Appendix A: Some Useful Mathematics

A.1 Trigonometric Identities

$$\exp(j\alpha) = \cos\alpha + j\sin\alpha \tag{A.1}$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{A.2}$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{A.3}$$

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{A.4}$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{A.5}$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{A.6}$$

$$\cos\theta - \cos\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right) \tag{A.7}$$

$$\sin\theta\sin\phi = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right) \tag{A.8}$$

$$\cos\theta\cos\phi = \frac{1}{2}\left(\cos(\theta-\phi) + \cos(\theta+\phi)\right) \tag{A.9}$$

$$\sin\theta\cos\phi = \frac{1}{2}\left(\sin(\theta-\phi) + \sin(\theta+\phi)\right) \tag{A.10}$$

$$\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta \tag{A.11}$$

$$\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta \tag{A.12}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{A.13}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{A.14}$$

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta) \tag{A.15}$$

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta) \tag{A.16}$$

$$\cos^2\theta + \sin^2\theta = 1 \tag{A.17}$$

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$$\cos(-\theta) = \cos(\theta) \tag{A.18}$$

$$\sin(-\theta) = -\sin(\theta) \tag{A.19}$$

A.2 Taylor Series

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{1}{2}f''(\alpha)(x-\alpha)^2 + \frac{1}{6}f'''(\alpha)(x-\alpha)^3 + \cdots, \tag{A.20}$$

where f', f'' and f''' denote the first, second and third derivatives of function f, respectively. We can approximate f by the first few terms of the series in the limit that $x \to \alpha$. As $x \to 0$, we have

$$\sin x = x - \frac{x^3}{6} + \cdots, \tag{A.21}$$

$$\cos x = 1 - \frac{x^2}{2} + \cdots, \tag{A.22}$$

$$(1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)\frac{x^2}{2} + \alpha(\alpha-1)(\alpha-2)\frac{x^3}{6} + \cdots, \tag{A.23}$$

$$\exp x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \tag{A.24}$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots. \tag{A.25}$$

In a similar fashion

$$f(x,y,z) \approx f(\alpha,\beta,\gamma) + f_x(\alpha,\beta,\gamma)(x-\alpha) + f_y(\alpha,\beta,\gamma)(y-\beta) + f_z(\alpha,\beta,\gamma)(z-\gamma) \tag{A.26}$$

in the limit $x \to \alpha$, $y \to \beta$ and $z \to \gamma$ where f_x, f_y and f_z denote the partial derivatives with respect to x, y and z respectively.

A.3 Fourier Series

For a function $f(\theta)$ with period 2π, i.e. $f(\theta + 2\pi) = f(\theta)$, it is possible to represent this as a series of trigonometric functions, i.e.

$$\begin{aligned} f(\theta) = \frac{a_0}{2} &+ a_1\cos(\theta) + b_1\sin(\theta) \\ &+ a_2\cos(2\theta) + b_2\sin(2\theta) \\ &+ a_3\cos(3\theta) + b_3\sin(3\theta) + \cdots. \end{aligned} \tag{A.27}$$

If f is a function with period 2π and

$$\begin{aligned} f(\theta) &= 1 \quad 0 < \theta < \pi \\ &= 0 \quad \pi < \theta < 2\pi \end{aligned} \tag{A.28}$$

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then

$$f(\theta) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin(\theta)}{1} + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + \cdots\right). \tag{A.29}$$

If f is a function with period 2π and

$$\begin{aligned} f(\theta) &= \sin(\theta) \quad 0 < \theta < \pi \\ &= 0 \quad \pi < \theta < 2\pi \end{aligned} \tag{A.30}$$

then

$$f(\theta) = \frac{1}{\pi} + \frac{\sin(\theta)}{2} - \frac{2}{\pi}\left(\frac{\cos(2\theta)}{1.3} + \frac{\cos(4\theta)}{3.5} + \frac{\cos(6\theta)}{5.7} + \cdots\right). \tag{A.31}$$

If f is a function with period 2π and

$$\begin{aligned} f(\theta) &= 1 \quad 0 < \theta < \frac{\pi}{2} \\ &= 0 \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2} \\ &= 1 \quad \frac{3\pi}{2} < \theta < 2\pi \end{aligned} \tag{A.32}$$

then

$$f(\theta) = \frac{1}{2} + \frac{2}{\pi}\cos(\theta) + \frac{2}{3\pi}\cos(3\theta) + \frac{2}{5\pi}\cos(5\theta) + \cdots. \tag{A.33}$$

A.4 Forced Oscillator

We will consider the ordinary differential equation (ODE)

$$A\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + B\frac{\mathrm{d}y}{\mathrm{d}t} + Cy = F\cos(\omega t). \tag{A.34}$$

This will have a solution of the form

$$y(t) = y_H(t) + y_I(t), \tag{A.35}$$

where y_H is the solution to the homogeneous equation (i.e. $F = 0$) and y_I is any function that satisfies the inhomogeneous equation (i.e. $F \neq 0$). The solution to the homogeneous equation has the form

$$y_H(t) = K_1\exp(-\alpha t + \beta t) + K_2\exp(-\alpha t - \beta t) \text{ if } B^2 > 4AC \tag{A.36}$$

$$y_H(t) = (K_1 + K_2 t)\exp(-\alpha t) \text{ if } B^2 = 4AC \tag{A.37}$$

$$y_H(t) = (K_1\cos(\tilde{\omega}t) + K_2\sin(\tilde{\omega}t))\exp(-\alpha t) \text{ if } B^2 < 4AC, \tag{A.38}$$

where $\alpha = B/2A$, $\beta = \sqrt{B^2 - 4AC}/2A$ and $\tilde{\omega} = \sqrt{4AC - B^2}/2A$. Arbitrary constants K_1 and K_2 are determined by the initial conditions. The solution to the inhomogeneous equation is given by

$$y_I(t) = \frac{F}{A^2(\omega_0^2 - \omega^2)^2 + \omega^2 B^2}\Big(A(\omega_0^2 - \omega^2)\cos(\omega t) + \omega B\sin(\omega t)\Big), \tag{A.39}$$

where $\omega_0 = \sqrt{C/A}$.

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In Chapter 1, Eq. (1.31) describes a tuned circuit that is driven by a harmonic source and is of the form of (A.34). Coefficient B arises as the result of resistance in the circuit and gives rise to damping since it is positive. This damping will cause the homogeneous solution to decay with time and so we will be left with the inhomogeneous solution y_I. This represents the steady state that the circuit will eventually attain.

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