the Laplace transform of the output voltage $v_2(t)$ is of the form:

$$V_2(s) = \frac{I(s) + as^2 + bs + c}{(s+1)\,(s^2+s+1)}$$

where $I(s)$ is the Laplace transform of the input $i(t)$. Derive expressions for the coefficients a, b and c in terms of the initial conditions $v_1(0)$, $v_2(0)$, and $i_L(0)$.

(b) If $i(t)$ is a unit step function of current applied at $t = 0$, what initial conditions are required to produce an output of 1 V for $0 < t < \infty$? Show that your answer is consistent with the *initial value* and *final value* theorems.

(c) If $i(t)$ is a unit step function of current and the circuit is initially quiescent, find an expression for the time-varying voltage for $0 < t < \infty$.

(Cambridge University)

Fig. 6.80. Circuit for problem 16.

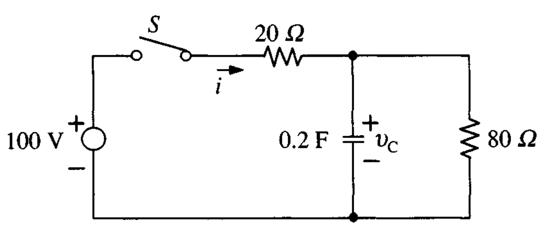
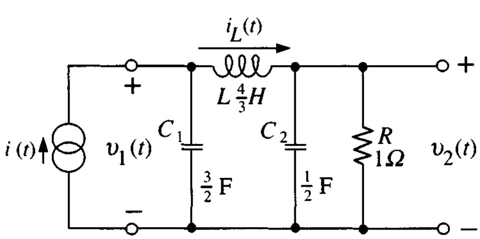


Fig. 6.81. Circuit for problem 17.



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18. A voltage waveform which starts at $t=0$, rises linearly at the rate of 10 V/s until it reaches an amplitude of 1 V and then instantaneously falls to zero, is applied to a resonant circuit consisting of resistance R, inductance L, and capacitance C, all connected in series. Determine the Laplace transform of the voltage waveform and the current $i(t)$ flowing in the series resonant circuit for $t>0$. Assume the Q factor of the circuit to be greater than 100 and that it is at rest when the waveform is applied.
(Newcastle University)

19. In the circuit of fig. 6.82 the switch is opened at $t=0$. Show that for $t>0$:

$$i=\frac{\mathrm{j}\omega CI}{1-(RC\omega)^2-\mathrm{j}3\omega CR}$$
$$\times\left[\mathrm{e}^{\mathrm{j}\omega t}-\mathrm{e}^{-\alpha t}\left\{\cosh\beta t-\frac{2-\mathrm{j}3\omega CR}{\mathrm{j}\omega CR\sqrt{5}}\sinh\beta t\right\}\right]$$

where

$$\alpha=\frac{3}{2RC}\qquad\beta=\frac{\sqrt{5}}{2RC}$$

(Manchester University)

20. (a) Find the Laplace transform of each of the following time functions:

$$\mathrm{e}^{\alpha t},\ t\mathrm{e}^{\alpha t},\ \sin\omega t,\ \sin(\omega t+\phi)$$

Fig. 6.82. Circuit for problem 19.

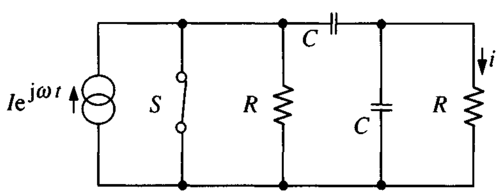
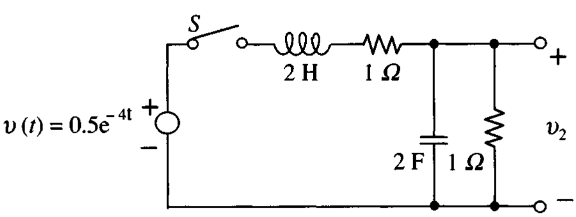


Fig. 6.83. Circuit for problem 20.



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(b) The switch S in the circuit shown in fig. 6.83 is closed at $t=0$. If all the initial conditions in the circuit are zero, find the output voltage v_2 as a function of the complex frequency s and as a function of time t.
(University of Kent)

21. Calculate the output voltage $v_0(t)$ in the circuit of fig. 6.84 if the current source delivers a pulse of 1 mA lasting 50 μsec. $R_1=R_2=10\,\text{k}\Omega$, $C_1=100\,\text{pF}$, $C_2=0.01\,\mu\text{F}$.
(Oxford University)

22. Show that the voltage transfer function $G(s)$ of the network of fig. 6.85 may be expressed as:

$$G(s)=\frac{E_0(s)}{E_{\text{in}}(s)}=\frac{1+2s\tau+\alpha s^2\tau^2}{1+2\beta+s\tau(2+\alpha+\alpha\beta)+\alpha s^2\tau^2}$$

where $\tau=CR$, $\alpha=C_1/C$, $\beta=R/R_L$, and $E_{\text{in}}(s)$ and $E_0(s)$ are the Laplace Transforms of the input voltage $e_{\text{in}}(t)$ and output voltage $e_0(t)$ respectively.

Sketch the frequency response of the network for the case where R_L is infinite and $C_1>C$. Indicate the asymptotic values of the response, and any maxima or minima, with their corresponding frequencies.

Show how this network might be used as the frequency-determining element of a sinusoidal oscillator.
(University of Kent)

23. Find the transfer function $H(s)=V_2(s)/V_1(s)$ of the third-order Butterworth filter shown in fig. 6.86. Verify that the amplitude response

Fig. 6.84. Circuit for problem 21.

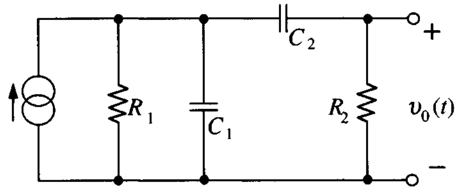
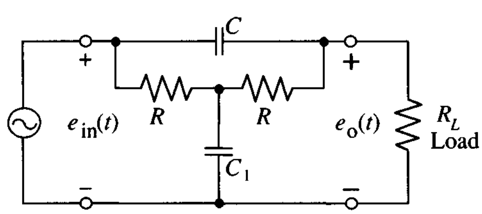


Fig. 6.85. Circuit for problem 22.



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function for a sinusoidal input at frequency ω is given by

$$|H(\mathrm{j}\omega)| = \frac{1}{\sqrt{(1+\omega^6)}}$$

Show that the poles of the transfer function lie on a circle of unity radius.
(Cambridge University)

24. Obtain an expression for the input impedance (in operational form) of the circuit shown in fig. 6.87. Find the zeros of this expression and hence find the minimum resistance required if the excitation of current oscillations by a step voltage input is to be avoided.
(Oxford University)

25. The impulse response of a potential divider is

$$f(t) = \mathrm{e}^{-\alpha t}[\alpha\cosh\beta t - \beta\sinh\beta t], \qquad \alpha > \beta$$

Find the frequency response function of the divider, and from it devise a possible circuit.
(Manchester University)

26. The equivalent circuit of an L–C surge absorber interposed at the junction between two transmission lines is shown in fig. 6.88.

An 'impulse' voltage (VT) volt-secs, expressible as $e_1 = (VT)s.1$, is applied at time $t = 0$ to the input terminals as shown. Show that the resultant output voltage e_2 may be expressed as

Fig. 6.86. Circuit for problem 23.

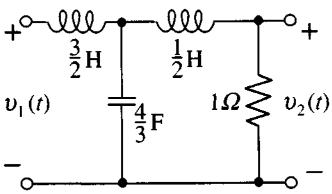
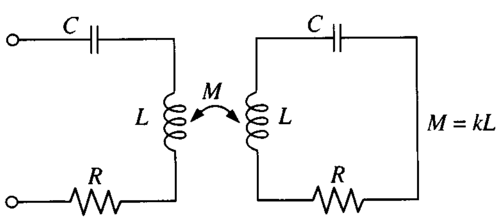


Fig. 6.87. Circuit for problem 24.



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$$e_2 = (VT)R_2 \frac{s}{LCR_2s^2 + (L + R_1R_2C)s + (R_1 + R_2)}.1$$

Show also that, if e_2 is to be non-oscillatory, the necessary condition which must be fulfilled is

$$(Z^2 - R_1R_2) > 2ZR_2$$

where

$$Z = \sqrt{\frac{L}{C}}$$

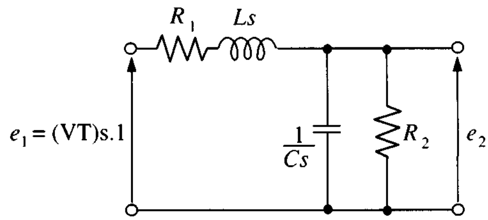
In a particular case, with numerical values inserted, the foregoing expression for e_2 reduces to

$$e_2 = 10^6 \left[\frac{s}{s+1} - \frac{s}{s+10}\right].1$$

where t is measured in micro-seconds.

Sketch the resultant wave-form of e_2, and show that it attains its maximum value of approximately 698 000 V at time $t \doteqdot 0.26\,\mu$s.
(Newcastle University)

Fig. 6.88. Circuit for problem 26.



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