7 Transmission Lines

In most radio systems, the radio signal spreads out from the source and can service many receivers. The downside is that the power of the radio waves falls away as the inverse square of distance. Transmission lines, however, are the means by which electromagnetic waves are directly transferred between transmitter and receiver with very little loss in power. It is obvious that such transmissions are needed under special circumstances and, in particular, in transferring power between a radio and its antenna. Long before the advent of radio, however, transmission lines were a key element in communications through the telegraph. The telegraph transmitted information as electromagnetic waves along a transmission line that consisted of a wire strung above the ground between poles. Telegraph systems were developed, around the year 1837, by William Fothergill Cooke and Charles Wheatstone in the UK and, independently, by Samuel Morse in the US. In the following chapter we will develop the basic theory of transmission lines and also look at some of the radio techniques that are a direct result of this theory.

7.1 Transmission-Line Theory

Some common types of transmission lines are depicted in Figure 7.1. A *coaxial transmission line* is shown in Figure 7.1a and consists of a central conductor that is surrounded by a cylindrical conductor (the space between the conductors can often be filled by a dielectric). Figure 7.1b shows a *parallel-wire transmission line*, i.e. two parallel conductors. The major thing to note is that the conductors possess inductance and that there will be capacitance between the conductors. Consequently, a simple model of the form shown in Figure 7.2 can be used. This model consists of short segments in which the conductors are replaced by inductors and a capacitor is added to represent the capacitance between the conductors ($L1$ and $L2$ are the inductances per unit length and C is the capacitance per unit length). The circuit represents a segment of transmission line that has length dz over which the voltage between the conductors changes from V to $V + \mathrm{d}V$ and the current through the conductors from I to $I + \mathrm{d}I$. (The return current in the lower conductor has the same magnitude as that in the upper conductor, but in the opposite direction.) We can apply the Kirchhoff circuit laws to each section of transmission line and obtain

$$\mathrm{d}\mathcal{V} = -\frac{\partial \mathcal{I}}{\partial t} L1\mathrm{d}z - \frac{\partial \mathcal{I}}{\partial t} L2\mathrm{d}z \text{ and } \mathrm{d}\mathcal{I} = -\frac{\partial \mathcal{V}}{\partial t} C\mathrm{d}z, \tag{7.1}$$

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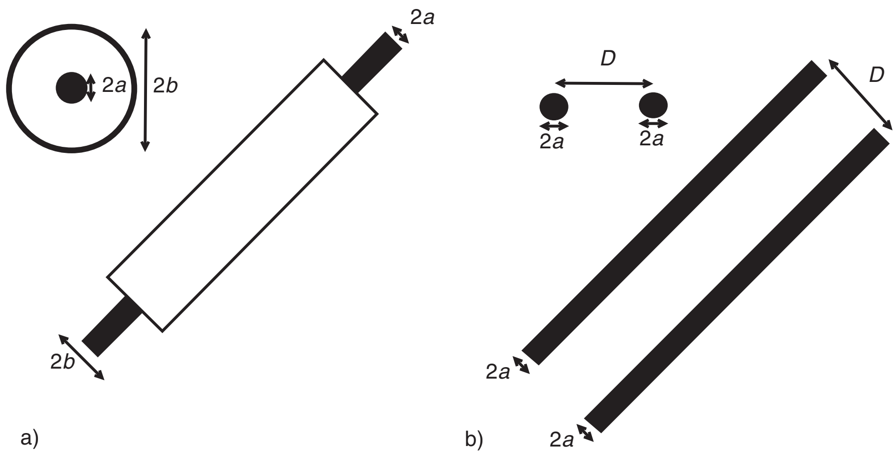

Fig. 7.1 Coaxial and parallel-wire transmission lines.

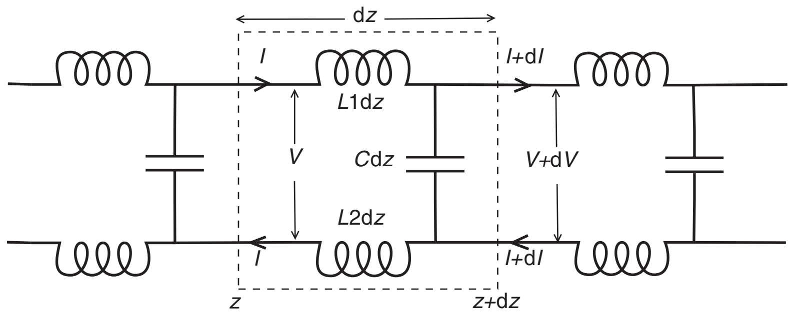

Fig. 7.2 Circuit model of a transmission line.

which can then be rewritten as

$$\frac{\partial \mathcal{V}}{\partial z} = -L\frac{\partial \mathcal{I}}{\partial t} \text{ and } \frac{\partial \mathcal{I}}{\partial z} = -C\frac{\partial \mathcal{V}}{\partial t}, \tag{7.2}$$

where $L = L1 + L2$ (i.e. all the inductance can be lumped into one inductor). From (7.2), we have that $\mathcal{V}$ and $\mathcal{I}$ both satisfy the same wave equation

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{V}}{\partial t^2} = 0 \text{ and } \frac{\partial^2 \mathcal{I}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{I}}{\partial t^2} = 0, \tag{7.3}$$

where $c = 1/\sqrt{LC}$. It is clear from the above equations that energy travels down a transmission line in the form of a wave, albeit a one-dimensional wave. The above wave equations have the solutions

$$\mathcal{V} = \mathcal{V}^+(z-ct) + \mathcal{V}^-(z+ct) \text{ and } \mathcal{I} = \mathcal{I}^+(z-ct) + \mathcal{I}^-(z+ct). \tag{7.4}$$

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It should be noted, however, that $\mathcal{V}$ and $\mathcal{I}$ are not independent solutions of these wave equations and must also satisfy relations (7.2). These relations will then imply that

$$\mathcal{V}^{\pm} = \pm Z_0 \mathcal{I}^{\pm}, \tag{7.5}$$

where $Z_0 = \sqrt{L/C}$ is known as the *characteristic impedance* of the transmission line (this relationship has the characteristics of Ohm's law). As a consequence, we find that

$$\mathcal{I} = \frac{1}{Z_0}\left(\mathcal{V}^{+}(z-ct) - \mathcal{V}^{-}(z+ct)\right). \tag{7.6}$$

For a coaxial transmission line $L = \mu \ln(b/a)/2\pi$ and $C = 2\pi\epsilon/\ln(b/a)$. In the case of a parallel-line transmission line, $L = \mu \cosh^{-1}(D/2a)/\pi$ and $C = \pi\epsilon/\cosh^{-1}(D/2a)$. Another type of transmission line is the *parallel-strip line* shown in Figure 7.3a. This line is almost a parallel-plate capacitor and so $C = \epsilon w/h$. The inductance per unit length is given by $L = \mu h/w$ and so speed of propagation is that of the medium between the plates. A related transmission line is the *microstrip* (see Figure 7.3b). This is extremely useful as it can be easily incorporated into a printed circuit design. Providing that $h \ll W$ the expressions for C and L are well approximated by those for the parallel-strip line. Outside this limit, however, the design of a microstrip becomes far more complex.

Real transmission lines will suffer losses and the model of Figure 7.4 provides a more realistic representation of the transmission line. The resistances per unit length ($R1$

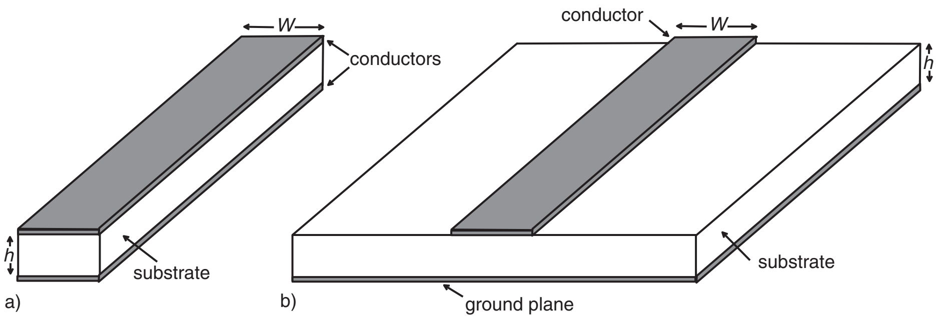

Fig. 7.3 Parallel-strip and microstrip transmission lines.

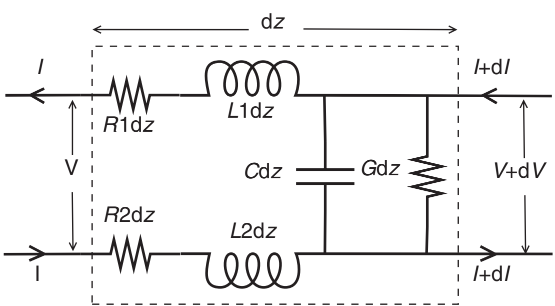

Fig. 7.4 Realistic model of a transmission line.

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and $R2$) represent the ohmic losses in the conductors and the conductance (the inverse of resistance) per unit length G represents the losses in the dielectric separating the conductors. The transmission line equations will then become

$$\frac{\partial \mathcal{V}}{\partial z} = -R\mathcal{I} - L\frac{\partial \mathcal{I}}{\partial t} \text{ and } \frac{\partial \mathcal{I}}{\partial z} = -G\mathcal{V} - C\frac{\partial \mathcal{V}}{\partial t}, \tag{7.7}$$

where $R = R1 + R2$. We have from (7.7) that $\mathcal{V}$ and $\mathcal{I}$ both satisfy the same equations

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{V}}{\partial t^2} - (RC + GL)\frac{\partial \mathcal{V}}{\partial t} - GR\mathcal{V} = 0$$

$$\text{and } \frac{\partial^2 \mathcal{I}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{I}}{\partial t^2} - (RC + GL)\frac{\partial \mathcal{I}}{\partial t} - GR\mathcal{I} = 0. \tag{7.8}$$

The above form of equation is known as *telegrapher's equation* since it describes the propagation of Morse code pulses down a telegraph line, one of the first major uses of transmission lines. Indeed, the equation was first developed by Oliver Heaviside around 1885 in order to explain such propagation. The telegrapher's equation is difficult to solve and so we merely quote the solution for a right-travelling harmonic wave at frequency ω, i.e.

$$\mathcal{V} = V_0 \exp(-t(R/L + G/C)/2) \sin\left(\omega t - \frac{\sqrt{\omega^2 + \alpha^2}}{c} z\right), \tag{7.9}$$

where $\alpha = (R/L - G/C)/2$ and V_0 is the amplitude when $t = 0$. The first thing to note is that the amplitude of the wave reduces as it propagates, i.e. energy is dissipated in the conductance and resistance of the line. Just as important, however, is that the speed of the wave is given by $v = c\omega/\sqrt{\omega^2 + \alpha^2}$ which, it will be noted, is frequency-dependent. Normally, the wave of interest will consist of a modulated harmonic wave (the pulses of Morse code for instance) and therefore it will be a mixture of components at many different frequencies. Since these components will travel at different speeds, the signal will lose integrity over time (see Figure 7.5) with a loss of information. This phenomenon, known as *dispersion*, is every bit as problematic as signal loss for long-distance telegraph lines. The solution, suggested by Heaviside, was to make a cable such that $R/G = L/C$, i.e. a cable free of dispersion.

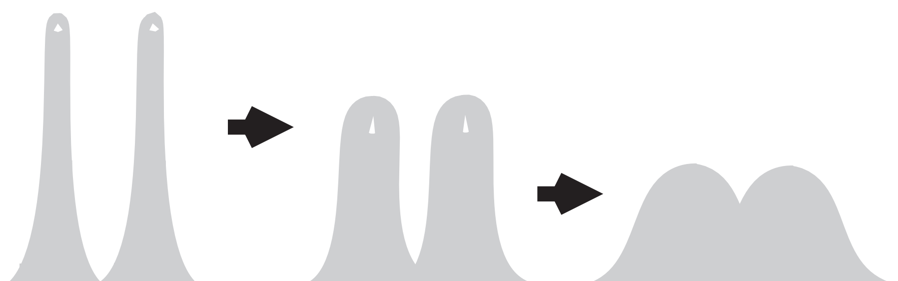

Fig. 7.5 The effect of dispersion.

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7.2 The Termination of Transmission Lines

We consider the situation where a wave is travelling down a transmission line to a resistive load R where it is then reflected back down the transmission line (see Figure 7.6). The incoming wave will have the form $\mathcal{V}^+(z-ct)$ and the reflected wave the form $\mathcal{V}^-(z+ct)$. At the load, Ohm's law will imply $\mathcal{V}=R\mathcal{I}$ and this, together with (7.6), will then imply that

$$\mathcal{V}^+(l-ct)+\mathcal{V}^-(l+ct)=\frac{R}{Z_0}\left(\mathcal{V}^+(l-ct)-\mathcal{V}^-(l+ct)\right). \tag{7.10}$$

Rearranging this expression

$$\mathcal{V}^-(l+ct)=\frac{R-Z_0}{R+Z_0}\mathcal{V}^+(l-ct)=\Gamma\mathcal{V}^+(l-ct), \tag{7.11}$$

where $\Gamma=(R-Z_0)/(R+Z_0)$ is known as the *reflection coefficient*. It should be noted that, when the load R is equal to the characteristic impedance Z_0, $\Gamma=0$ and there is no reflection. In this situation, the incoming wave is totally absorbed by the load and the load is said to be *matched* to the transmission line.

We now look at the case of a harmonic wave travelling down the transmission line, i.e.

$$\mathcal{V}=\Re\{V\exp(\mathrm{j}\omega t)\} \text{ and } \mathcal{I}=\Re\{I\exp(\mathrm{j}\omega t)\}, \tag{7.12}$$

then

$$V=V^+\exp(-\mathrm{j}\beta z)+V^-\exp(\mathrm{j}\beta z) \text{ and } I=\frac{1}{Z_0}\left(V^+\exp(-\mathrm{j}\beta z)-V^-\exp(\mathrm{j}\beta z)\right), \tag{7.13}$$

where $\beta=\omega/c$ is the *propagation constant* (subscript $+$ refers to the incoming wave and $-$ refers to the reflected wave). At the load, the time-harmonic Ohm's law will imply that $V=IZ_L$ and so

$$V^+\exp(-\mathrm{j}\beta l)+V^-\exp(\mathrm{j}\beta l)=\frac{Z_L}{Z_0}\left(V^+\exp(-\mathrm{j}\beta l)-V^-\exp(\mathrm{j}\beta l)\right) \tag{7.14}$$

from which

$$V^-=\frac{Z_L-Z_0}{Z_L+Z_0}\exp(-2\mathrm{j}\beta l)V_+=\Gamma_{\mathrm{in}}V^+ \tag{7.15}$$

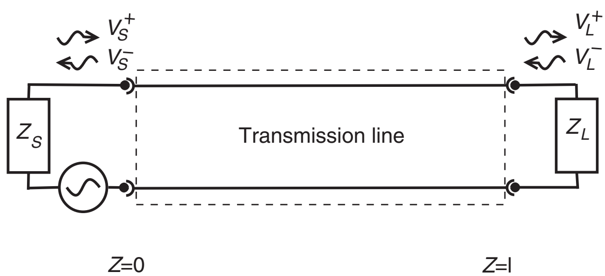

Fig. 7.6 A terminated transmission line.

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(we have implicitly assumed that the source is matched to the transmission line and so all of the reflected wave is absorbed by the source). $\Gamma_{\rm in}$ is the reflection coefficient looking into the transmission line (i.e. where $z = 0$) and it is obvious that $\Gamma_{in} = \Gamma_L \exp(-2j\beta l)$ where $\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)$ is the reflection coefficient looking into the load. We will now have that

$$V = V^+ \left(\exp(-j\beta z) + \Gamma_L \exp(-2j\beta l) \exp(j\beta z)\right) \tag{7.16}$$

and

$$I = \frac{V^+}{Z_0} \left(\exp(-j\beta z) - \Gamma_L \exp(-2j\beta l) \exp(j\beta z)\right). \tag{7.17}$$

From (7.16), we have that

$$|V| = |V_L^+|\,|1 + \Gamma_L \exp(2j\beta(z - l))| \tag{7.18}$$

and from which we see that, along the transmission line, the voltage amplitude varies from a minimum of $V_{\rm min} = |V_L^+|(1 - |\Gamma_L|)$ to a maximum of $V_{\rm max} = |V_L^+|(1 + |\Gamma_L|)$. An important measure of the degree to which a transmission line is matched to a load is the *voltage standing-wave ratio* (VSWR) which is defined to be

$$\text{VSWR} = \frac{V_{\rm max}}{V_{\rm min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}. \tag{7.19}$$

There will be a perfect match for a VSWR of 1 and the higher the value of VSWR the greater the mismatch. VSWR is not a full description of the load, but the magnitude of the reflection coefficient can be derived from it.

The reflection coefficient Γ provides an alternative description of a device when it is connected to other devices through a transmission line. The reflection coefficient is related to the impedance of the device through $Z = Z_0(1 + \Gamma)(1 - \Gamma)$ where Z_0 is the characteristic impedance of the transmission line. The resistive part R of the impedance can be expressed in terms of the reflection coefficient as

$$R = Z_0 \Re\left\{\frac{1 + \Gamma}{1 - \Gamma}\right\} = \frac{1 - |\Gamma|^2}{|1 - \Gamma|^2}. \tag{7.20}$$

From the above expression, it is clear that the resistance of the device will be positive when $|\Gamma| < 1$ and negative when $|\Gamma| > 1$. As we have seen in Chapter 5, a device that exhibits negative resistance has the ability to cause oscillations and is thus unstable. Passive devices are inherently stable, but active devices can often exhibit instability. Hence, to ensure the stability of a device we need to have a reflection coefficient with $|\Gamma| < 1$. If we load an unstable device with a passive impedance Z_L, the resulting system will oscillate if $Z_L + Z = 0$ (this is the condition that allows an arbitrary current to flow in the circuit), i.e. $R_L + R = 0$ and $X_L + X = 0$. The device resistance will need to be sufficiently negative to cancel the effect of the load resistance and the oscillation frequency will be determined by $X_L + X = 0$. If $R_L + R < 0$, the amplitude of oscillation will rise until the inherent nonlinearities of the device have increased the resistance R to a point where $R_L + R = 0$.

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7.3 Transmission-Line Resonators

From (7.16) and (7.17), the impedance looking into a transmission line of length l, terminated by a load Z_L, is given by

$$Z_{\text{in}} = \frac{V(0)}{I(0)} = Z_0 \frac{1+\Gamma_L \exp(-2\mathrm{j}\beta l)}{1-\Gamma_L \exp(-2\mathrm{j}\beta l)}, \tag{7.21}$$

where $\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)$. This can be rearranged into

$$Z_{\text{in}} = Z_0 \frac{Z_L \cos(\beta l) + \mathrm{j}Z_0 \sin(\beta l)}{Z_0 \cos(\beta l) + \mathrm{j}Z_L \sin(\beta l)} \tag{7.22}$$

and, from the above expression, it can be seen that a length of transmission line can be used to transform an impedance. There are two important special cases, the *open-circuit transmission line* ($Z_L = \infty$) and the *short-circuit transmission line* ($Z_L = 0$). For both cases, the impedance looking into the transmission line is entirely reactive with

$$Z_{\text{in}} = -\mathrm{j}Z_0 \cot(\beta l) \tag{7.23}$$

for the open-circuit line and

$$Z_{\text{in}} = \mathrm{j}Z_0 \tan(\beta l) \tag{7.24}$$

for the short-circuit line. Figure 7.7 shows how the reactance varies with line length for a given frequency ω (the length is measured in terms of wavelength $\lambda = 2\pi c/\omega$ on the line). It will be noted that the reactances for both cases vary between capacitative reactance and inductive reactance. Importantly, for lengths less than a quarter wavelength, the open-circuit line has capacitative reactance (it is negative) and the short-circuit line has inductive reactance (it is positive). Both lines have a resonance at a quarter wavelength, the open circuit exhibiting a series resonance and the short circuit exhibiting a parallel resonance.

From the above discussion, it is clear that an open-circuit transmission line behaves like a series resonant tuned circuit with a resonance at a frequency ω_0 for which the length

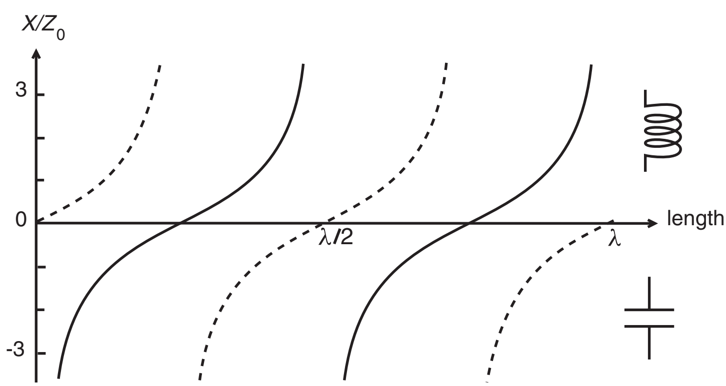

Fig. 7.7 Impedance of an open-circuit (unbroken-line) and a short-circuit (broken-line) transmission line (TL).

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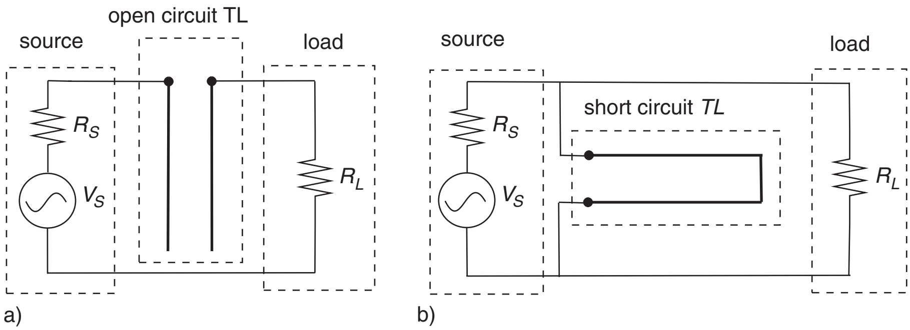

Fig. 7.8 Filtering using open-circuit and short-circuit quarter-wave transmission lines.

l corresponds to a quarter wavelength ($l = \pi c/2\omega$). The impedance of this resonator is given by $Z = -\mathrm{j}Z_0 \cot\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right)$. If we use the resonator in the configuration shown in Figure 7.8a, the voltage drop across the load resistor is then given by

$$V_L = V_S \frac{R_L}{R_S + R_L + Z} = V_S \frac{R_L}{R_S + R_L} \frac{1}{1 - \mathrm{j}\frac{Z_0}{R_S+R_L} \cot\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right)} \tag{7.25}$$

and, around the resonance at ω_0, this can be approximated by

$$V_L = V_S \frac{R_L}{R_S + R_L} \frac{1}{1 - 2\mathrm{j}Q\frac{\delta\omega}{\omega_0}}, \tag{7.26}$$

where $Q = \pi Z_0/4(R_L + R_S)$ and $\delta\omega = \omega - \omega_0$. It is possible to make transmission-line resonators with very high unloaded Q (much higher than is possible with inductor- and capacitor-based tuned circuits) and so the above tuned circuit can have an extremely sharp response if the source and/or load resistances are large. Figure 7.9 shows the insertion gain of the filter for various values of Q (solid line for a Q of 5, broken line for a Q of 2 and dotted line for a Q of 1). As to be expected, the filtering action is sharper for the higher Q. However, unlike inductor and capacitor combinations, it will be noted that the transmission-line resonators have a whole series of resonances (these occur at ω_0, $3\omega_0$, $5\omega_0$, etc.). Consequently, in using transmission-line resonators one needs to ensure that these higher-frequency responses are not an issue.

A short-circuited transmission line behaves like a parallel combination of an inductor and a capacitor with a resonance at a frequency ω_0 for which the line is a quarter wavelength. The impedance of such a resonator is given by $Z = \mathrm{j}Z_0 \tan\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right)$. If we use the resonator in the configuration shown in Figure 7.8b, the voltage drop across the load resistor is given by

$$V_L = V_S \frac{ZR_L}{R_S Z + R_L Z + R_S R_L} = V_S \frac{R_L}{R_S + R_L} \frac{1}{1 - \mathrm{j}\frac{R_S \| R_L}{Z_0} \cot\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right)}. \tag{7.27}$$

Around resonance at ω_0, the behaviour of V_L can be approximated by (7.26), but with $Q = (\pi R_S \parallel R_L)/4Z_0$. Further, like the open-circuit transmission line, the short-circuit line will have additional resonances at $3\omega_0$, $5\omega_0$, etc.

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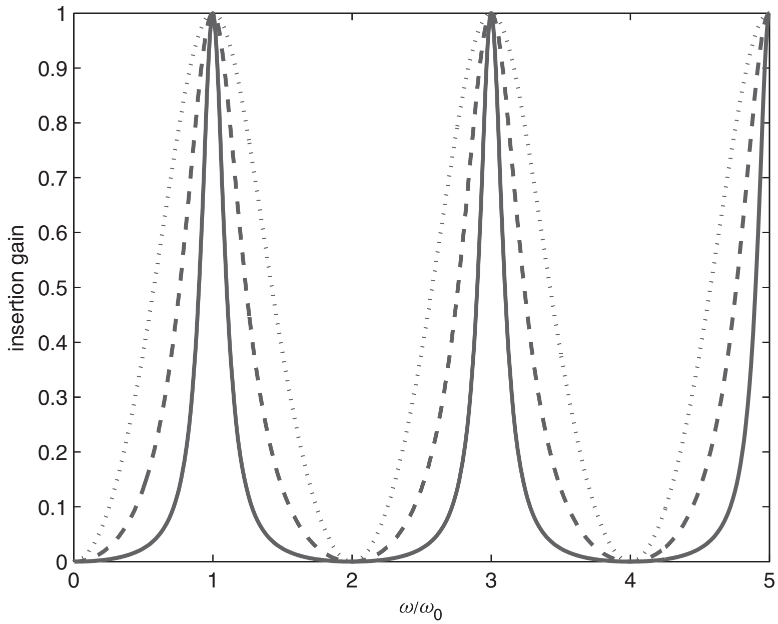

Fig. 7.9 Insertion gain for an open-circuit transmission line with $Q = 5$ (solid line), $Q = 2$ (broken line) and $Q = 1$ (dotted line).

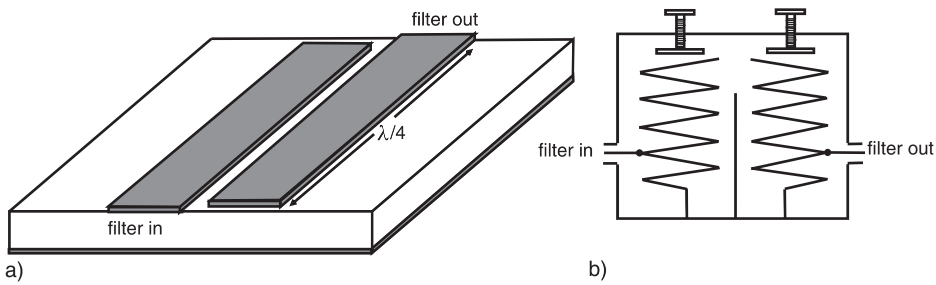

Fig. 7.10 Coupled transmission-line filters (microstrip and helical).

In Chapter 3 we saw that series and parallel resonant circuits could be combined to form complex band-pass filters. It is therefore possible to build transmission line versions of these filters by replacing the series and parallel resonant circuits of these filters by open- and short-circuit quarter-wavelength transmission lines. An alternative filter is the coupled microstrip filter shown in Figure 7.10a. In this filter we have two quarter-wavelength resonators that are close enough for power to be transferred through their mutual capacitance and inductance. Using such an approach, effective filters can be made for frequencies above 500 MHz, but below these frequencies the physical size of a quarter-wavelength line can become an issue. The physical size can be reduced by slowing down the wave on the transmission line. In the case of the microstrips, this can be done by increasing the permittivity of the substrate dielectric (substrates with $\epsilon = 10$

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are certainly available). Another approach, however, is to increase the inductance of the conductors. This is the case in the helical filter shown in Figure 7.10b. This filter consists of two transmission-line resonators, each consisting of a metal chamber containing a central conductor that is a helix; this helical shape increases the inductance of the central conductor. The two windings are then coupled through a window between the chambers (note the adjustable capacitance at the top of a chamber that allows the filter frequency to be fine-tuned). In order to ensure that the filter input and output present the correct impedance to the source and load, these are fed into a tap at a suitable point along the helix.

Another application of transmission lines is the matching of a source Z_S to a load Z_L. In the case that the source and load have real impedances, this is simply achieved using a quarter-wave transmission line. From (7.22), we have that the impedance looking into the loaded line will be $Z_{\text{in}} = Z_0^2/Z_L$. Consequently, if we choose $Z_0 = \sqrt{Z_L Z_S}$, we will have that $Z_{\text{in}} = Z_S$, i.e. we have matched the load to the source. In the case of general source and load impedances, the matching is far more difficult. Consider the situation shown in Figure 7.11. We will consider the problem in terms of the admittances $Y_L = 1/Z_L$ and $Y_S = 1/Z_S$ (the admittance of parallel impedances is additive). From (7.22), the admittance transforms along the transmission line according to

$$Y_{in} = Y_0 \frac{Y_L \cos(\beta l) + \mathrm{j} Y_0 \sin(\beta l)}{Y_0 \cos(\beta l) + \mathrm{j} Y_L \sin(\beta l)}, \tag{7.28}$$

where $Y_0 = 1/Z_0$. For a matching to occur, we need the admittances Y_{in} and Y_S to be conjugate and so first choose the length l such that $\Re\{Y_{\text{in}}\} = \Re\{Y_S\}$. Then, if $\Im\{Y_{\text{in}}\} \neq -\Im\{Y_S\}$, we manufacture a stub reactance X such that

$$\Im\{Y_{\text{in}}\} - \frac{1}{X} = -\Im\{Y_S\}. \tag{7.29}$$

Although we have shown the stub as open circuit, it could equally well be a shorted stub. At the end of the day, however, it is best to choose the type of stub that has the shortest length that gives the desired reactance X.

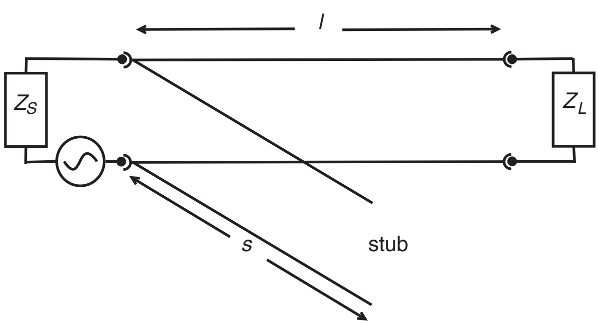

Fig. 7.11 Stub matching a source to a load.

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7.4 Scattering Matrices

One of the problems with the description of a circuit in terms of voltages and currents is that the measurement of these quantities becomes more and more problematic as frequency rises. In particular, the probes used to measure these quantities can severely disturb the operation of the circuit. We have hitherto described our circuits in terms of impedances and sources, but we have already seen that a reflection coefficient Γ can provide an alternative description of the load (the impedance can be related to the reflection coefficient through the relation $Z = Z_0(1+\Gamma)/(1-\Gamma)$). In the impedance description of RF electronics, we analyse circuits in terms of sources (both current and voltage) and impedances that are modelled by Ohm's law, i.e.

$$V = IZ, \tag{7.30}$$

where V is the voltage drop across an impedance Z through which a current I flows. In the reflection coefficient description of RF circuits, we deal with sources and devices that are connected through transmission lines. We describe the circuit in terms of voltage waves that travel back and forth along the transmission lines and model a device through its reflection coefficient, i.e.

$$V^r = \Gamma V^i, \tag{7.31}$$

where V^i is the voltage wave incident at the impedance, V^r is the voltage wave reflected at the impedance and Γ is the reflection coefficient at the impedance. It should be noted, however, that this description depends on the characteristic impedance Z_0 of the transmission lines and that Γ can change if Z_0 changes. The Smith chart is a graphical representation of the reflection coefficient and is particularly useful for illustrating the behaviour of an impedance with respect to variations of a parameter such as frequency. The vertical axis represents the imaginary component of the reflection coefficient and the horizontal axis the real component.

Figure 7.13a shows curves of constant $|\Gamma|$ plotted on a Smith chart, Figure 7.13b curves of constant resistance (normalised on Z_0) and Figure 7.13c curves of constant reactance (normalised on Z_0). Figure 7.14 shows a complex tuned circuit with its Smith chart. The variation with frequency shows two resonant frequencies (these are points where the reactance is zero). There is a series resonance at 7.16 MHz where the value of resistance is around $R = 50\,\Omega$ and parallel resonance at 10 MHz where the resistance is large.

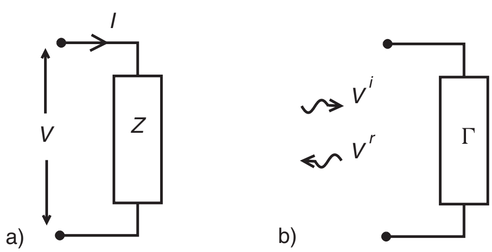

Fig. 7.12 Impedance and reflection coefficient models of a one-port device.

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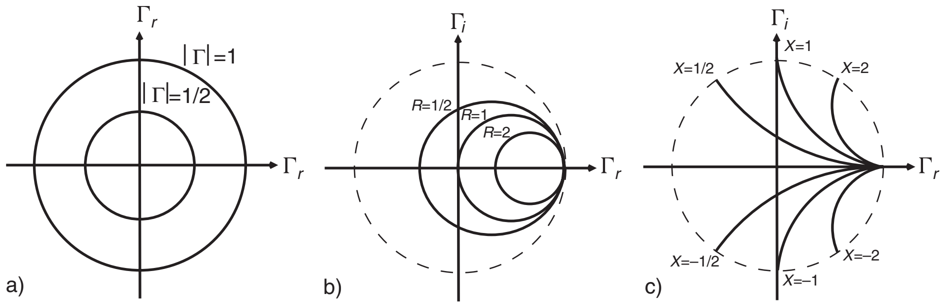

Fig. 7.13 Smith-charts representation of the reflection coefficient $\Gamma = \Gamma_r + j\Gamma_i$, resistance R and reactance X.

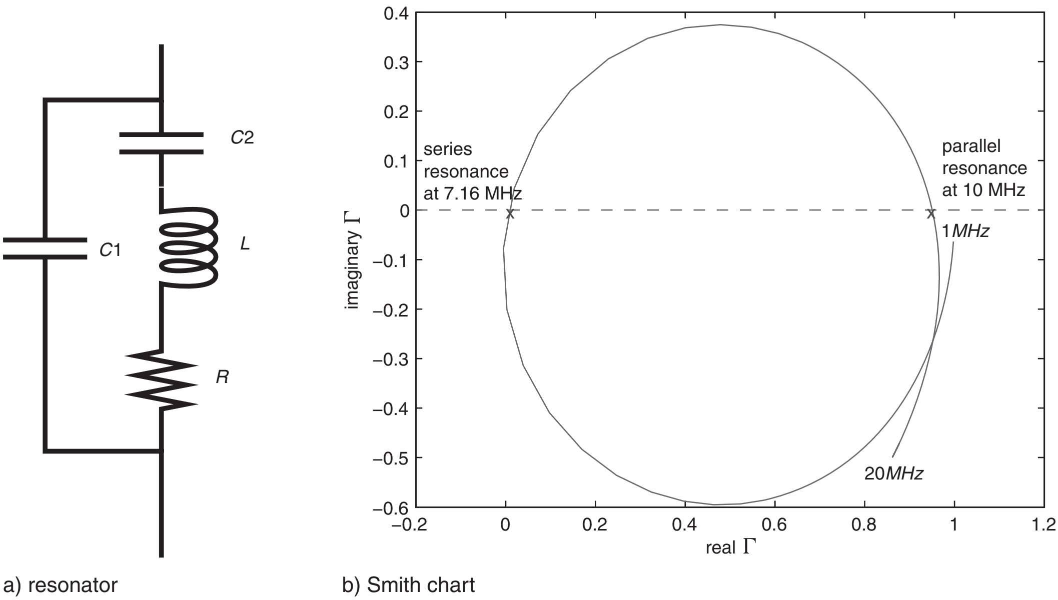

Fig. 7.14 Smith chart of a resonator with both parallel and series resonances ($C1 = C2 = 50\,\text{pF}$, $L = 10\,\mu\text{H}$ and $R = 50\,\Omega$).

Impedance is a concept that applies to a one port device, but can be extended to multiport devices through the concept of an *impedance matrix*. An amplifier is an example of a two-port device, the input and the output of the amplifier being the two ports. In the world of voltage and current, the currents flowing into the ports are related to the voltages applied to the ports through an impedance matrix Z. Figure 7.15a illustrates the situation for a two-port device. The relationship between the voltages V_1 and V_2 at the ports and the currents I_1 and I_2 flowing into these ports is then given by

$$\begin{aligned} V_1 &= Z_{11}I_1 + Z_{12}I_2 \\ V_2 &= Z_{21}I_1 + Z_{22}I_2, \end{aligned} \tag{7.32}$$

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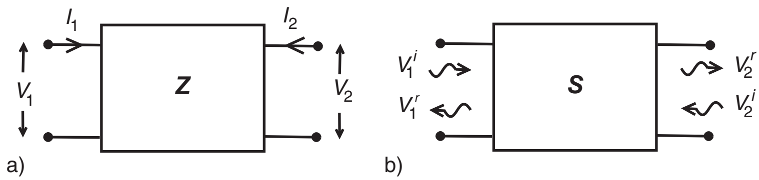

Fig. 7.15 Impedance-matrix and scattering-matrix models of a two-port device.

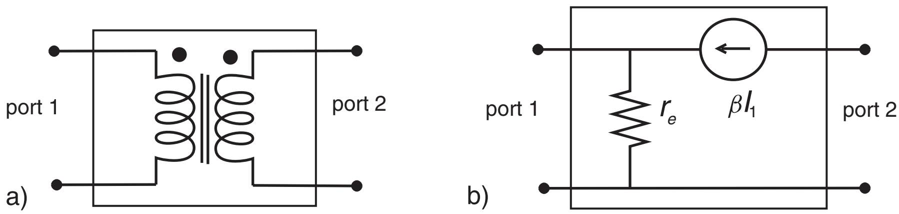

Fig. 7.16 Examples of two-port networks.

where Z_{11}, Z_{12}, Z_{21} and Z_{22} are the coefficients of the impedance matrix. In the language of matrices

$$\mathbf{V} = Z\mathbf{I}, \tag{7.33}$$

where

$$\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \text{ and } \quad Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}.$$

We have already seen such a relationship of the form of (7.32) when studying transformers in Chapter 3 (see Figure 7.16a). In this case, the impedance matrix is given by

$$Z = \begin{pmatrix} \mathrm{j}\omega L_{11} & \mathrm{j}\omega L_{12} \\ \mathrm{j}\omega L_{21} & \mathrm{j}\omega L_{22} \end{pmatrix}$$

where $L_{12} = L_{21}$ is the mutual impedance between the transformer windings and L_{11} and L_{22} are the self impedance of these windings. (A more general schematic of a transformer is used in Figure 7.16a with the dots indicating 'in phase' terminals and the bars indicating the presence of an iron core.)

In a similar fashion to impedance, the concept of a reflection coefficient can be extended to multiport devices through the *scattering matrix*. In the scattering-matrix approach, the incident and reflected voltage waves at the ports of a device are related to each other through the scattering matrix. Figure 7.15b illustrates the situation for a two-port device. The relationship between the incident waves V_1^i and V_2^i and the reflected waves V_1^r and V_2^r is then given by

$$\begin{aligned} V_1^r &= S_{11}V_1^i + S_{12}V_2^i \\ V_2^r &= S_{21}V_1^i + S_{22}V_2^i, \end{aligned} \tag{7.34}$$

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where S_{11}, S_{12}, S_{21} and S_{22} are the coefficients of the scattering matrix. In the language of matrices

$$\mathbf{V}^r = S\mathbf{V}^i, \tag{7.35}$$

where

$$\mathbf{V}^i = \begin{pmatrix} V_1^i \\ V_2^i \end{pmatrix}, \quad \mathbf{V}^r = \begin{pmatrix} V_1^r \\ V_2^r \end{pmatrix} \text{ and } \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

The above description might seem incomplete (there is no mention of current) but we must remember that, in the scattering-matrix world, devices are connected through transmission lines. As a consequence, the current and voltage waves are related through $V_1^i = Z_0 I_1^i$, $V_1^r = -Z_0 I_1^r$, $V_2^i = -Z_0 I_2^i$ and $V_2^r = Z_0 I_2^r$, where Z_0 is the characteristic impedance of the transmission lines. In the same way that the reflection coefficient and impedance of a one-port device are related, the scattering matrix and the impedance matrix of a multiport device are also related (a reflection coefficient is merely scattering matrix for a one-port device). The relation is given by $S = (Z - Z_0 I)(Z + Z_0 I)^{-1}$ where I is the unit matrix. For an ideal transformer, this then implies that

$$S = \frac{1}{\mathrm{j}\omega(L_{11} + L_{22}) + Z_0} \begin{pmatrix} \mathrm{j}\omega(L_{11} - L_{22}) - Z_0 & 2\mathrm{j}\omega L_{12} \\ 2\mathrm{j}\omega L_{21} & \mathrm{j}\omega(L_{22} - L_{11}) - Z_0 \end{pmatrix}. \tag{7.36}$$

We now consider the transformer shown in Figure 7.16a to be ideal, i.e. $V_1 = nV_2$ and $I_1 = -I_2/n$, where $n = N_1/N_2$ is the turns ratio. Since $V_1 = V_1^i + V_1^r$ and $V_2 = V_2^i + V_2^r$, we have that

$$V_1^r - nV_2^r = nV_2^i - V_1^i. \tag{7.37}$$

Further, since $I_1 = V_1^i/Z_0 - V_1^r/Z_0$ and $I_2 = -V_2^i/Z_0 + V_2^r/Z_0$ (note that the incident wave at port 2 is now leftward-travelling), we have that

$$V_1^r + \frac{V_2^r}{n} = \frac{V_2^i}{n} + V_1^i. \tag{7.38}$$

Adding (7.37) to n^2 times (7.38), we obtain that

$$V_1^r(n^2 + 1) = (n^2 - 1)V_1^i + 2nV_2^i \tag{7.39}$$

and subtracting (7.37) from (7.38), we obtain that

$$V_2^r(n + \frac{1}{n}) = 2V_1^i + (\frac{1}{n} - n)V_2^i. \tag{7.40}$$

Bringing this all together, we have a scattering matrix of the form

$$S = \begin{pmatrix} \frac{n^2-1}{n^2+1} & \frac{2n}{n^2+1} \\ \frac{2n}{n^2+1} & -\frac{n^2-1}{n^2+1} \end{pmatrix}. \tag{7.41}$$

In the limit that the transformer is strongly coupled, and the windings have large reactance, (7.36) reduces to the above expression.

Now consider the S matrix of the BJT shown in Figure 7.16b. The current flowing through r_e will be $(1+\beta)I_1$ and so $V_1 = r_e(1+\beta)I_1$. At the input, we have $V_1 = V_1^i + V_1^r$

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and $I_1 = V_1^i/Z_0 - V_1^r/Z_0$. As a consequence, $V_1 = V_1^i + V_1^r = r_e(1+\beta)(V_1^i - V_r)/Z_0$ and from this

$$V_1^r = \frac{(\beta+1)r_e - Z_0}{(\beta+1)r_e + Z_0} V_1^i. \tag{7.42}$$

Comparing this with (7.34) we have that $S_{12} = 0$ and

$$S_{11} = \frac{(\beta+1)r_e - Z_0}{(\beta+1)r_e + Z_0}. \tag{7.43}$$

If we load port 2 with the characteristic impedance Z_0, there will be no reflection from this load and hence $V_2^i = I_2^i = 0$. The current flowing through the load will be βI_1 from which $V_2^r = V_2 = \beta I_1 Z_0$ and so

$$V_2^r = \beta(V_1^i - V_1^r). \tag{7.44}$$

Eq. (7.34) will also imply that $V_2^r = S_{21}V_1^i$ and so this, together with (7.44), will imply that

$$S_{21} = \beta\left(1 - \frac{V_1^r}{V_1^i}\right) = \beta(1 - S_{11}). \tag{7.45}$$

Finally, looking into port 2, the current source will present an open circuit (i.e. infinite impedance) and so $S_{22} = 1$. Bringing this all together, we have a scattering matrix of the form

$$S = \begin{pmatrix} \frac{(\beta+1)r_e - Z_0}{(\beta+1)r_e + Z_0} & 0 \\ \frac{2\beta Z_0}{(\beta+1)r_e + Z_0} & 1 \end{pmatrix}. \tag{7.46}$$

For the simple devices above, we have the luxury of being able to obtain an analytic expression for the S matrix. For more complex devices, however, the S matrix is usually obtained through measurements. Such measurements are important due to the fact that most RF devices are now characterised by their manufacturers in terms of S parameters. To do the necessary measurements, we need a device which we can sample the incoming and reflected voltage waves at the device ports, a device known as a *directional coupler*.

Figure 7.17 shows a coupler that is based around two transformers with turns ratio of n. Ports 3 and 4 are loaded by the impedances Z_0 (the characteristic impedance of the transmission lines). As a consequence, the left-hand transformer causes a current $(V^i - V^r)/nZ_0$ to flow into the line joining ports 3 and 4, half of which flows into the left load and half into the right load. The right-hand transformer will generate a voltage $(V^i + V^r)/n$ in the line which causes an additional current $(V^i + V^r)/2nZ_0$ to flow into the left load and $-(V^i + V^r)/2nZ_0$ into the right load. As a consequence, we will have voltages V^i/n and $-V^r/n$ across the left and right loads respectively. We now have a device which, if placed in line, allows us to sample left- and right-travelling waves. Figure 7.18 shows the setup for measuring the S parameters of a two-port device. Since the output port is loaded with the characteristic impedance Z_0, we will have $V_2^i = 0$ and so $V_1^r = S_{11}V_1^i$ and $V_2^r = S_{21}V_1^i$. Consequently, we can obtain S_{11} and S_{21} by measuring the input and output voltage waves. To measure S_{12} and S_{22}, we simply reverse the DUT (device under test).

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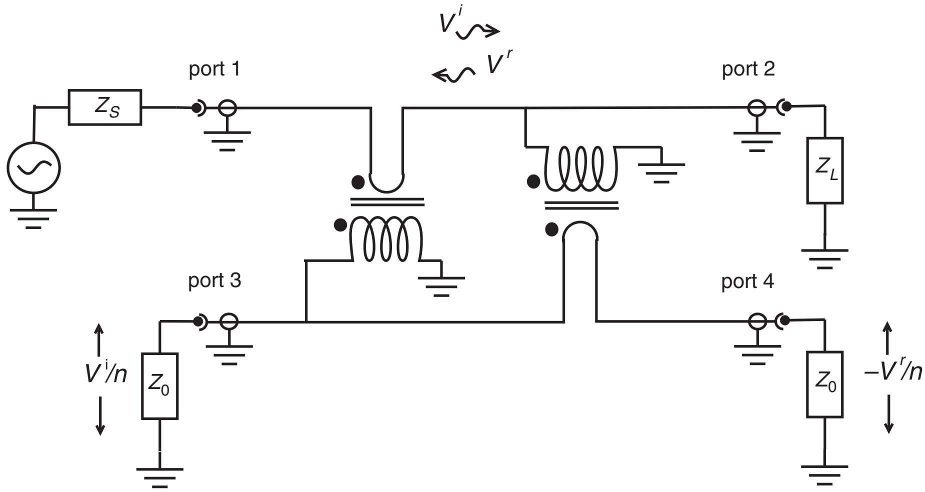

Fig. 7.17 Directional coupler.

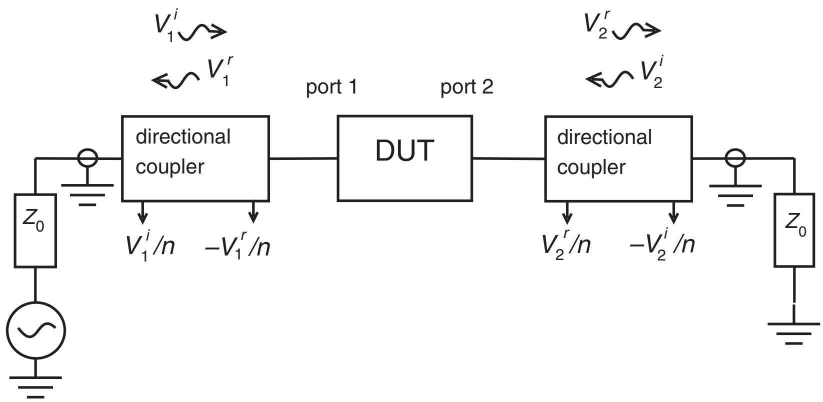

Fig. 7.18 The measurement of S parameters.

An amplifier is an important example of a two-port device and we now look at how its S parameters relate to its gain and stability. We first consider the device to be loaded at the output and so $V_2^i = \Gamma_L V_2^r$. Then, from (7.34),

$$V_1^r = S_{11} V_1^i + S_{12} \Gamma_L V_2^r$$
$$V_2^r = S_{21} V_1^i + S_{22} \Gamma_L V_2^r \tag{7.47}$$

and eliminating V_2^r between these equations,

$$\frac{V_1^r}{V_1^i} = S_{11} + \frac{S_{21} S_{12} \Gamma_L}{1 - S_{22} \Gamma_L}. \tag{7.48}$$

Consequently, the reflection coefficient looking in the device input, i.e. V_1^r/V_1^i, is given by

$$\Gamma_{\text{in}} = S_{11} + \frac{S_{21} S_{12} \Gamma_L}{1 - S_{22} \Gamma_L}. \tag{7.49}$$

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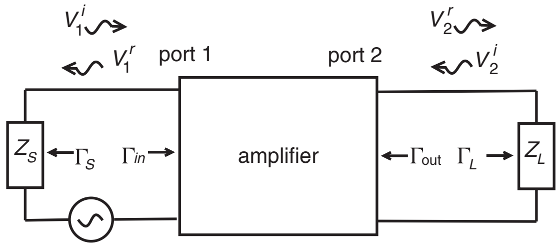

Fig. 7.19 Reflection coefficients for a loaded two-port device.

In a similar fashion, the reflection coefficient looking in the device output, i.e. V_2^r/V_2^i, is given by

$$\Gamma_{\text{out}} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S} \tag{7.50}$$

As we have seen, in the case of a one-port device, we need the reflection coefficient at a device port to be less than one in order to ensure stability. In the case of our two-port device, this will require that $\Gamma_{\text{in}} < 1$ and $\Gamma_{\text{out}} < 1$. Obviously, if there is feedback (i.e. if $S_{12} \neq 0$), there is possibility of instability. However, even if $S_{12} = 0$, there is still the possibility of instability through $S_{22} > 1$ and/or $S_{11} > 1$. As for a one-port active device, this can arise through negative resistance. As can be seen from (7.49) and (7.50), the stability of a device will depend on the source and load (both assumed to be stable, i.e. $|\Gamma_S| < 1$ and $|\Gamma_L| < 1$). Consequently, it is sometimes possible to stabilise a potentially unstable amplifier with the correct choice of load and/or source. Obviously, it would be preferable to have a device that is stable for all sources and loads. Such a device is said to be *unconditionally stable* and this will be the case if

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}||S_{21}|} > 1, \tag{7.51}$$

where $\Delta = S_{11}S_{22} - S_{21}S_{21}$ and K is known as the Rollett stability factor.

We now consider the *transducer power gain* of the amplifier, i.e. the power delivered to the load divided by the maximum power available from the source. Firstly, note that the voltage at port 1 is given by $V_1 = V_S Z_{\text{in}}/(Z_{\text{in}} + Z_S)$ where Z_{in} is the input impedance of the device (Z_S and Z_{in} act as a voltage divider). We note that $V_1 = V_1^r + V_1^i = V_1^i(1 + \Gamma_{\text{in}})$ and that $Z_{\text{in}} = Z_0(1 + \Gamma_{\text{in}})/(1 - \Gamma_{\text{in}})$. We therefore have that

$$V_1^i(1 + \Gamma_{\text{in}}) = V_S \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_S} = V_S \frac{Z_0(1 + \Gamma_{\text{in}})}{Z_0(1 + \Gamma_{\text{in}}) + Z_S(1 - \Gamma_{\text{in}})}. \tag{7.52}$$

We now note that $Z_L = Z_0(1 + \Gamma_S)/(1 - \Gamma_S)$ and, substituting this into (7.52), we obtain that

$$V_1^i = \frac{V_S}{2} \frac{1 - \Gamma_S}{1 - \Gamma_S \Gamma_{\text{in}}}. \tag{7.53}$$

From (7.47), the voltage wave arriving at the load will be

$$V_2^r = V_1^i \frac{S_{21}}{1 - S_{22}\Gamma_L} = \frac{V_S}{2} \frac{1 - \Gamma_S}{1 - \Gamma_S \Gamma_{\text{in}}} \frac{S_{21}}{1 - S_{22}\Gamma_L} \tag{7.54}$$

and so the power absorbed by the load will be

$$P_L = \frac{1}{8Z_0} \frac{|V_S|^2 |1-\Gamma_S|^2 S_{21}^2 (1-|\Gamma_L|^2)}{|1-\Gamma_S\Gamma_{\text{in}}|^2 |1-S_{22}\Gamma_L|^2} \tag{7.55}$$

i.e. the difference between the power arriving at the load and that reflected. The maximum power will be delivered by the source when there is a conjugate match between the source and the device input. Consequently, the maximum available power will be $P_{\text{av}} = |V_S|^2/8R_S$ where

$$R_S = \Re\{Z_S\} = \Re\left\{Z_0 \frac{1+\Gamma_S}{1-\Gamma_S}\right\} = Z_0 \frac{1-|\Gamma_S|^2}{|1-\Gamma_S|^2}. \tag{7.56}$$

As a consequence, the transducer gain will be given by

$$G_T = \frac{P_L}{P_{\text{av}}} = \frac{(1-|\Gamma_S|^2) S_{21}^2 (1-|\Gamma_L|^2)}{|1-\Gamma_S\Gamma_{\text{in}}|^2 |1-S_{22}\Gamma_L|^2}. \tag{7.57}$$

It is clear that the gain is maximum when $\Gamma_L = \Gamma_S = 0$, i.e. when input and output are matched for zero reflection.

7.5 Transmission-Line Transformers

From our various examples of transmission lines, we see that there are two distinct groups. The first group, known as unbalanced lines, use the ground (or a ground plane) for the return current and the second group, known as balanced lines, use an identical conductor that is quite separate from the ground. Telegraph, microstrip and coaxial lines are examples of unbalanced lines. Parallel-wire and parallel-strip lines are examples of balanced lines. We can regard balanced lines as pairs of unbalanced lines with the return currents through the ground cancelling each other out. If we consider the situation of Figure 7.20a, the currents flowing through the two conductors of the transmission line will be equal, provided that the loads are balanced (i.e. $Z_1 = Z_2$). There will be no flow down to the ground and this connection can be removed without any effect. If the loads are unbalanced (i.e. $Z_1 \neq Z_2$), however, the currents through the conductors will be unequal (i.e. $I_1 \neq I_2$) and there will be a net flow into the ground. Unfortunately, this will have a highly undesirable impact upon the operation of the transmission line. With a balanced load the radiation from the two conductors will cancel since they are closely spaced (the spacing is usually much less than a wavelength) and $I_2 = I_1$. With an unbalanced load, however, $I_1 \neq I_2$ and there will be net radiation from transmission line.

In general, the currents on a transmission line can be divided into a differential mode (equal magnitude currents I_d on the conductors in opposite directions) and a common mode (equal magnitude currents I_c on the conductors in the same direction). It is the common mode currents I_c that cause the radiation and which need to be removed for the proper operation of a transmission line (i.e. the transference of energy with minimum losses).

Now consider the case of an unbalanced transmission line that is connected to a balanced load (see Figure 7.20b). By its construction, the inside of the coaxial line can only support the differential mode. There is, however, a route for a common mode current I_c along the outside of the transmission line and this will result in radiation. From our

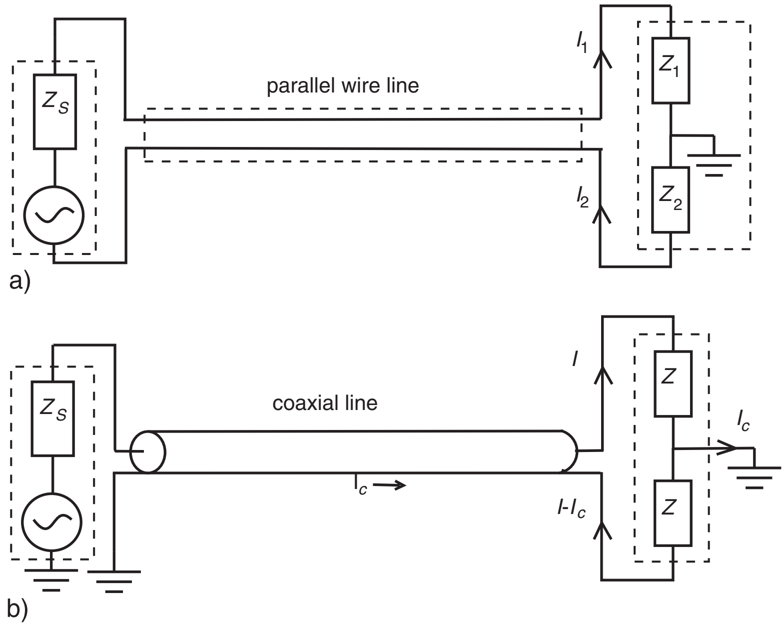

Fig. 7.20 Balanced and unbalanced loads.

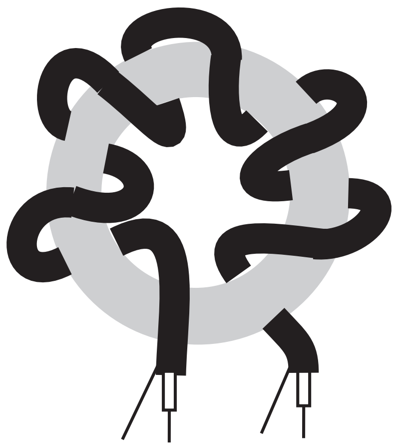

Fig. 7.21 Transmission line BALUN transformer using a coaxial transmission line.

above considerations, we see that the common mode currents arise if we try to connect a balanced transmission line to an unbalanced load, or a balanced load (a dipole antenna is an example of such a load) to an unbalanced transmission line. The solution to this problem is what is known as a BALUN (balanced-to-unbalanced) transformer. Such a transformer is shown in Figure 7.21 and consists of a short section of transmission line (usually less than 1/10 of a wavelength) that is wound on a low-loss iron core (a ferrite or iron-dust toroid is usually preferred). The common-mode current will now encounter

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a large reactance and so will be choked off by the BALUN. Through such a device, an unbalanced load can be connected to a balanced transmission line or a balanced load to an unbalanced transmission line.

A BALUN does not transform impedance and its main purpose is to provide a blockage to all but the differential mode. This property, however, can be used to make a transformer that does transform impedance. Consider the circuit of Figure 7.22 in which there is a transmission line that only allows the differential mode. We will calculate the average power P_L dissipated in the load R_L. From Kirchhoff's voltage law we will have that $v_S = v_1 + R_S(i_1 + i_2)$ and $v_1 + v_2 = i_2 R_L$. If the length of the transmission line l is much smaller than a wavelength, then $i_2 \approx i_1$ and $v_2 \approx v_1$. Consequently, $v_S \approx v_1 + 2R_S i_2$ and $v_1 \approx i_2 R_L/2$. Eliminating V_1, we then obtain that $i_2 \approx 2V_S/(R_L + 4R_S)$ and so power

$$\begin{aligned} P_L &= i_2^2 R_L \\ &\approx \frac{2v_S^2 R_L}{(4R_S + R_L)^2} \end{aligned} \tag{7.58}$$

will be dissipated in the load R_L (note that this expression is frequency-independent). Solving $\partial P_L/\partial R_L = 0$, we find that this power will be the maximum when $R_L \approx 4R_S$ (i.e. the transformer matches source R_S to load $4R_S$). The important thing to note is that we now have a 4:1 transformer that works at frequencies for which the transmission line is much shorter than a wavelength (less than $\lambda/10$). The lower frequency limit is set by the requirement that the inductance of the line be large enough to choke off the common mode. At the high-frequency end, the range can be extended by choosing a transmission-line impedance that is the geometric average of the source and load (i.e. $Z_0 = 2R$). Further examples of transformers that are based on the BALUN are shown in Figure 7.24. Figure 7.24a shows a 9:1 broadband transformer and Figure 7.24b shows a power combiner whose output is the sum of the two input signals (note that the input sources need to have twice the impedance of the load).

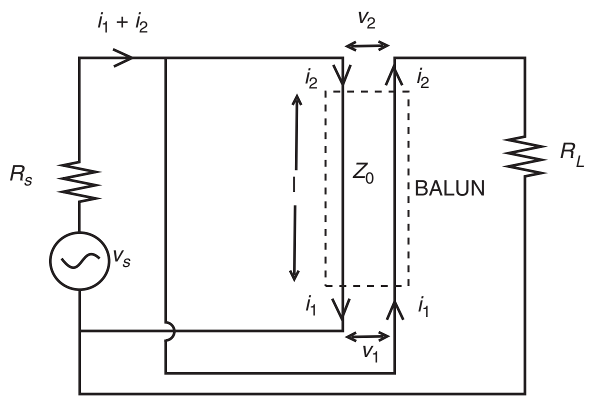

Fig. 7.22 Circuit of a transmission-line transformer.

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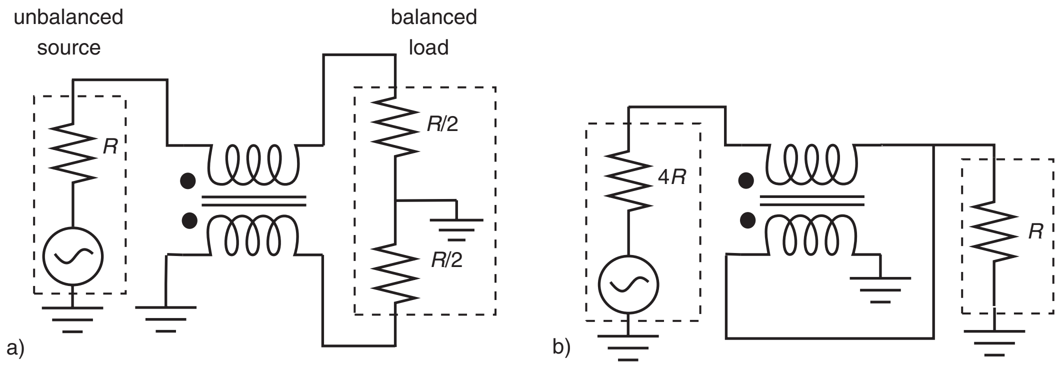

Fig. 7.23 A BALUN and a 4:1 broadband transformer.

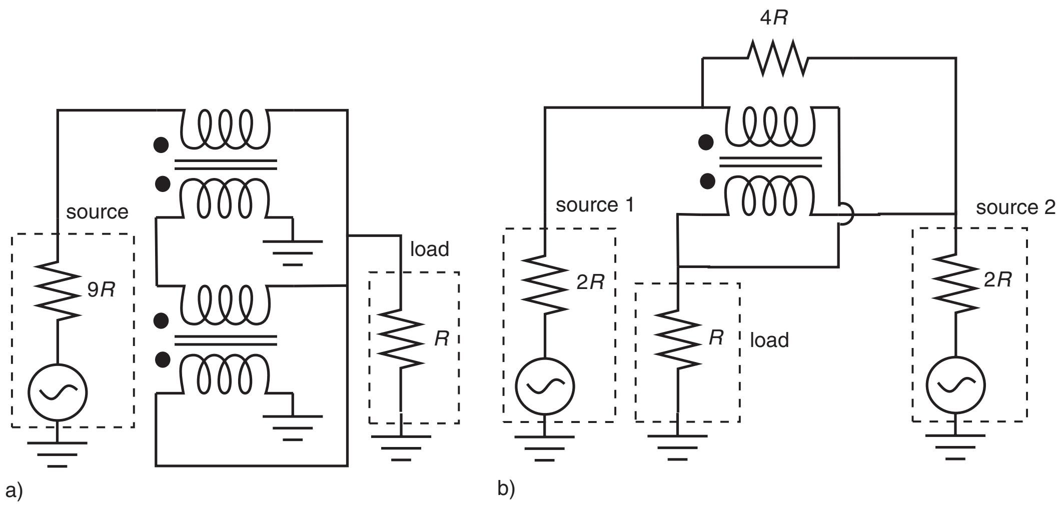

Fig. 7.24 A 9:1 broadband transformer and a combiner.

7.6 Conclusion

In the present chapter, we have studied transmission lines. These are the means by which RF power can be efficiently transferred over long distances and were the basis of the telegraph, the major means of communication before radio. We have found transmission lines have properties that allow us to use them to manufacture high-quality components, especially suitable for frequencies in the GHz range. In radio, the electromagnetic energy spreads out from an antenna that is usually fed by a transmission line, i.e. an antenna is the connector that joins a one-dimensional propagation medium to the three-dimensional propagation medium that is free space. In the next chapter we will study antennas in considerable detail.

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