CHAPTER

10 Transmission lines

We draw circuit diagrams with "lumped"components: ideal R's, C's, L's, transistors, etc., connected by lines that represent zero-length wires. But all real wires, if not much shorter than the shortest relevant wavelength, are themselves complicated circuit elements; the current is not the same everywhere along such a wire, nor is voltage uniform, even if the wire has no resistance. On the other hand, when interconnections are made with transmission lines, which are well-understood circuit elements, we can accurately predict circuit behavior. In this section we will consider two-conductor lines such as coaxial cables and open parallel wire lines. "Microstrip lines" (conducting metal traces on an insulation layer over a metal ground plane) behave essentially in the same way, but they have some subtle complications, which are mentioned in Appendix 10.1.

10.1 Characteristic impedance

The first thing one learns about transmission lines is that they have a parameter known as *characteristic impedance*, denoted Z_0. How "real" is characteristic impedance? If we connect an ordinary dc ohmmeter to the end of a 50-ohm cable will it indicate 50 ohms? Yes, if the cable is *very* long, so that a reflection from the far end does not arrive back at the meter before we finish the measurement. Otherwise, the meter will simply measure whatever is connected to the far end, which could be short, an open circuit, or a resistance. However, using a pulse generator and an oscilloscope, you can easily make an ohmmeter set-up that is fast enough that, even for a short cable, you can determine V_{in} and I_{in} and then calculate $V_{in}/I_{in} = Z_0$.

To make a theoretical determination of Z_0, we first model the transmission line as a ladder network made of shunt capacitors and series inductors, as shown in Figure 10.1.

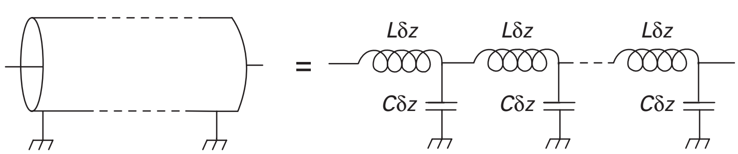

Figure 10.1. Transmission line model – a ladder network of infinitesimal LC sections.

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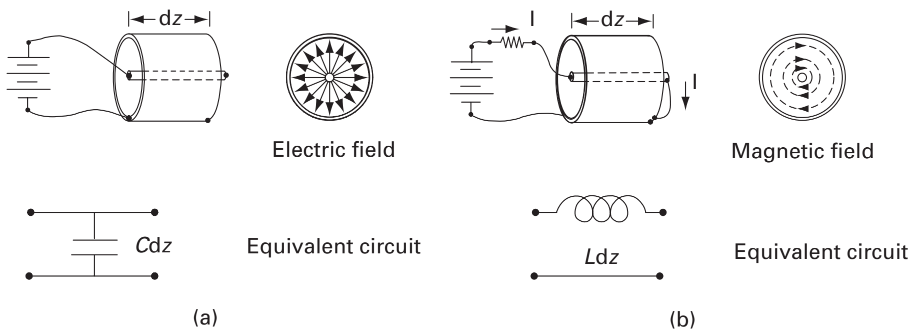

Figure 10.2. Capacitance and inductance per unit length.

To see that this model is reasonable, consider Figure 10.2(a), which shows the electric field lines in a length of coaxial cable connected to a voltage source. The field lines are radial and their number is obviously proportional to the length of the cable, so that capacitance per unit length is a constant. Likewise, a current through the cable (b) sets up a magnetic field, so another characteristic of the cable is its inductance per unit length. We will follow common convention and use the symbols C and L to denote capacitance and inductance per unit length. That convention is obvious when capacitors and inductors are labeled, respectively, $C\delta z$ and $L\delta z$, where δz is a short increment of length along the z-axis, i.e., parallel to the cable.

Every increment of a transmission line contributes series inductance and shunt capacitance; the ladder network shown in Figure 10.1 models a real transmission line in the limit that δz goes to zero. For some situations, e.g., baseband telephony and digital data transmission through long cables, the model must also include series and shunt resistance. At radio frequencies, however, the series reactance is usually much greater than the series resistance and the shunt reactance is usually much less than the shunt resistance so both resistances can be neglected. (See Problem 10.3.)

To see that $Z_0^2 = L/C$, consider the circuit of Figure 10.3, where we have added another infinitesimal LC section to the model transmission line, which is either infinitely long or terminated with a resistance equal to the characteristic impedance, so as to appear infinitely long. After adding the section, the line is still infinitely long and the impedance looking into it must still be Z_0. If the voltage and current at the input of the line were V and I, they will be modified to become $V+\delta V$ and $I+\delta I$ at the input to the new section. (This does not imply an increase in power; $V+\delta V$ and $I+\delta I$ are merely phase-shifted versions of V and I.)

Since the impedance looking into the line must stay the same, we have

$$\frac{V + \delta V}{I + \delta I} = \frac{V}{I}, \tag{10.1}$$

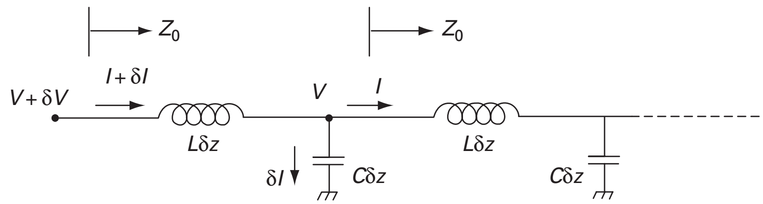

Figure 10.3. Adding another infinitesimal section must leave Z_0 unchanged.

from which $\delta V/\delta I = V/I = Z_0$.

Using this, and substituting $\delta I = (C\delta z)\ \mathrm{d}V/\mathrm{d}t$ and $\delta V = (L\delta z)\ \mathrm{d}/\mathrm{d}t(I + \delta I)$ and ignoring the vanishingly small $\delta z\ \delta I$ term, we have

$$Z_0 = \frac{\delta V}{\delta I} = \frac{L\delta z\,(\mathrm{j}\omega I)}{C\delta z\,(\mathrm{j}\omega V)} = \frac{L}{C}\,Z_0^{-1}. \tag{10.2}$$

Looking at the first and last terms of this equation, we see that $Z_0 = (L/C)^{1/2}$. Note: you can verify that, because δV and δI are infinitesimal, the above equations are the same if the network starts with a capacitor instead an inductor.

To evaluate Z_0, it is sufficient to know either L or C, since it follows from electrodynamics that they are related by $LC = \varepsilon_r/c^2$ where ε_r is the dielectric constant (relative to vacuum), and c is the speed of light. This relation between L and C holds for any two-conductor structure with translational symmetry such as an unlikely transmission line consisting of a square inner conductor inside a triangular outer conductor.

For a coaxial transmission line, $C = 2\pi\varepsilon_r\varepsilon_0 / \ln(b/a)$ farads/meter, where a and b are the inner and outer radii and ε_0, the "permittivity of free space," is equal to $(4\pi \times 10^{-7}c^2)^{-1}$. Using this, together with the relation $LC = \varepsilon_r/c^2$, gives us $Z_0 = (\varepsilon_r)^{-1/2}\ 60\ \ln\ (b/a)$. Note that Z_0 depends on the ratio a/b, but not on the size of the cable.

10.2 Waves and reflected waves on transmission lines

We will use a simple ac analysis to show that an applied sinusoidal voltage causes a spatial voltage sine wave to propagate down the line: Let us apply a voltage $e^{\mathrm{j}\omega t}$ and find the voltage drop across an incremental length of line (see Figure 10.4).

Since we already know the input impedance is Z_0, the input current must be V/Z_0 and the voltage across the inductor can be written $\delta V = -(V/Z_0)\ (\mathrm{j}\omega L\delta z)$. But this is just the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -\mathrm{j}\omega\frac{L}{Z_0}V = -\mathrm{j}\omega\sqrt{LC}\,V. \tag{10.3}$$

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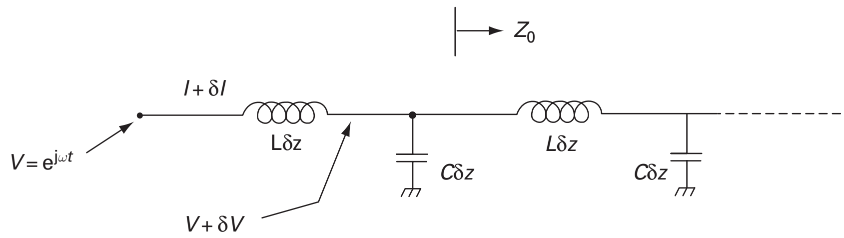

Figure 10.4. Finding the change in voltage, δV, over a distance δz.

The solution to this familiar equation is

$$V = V_f e^{-j\omega\sqrt{LC}z} = V_f e^{-jkz} \quad \text{where} \quad k = \omega\sqrt{LC}, \tag{10.4}$$

where V_f is a constant, the amplitude. The constant k is known as the *propagation constant* and is the number of radians the wave progresses per unit length. The wave therefore repeats in a distance (the wavelength) given by $\lambda = 2\pi/k$. Since $V/I = Z_0$, the current along the line is also a wave: $I = (V_f/Z_0)e^{-jkz}$. If we include the otherwise implicit multiplicative time dependence factor $e^{j\omega t}$, the voltage is

$$V = V_f e^{j\omega t} e^{-jkz} = V_f e^{j(\omega t - kz)}. \tag{10.5}$$

This is just a sine wave running in the forward z-direction. The complex exponential now contains space as well as time but, as always, the physical voltage is the real part, i.e., Re[$V_f e^{j(\omega t - kx)}$] which is a weighted superposition of $\sin(\omega t - kz)$ and $\cos(\omega t - kz)$. For a point of constant phase, $\omega t - kz =$ constant, we have $\delta z/\delta t = \omega/k$. This velocity, $\omega/k = c/\sqrt{\varepsilon_r}$, is known as the *phase velocity,* v_{phase}. Figure 10.5 shows a forward-running wave on a coaxial cable. The electric and magnetic field lines are drawn only at the points where they reach their peak values. A graph shows the spatial distribution. Everything has the same phase, i.e., the voltage, current, and charge density all rise and fall together along the z-axis. Note that a wave of amplitude V transfers power at a rate $|V|^2/(2Z_0)$.

A transmission line can equally well support waves running in the negative z-direction. If we had assumed a current in the $(-z)$-direction, the phase would progress as $\omega t + kz$. A transmission line in a circuit operating at a frequency ω will, in general, have both a forward wave and a reverse wave. The waves have complex amplitudes, V_f and V_r, each containing magnitude and phase. Of course both waves have the same frequency and propagation constant. We regard current as positive when it is in the (+z)-direction, so the current of a forward wave is $I_f(z,t) = V_f(z,t)/Z_0$, but the current of a reverse wave is $I_r(z,t) = -V_r(z,t)/Z_0$, since the reverse wave is traveling in the $(-z)$-

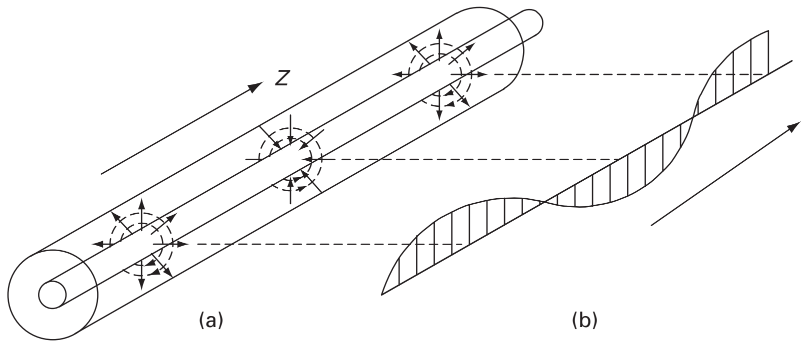

Figure 10.5. Forward wave on a transmission line.

direction. Together, the forward and reverse waves are, in general, equivalent to a stationary (*standing*) wave plus a single propagating wave.

Note also that the phase velocity is independent of ω; there is no dispersion in this kind of lossless transmission line. Therefore, if we apply an arbitrary voltage waveform, $V_{\text{arb}}(t)$, at the input to the line, this waveform, considered as a Fourier superposition of sine waves, will propagate down the line without distortion. At any point z, the voltage will be $V_{\text{arb}}(t-z/v_{\text{phase}})$, a delayed but undistorted version of the input signal. For example, if, at $t=0$, we connect a dc voltage to the line, a step function propagates down the line.

The *electrical length* of a line is the phase change imparted by the line. For example, a "quarter wave line" imparts a 90° phase shift, $kl=\pi/2$, and therefore $l=\pi/(2k)=\pi c/(2\omega\sqrt{\varepsilon_r})=(c/f)/(4\sqrt{\varepsilon_r})=1/4(\lambda_0/\sqrt{\varepsilon_r})$, where λ_0 is the wavelength in free space.

Standing waves

When both a forward and a reverse wave are present on a transmission line the voltage along the line, which is the sum of the contributions from the two waves, forms an interference pattern or standing wave. To see this, let $V(z,t)=V_f e^{j(\omega t-kz)}+V_r e^{j(\omega t+kz)}$. The real parts of these two rotating phasors will be in phase at points along the transmission line which are separated by $\lambda/2$. At these points, the magnitude of the sum will be $|V_f|+|V_r|$. Halfway between these points, the real parts of the phasors will be out of phase and the magnitude of the sum will be $||V_f|-|V_r||$. The ratio of these maximum and minimum voltage magnitudes is called the *voltage standing wave ratio*: $\text{VSWR}=(|V_f|+|V_r|)/||V_f|-|V_r||$. If $|V_f|=|V_r|$ there is only a standing wave and the VSWR is infinite. When $|V_f|\neq|V_r|$, the weaker one, along with an equal portion of the stronger one, form a standing wave, leaving the remainder of the stronger one as a travelling wave.

10.3 Modification of an impedance by a transmission line

From the discussion above, you can see that a transmission line terminated by a resistor of value Z_0 will always present an input impedance of Z_0. But a piece of transmission line that is terminated with an arbitrary impedance, $Z \neq Z_0$, as shown in Figure 10.6, will produce a modified ("transformed") impedance, Z'.

This figure shows a line of length l whose right-hand end ($z = 0$) is connected to some impedance Z_L (L denotes "load"). Assume that some constant ac source produces a constant incident wave traveling to the right, $V_f e^{-jkz}$ (we will not bother writing the always present factor $e^{j\omega t}$), and that Z_L causes a constant reflected wave, $\Gamma V_f e^{jkz}$, to travel to the left.[1] The factor Γ is known as the *reflection coefficient*. At any point, z, the voltage on the line is $V(z) = V_f e^{-jkz} + \Gamma V_f e^{jkz}$. The corresponding current is $I(z) = (V_f/Z_0)(e^{-jkz} - \Gamma e^{jkz})$. The minus sign occurs because the current in the reflected wave flows in the negative z-direction. At the right-hand end ($z = 0$), the load ensures that $V(0)/I(0) = Z_L$. This will give us Γ:

$$\frac{V(0)}{I(0)} = Z_L = \frac{(1+\Gamma)}{(1-\Gamma)/Z_0} \quad \text{so} \quad \frac{Z_L}{Z_0} = \frac{(1+\Gamma)}{(1-\Gamma)} \quad \text{and} \quad \Gamma = \frac{(Z_L - Z_0)}{(Z_L + Z_0)}. \tag{10.6}$$

Putting this expression in Equation (10.6) for Γ, together with the expressions for $V(z)$ and $I(z)$, we can immediately find $V(-l)/I(-l)$ which is what we are after, i.e., Z', the input impedance at a point l to the left of the load:

$$\begin{aligned} Z' = \frac{V(-l)}{I(-l)} &= \frac{e^{-jk(-l)} + \Gamma e^{jk(-l)}}{e^{-jk(-l)}/Z_0 - \Gamma e^{jk(-l)}/Z_0} \\ &= Z_0 \frac{(Z_L + Z_0)e^{jkl} + (Z_L - Z_0)e^{-jkl}}{(Z_L + Z_0)e^{jkl} - (Z_L - Z_0)e^{-jkl}} \end{aligned}$$

or

$$Z' = Z_0 \frac{Z_L + jZ_0 \tan(kl)}{Z_0 + jZ_L \tan(kl)}. \tag{10.7}$$

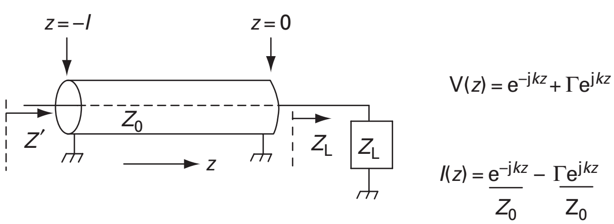

Figure 10.6. An impedance is modified when seen through a transmission line.

[1] Since everything is linear, superposition holds and the incident and reflected waves do not collide or interact in any way. They simply pass through one another unaltered. At any point, the current is the sum of their currents and the voltage is the sum of their voltages.

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This important result, the modification of an impedance Z_L by a length l of transmission line, is not hard to remember; it has no minus signs and is symmetric. Just remember (1+j tan)/(1+j tan). Once you have written this framework, you will remember how to put in the coefficients. Some important special cases are listed below:

- If $Z_L = Z_0$, then $Z' = Z_0$ for any length of line.
- If $Z_L = 0$ (a short) then $Z' = jZ_0\tan(kl)$, a pure reactance, which is inductive[2] for $kl < \pi/2$, then capacitive, etc.
- If $Z_L =$ infinity (an open circuit) then $Z' = Z_0/j\tan(kl)$ which is capacitive for $kl < \pi/2$, then inductive, etc.
- An impedance is left unchanged by a line of arbitrary Z_0 whose length is a half-wave ($kz = \pi$) or any integral multiple of a half-wave.
- A quarter-wave line ($kz = \pi/2$) or an odd multiple of a quarter-wave line, *inverts* an impedance: $Z' = Z_0^2/Z_L$. A short is transformed into an open and an open into a short, an inductor is transformed into a capacitor and vice versa, etc.

10.4 Transmission line attenuation

In a lossy transmission line, i.e., a line that causes attenuation of the signal, the e^{-jkz} or e^{jkz} spatial dependence of the wave is replaced by $e^{-jkz}\,e^{-\alpha z} = e^{-j(k-j\alpha)z}$ (forward wave) or $e^{jkz}\,e^{\alpha z} = e^{j(k-j\alpha)z}$ (reverse wave), where α is the attenuation constant. In a distance $1/\alpha$, the amplitude falls by a factor 1/e and the power falls by a factor $(1/e)^2$. Note that k for a lossless line is simply replaced by $k - j\alpha$, i.e., the propagation constant becomes complex. You can put this complex k into the "tan tan" formula to see how an impedance is modified by a lossy cable.

Transmission line attenuation is usually expressed in units of dB/meter. To find α for a line whose loss is A_L dB/m, note that, since the amplitude falls by a factor $e^{-\alpha\cdot 1}$ in 1 meter, we can write $-A_L = 10\log(e^{-\alpha\cdot 1})^2 = -20\alpha\log(e)$ from which $\alpha = A_L/(20\log(e))$.

10.5 Impedance specified by reflection coefficient

We have seen that an impedance Z produces a reflection coefficient given by $\Gamma = (Z - Z_0)\,/\,(Z + Z_0)$. This relation is easily inverted, $Z = Z_0(1+\Gamma)/(1-\Gamma)$, so there is a one-to-one mapping between Z and Γ. In antenna and microwave work, especially when using S-parameter analysis (Chapter 28), it is customary to think in terms of Γ, rather than Z.

One big advantage of working in the complex Γ-plane is that the modification of an impedance (represented by its equivalent Γ) is extremely simple. The

[2] Note that "inductive" does not mean equivalent to a lumped inductor since $Z_0\tan(kl) = Z_0\tan(\omega l/v_{phase})$ is not proportional to ω, except for small kl. Likewise, a short open-ended line is not equivalent to a lumped capacitor, except for small $\omega l/v_{phase}$.

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reflection coefficient for the given impedance as seen through a length l of transmission line is just

$$\Gamma' = \Gamma e^{-j2kl}, \tag{10.8}$$

which means we simply rotate the point clockwise around the origin ($\Gamma = 0$) by an angle $2kl$ to give Γ', the modified reflection coefficient. This is easy to see: when we add a length of cable, the incident wave's phase is delayed by kl getting to the end of the cable and the reflected wave is delayed by the same kl getting back again. The effect of a cable is therefore to rotate the complex number Γ clockwise by an angle $2kl$.[3] (Since the time dependence is $e^{j\omega t}$, the round-trip time delay is a clockwise displacement.) Keep in mind that the Γ-plane is a complex plane but that it is *not* the $R + jX$ plane. Let us look at a few special points in the Γ-plane.

1. The center of the plane, $\Gamma = 0$, corresponds to a reflected wave of zero amplitude, so this point represents the impedance $Z_0 + j0$.
2. The magnitude of Γ (radius from the origin) must be less than or equal to unity for passive impedances. Otherwise the reflected wave would have more power than the incident wave.
3. The point $\Gamma = -1+j0$ corresponds to $Z = 0$, a short circuit.
4. The point $\Gamma = 1+j0$ corresponds to $Z = \infty$, an open circuit.
5. Points on the circle $|\Gamma| = 1$ correspond to pure reactances, $Z = 0+jX$. All points inside this circle map to impedances with positive nonzero R.
6. The point $\Gamma = 0+j1$ corresponds to an inductance, $Z = 0+jZ_0$. All points in the top half of the Γ-plane are "inductive," i.e., $Z = R+j|X|$ or, equivalently, $Y = G-j|B|$.
7. The point $\Gamma = 0-j1$ corresponds to a capacitance, $Z = 0-jZ_0$. All points in the bottom half of the Γ-plane are "capacitive," i.e., $Z = R-j|X|$ or, equivalently, $Y = G+j|B|$.

These special cases of mapping of Z into Γ are shown in Figure 10.7.

In the Γ-plane, if you plot $\Gamma = R + jX$, where R is a constant and X varies, you will get a circle centered on the real axis and tangent to the line $\mathrm{Re}(\Gamma) = 1$. For every value of R there is one of these "resistance circles." The resistance circle for $R = 0$ is the unit circle in the Γ-plane. The resistance circle for $R = \infty$ is a circle of zero radius at the point $\Gamma = 1+j0$. Likewise, if you plot $\Gamma(R+jX)$ where X is a constant and R varies, you will get "reactance circles" centered on the line $\mathrm{Re}(\Gamma) = 1$ and tangent to the line $\mathrm{Im}(\Gamma) = 0$. These circles are shown in Figure 10.8.

If you now trim the circles to leave only the portions within the $|\Gamma| = 1$ circle (corresponding to passive impedances, i.e., impedances whose real part is

[3] If the line is lossy, the magnitude of Γ decreases as it rotates around the origin, forming a spiral. For a long enough length of lossy line, Γ spirals all the way into the origin producing $Z = Z_0$, no matter what value of Z terminates the far end of the cable.

Figure 10.7. Impedances mapped into the reflection plane.

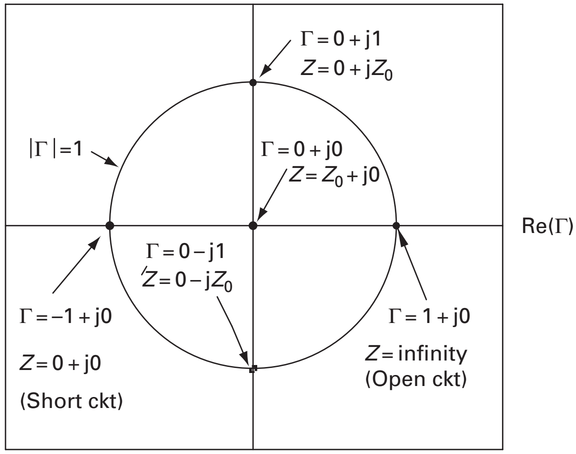
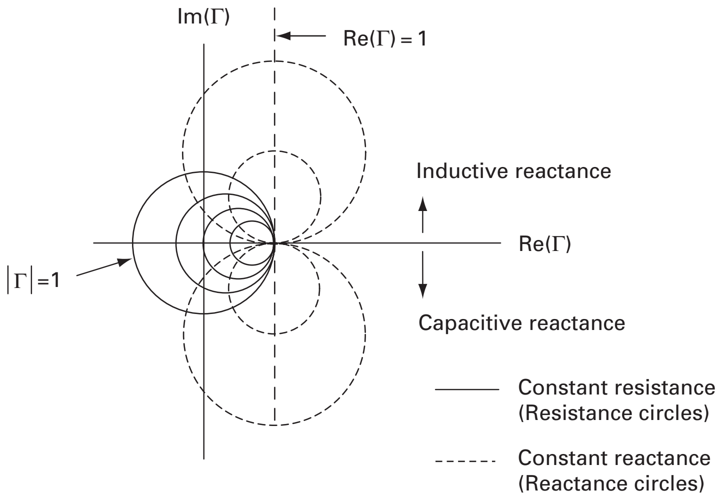


Figure 10.8. Loci of constant resistance and of constant reactance – circles in the Γ-plane.



positive) you are left with a useful piece of graph paper, the famous *Smith chart*, shown in Figure 10.9.

The circular R and X "axes" on the Smith chart allow you to locate the Γ-point that corresponds to $Z=R+jX$. We have already seen that when we have located an impedance on the Γ-plane, we can find how that impedance is modified by a length of transmission line (whose Z_0 is the same as the Z_0 used to draw the chart) by rotating the point clockwise around the origin. We simply rotate the point clockwise around the origin by an angle equal to twice the electrical length of the line. The values of R and X corresponding to the rotated point can be read

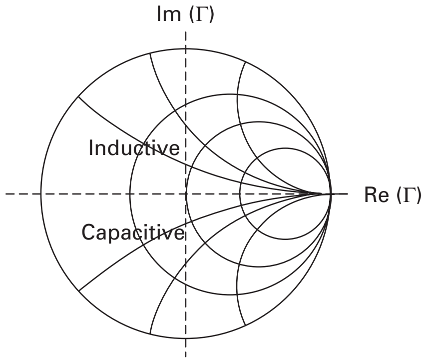


Figure 10.9. The Smith chart – resistance and reactance circles on the Γ-plane.

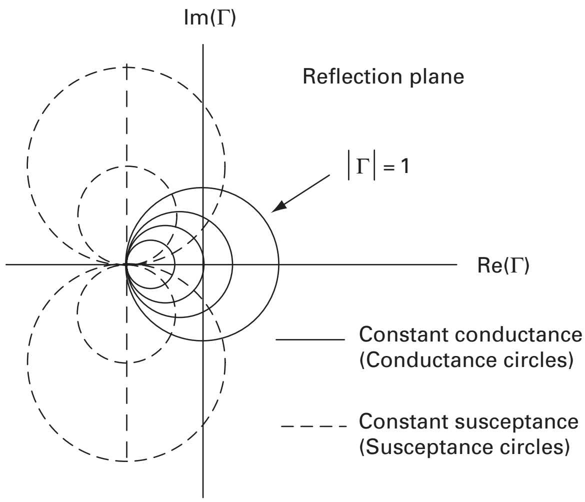


Figure 10.10. Conductance and susceptance circles.

off the chart's R and X "axes." We can also use the chart to find how an impedance is modified by adding a series R or series X. In this operations, the Smith chart can be considered something of a calculator. Note that the Smith chart can also be made with G and B "axes". As you might guess, these produce "G circles" and "B circles" as shown in Figure 10.10.

Sometimes the Smith chart contains G and B circles as well as R and X circles. This full-blown chart, which can be quite dense, is shown in Figure 10.11. Again, remember the Smith chart is actually a rectangular graph of Γ; the x-axis is Re(Γ) and the y-axis is Im(Γ). Because only the area inside the circle $|\Gamma| = 1$, i.e., $x^2+y^2 = 1$, is used, the Smith chart resembles a polar graph. And, indeed, when we rotate a point around the origin to how a transmission line modifies an impedance, we are using it in a polar fashion. Sometimes the Smith chart is scaled for a specific Z_0 (usually 50 ohms or 75 ohms). Other charts are normalized; the $R = 1$ circle would be the 50-ohm circle if we are dealing with 50-ohm cable, etc.

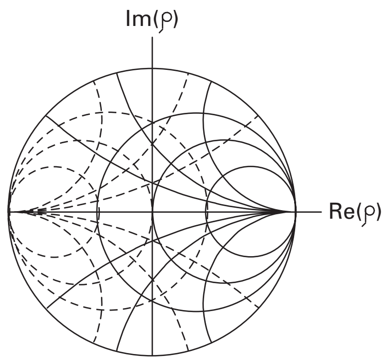


Figure 10.11. Smith chart with *R*, *X*, *G*, and *B* circles.

10.6 Transmission lines used to match impedances

Designing a matching network becomes an exercise in moving from a given Γ to a desired Γ′ in the reflection plane. Working graphically, it is often easy to find a matching strategy. Let us use the Smith chart and revisit the 1000-ohm-to-50-ohm matching circuit example of Chapter 2.

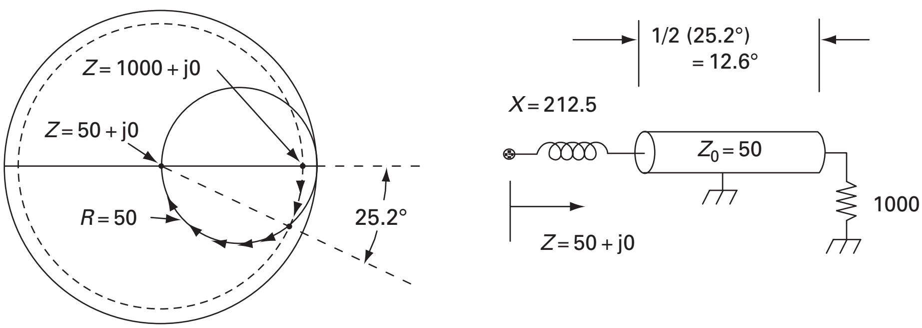


Figure 10.12. Conversion from 1000 ohms to 50 ohms – transmission line and inductor circuit.

I. The starting impedance, 1000 ohms, and the final target impedance, 50 ohms, are indicated on the chart in Figure 10.12. Also shown is the 50-ohm circle. We can use a (50-ohm) transmission line to move along the dashed circle until we reach the 50-ohm circle. Now we have $R = 50$ plus a capacitive reactance. A series inductor will cancel the capacitive reactance, taking us to $Z = 50+j0$ (the center of the chart).

II. Another solution (Figure 10.13) would be to use a longer piece of cable to circle most of the chart, hitting the 50-ohm circle in the top half of the plane. At this point we have $Z = 50 + jX$ where X is positive (inductive). We can add a series capacitor to cancel this X and again arrive at $Z = 50 + j0$.

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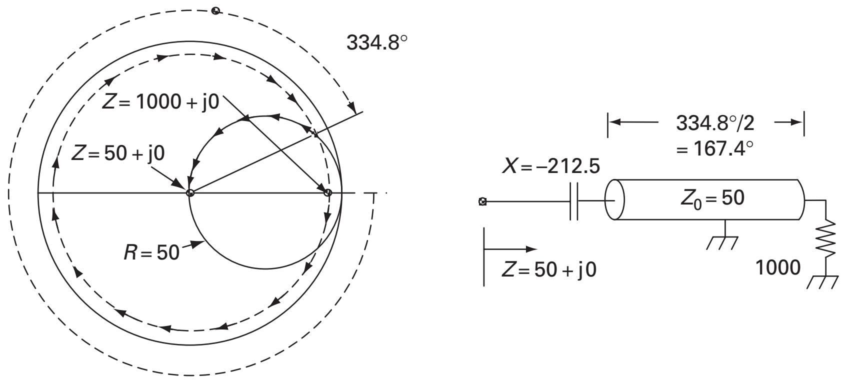


Figure 10.13. Transmission line and capacitor matching circuit.

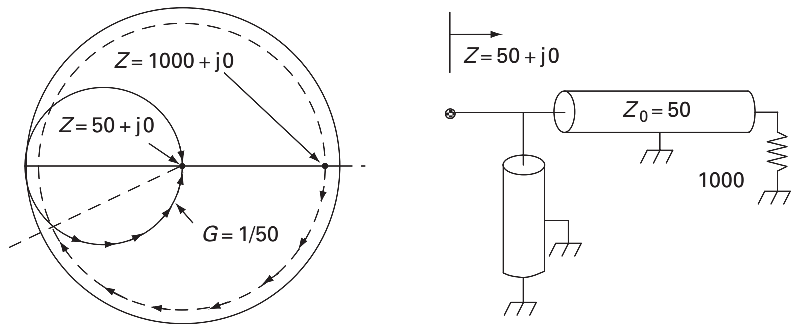


Figure 10.14. Series and shunt transmission line matching circuit.

III. So far we have only used series elements. Let us now start by traveling around to the $G = 1/50$ circle. Then we can add a shunt element to reach the center of the chart. The first intersection of the $G = 1/50$ circle is in the lower half-plane (capacitive) so, to get from this point to the center, we need a shunt inductor. Instead of a lumped inductor we might use a shorted length of transmission, as shown in Figure 10.14, to make a matching circuit using only transmission line elements.

IV. Figure 10.15 shows a solution that uses no transmission line. We start on the $G = 1/1000$ circle, at $G = 0$. If we apply shunt reactance we can move along this circle. Let us pick shunt inductance which will move us upward along the G circle to the 50-ohm circle. We now have $R = 50$, but there is inductive reactance. As in the above example, we can now cancel the inductance reactance with a series capacitor. This is just the L-network found in Chapter 2.

V. If we had used shunt capacitance rather than shunt inductance, we would have moved downward to the 50-ohm circle, as shown in Figure 10.16. The remaining series capacitance can be cancelled with an inductor. This produces an L-network where the positions of the L and C are reversed.

Figure 10.15. *LC* matching network.

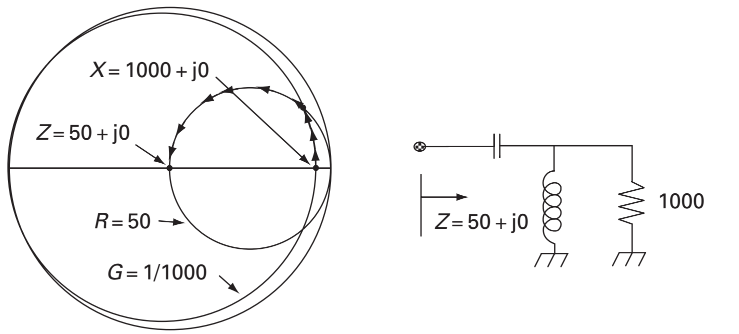


Figure 10.16. *CL* matching network.

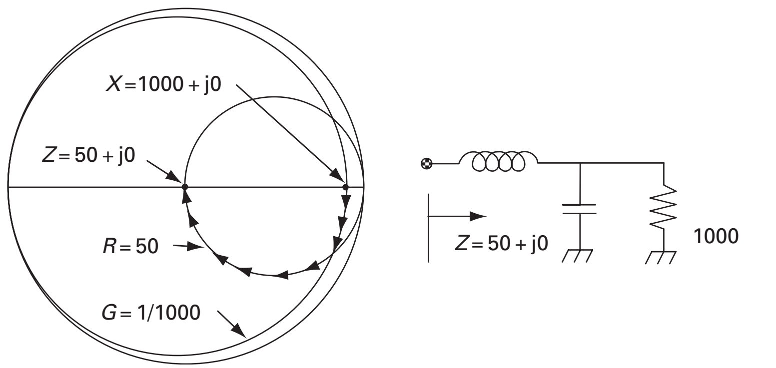
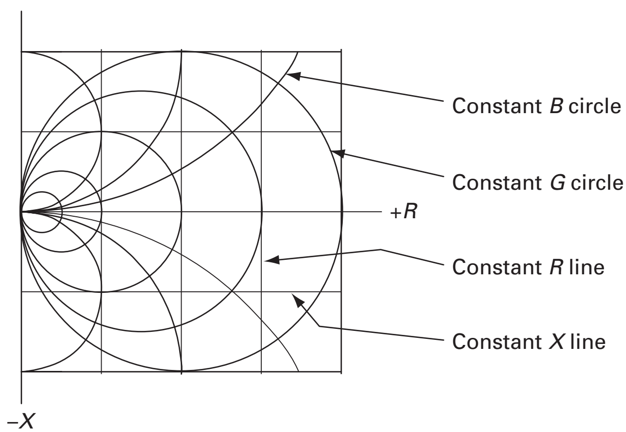


Figure 10.17. An impedance-admittance chart.



In these examples, our final impedance was at the center of the chart ($Z = 50 + j0$), but you can see that these techniques allow us to transform any point on the chart (i.e., any impedance) into any other point on the chart (any other impedance).

The Smith chart is a favorite because it handles networks that include transmission lines as well as inductors and capacitors. If we did not care about transmission lines, then any chart that maps R, X into G, B would do. For example, take the R, X plane (half-plane, since we will exclude negative R). Draw in the curves for G = constant and B = constant. The resulting chart, shown in Figure 10.17, can be used to design lumped element L, C, R ladder networks, such as the networks of Figures 10.15 and 10.16.

Appendix 10.1. Coaxial cable – Electromagnetic analysis

This chapter began with a derivation of Z_0 based on an equivalent lumped-element circuit model of a transmission line. That derivation required only elementary ac circuit theory, but is a rather indirect approach to what is really a problem in electromagnetics. Even then, some electromagnetic theory is needed to derive the expressions for capacitance and inductance per unit length.

An electromagnetic analysis of a coaxial transmission line is presented here for the reader who has some familiarity with Maxwell's equations. We make use of the fact that the propagation velocity of a *TEM wave*[4] is given by $v = (\mu\varepsilon)^{-1/2}$, where μ is the magnetic permeability and ε is the electrical permitivity of the material through which the fields propagate.[5]

To find the impedance of the coaxial line, we will first assume that the current on the inner conductor is given by $I = I_0 \cos(\omega t - kz)$, which is a wave traveling in the $(+z)$-direction. This is illustrated in Figure 10.18.

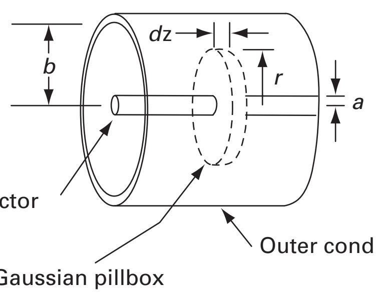


Figure 10.18. Transmission line element.

We will then proceed to find the charge density, the electric field, and then the voltage, which will have the form $V_0 \cos(\omega t - kz)$. Once we have the voltage, the

[4] In a TEM wave, by definition, both the electric field and the magnetic field are transverse, i.e., perpendicular to the direction along which the wave propagates. In most applications of coaxial cables and parallel-wire transmission lines, the wavelength is much greater than the transverse dimensions of the line and only TEM waves can propagate. Waves in free space are also TEM waves.

[5] To find the propagation velocity of a TEM wave: The variables t and z appear in E_x, E_y, B_x, and B_y only in the factor $e^{j(\omega t - kz)}$. Using the condition $E_z = 0$, the x-component of the Maxwell equation $\mathrm{curl}(E) = -\partial B/\partial t$ gives us $B_x = -(k/\omega)E_y$. Likewise, using the condition $B_z = 0$, the y-component of the Maxwell equation $\mathrm{curl}(B/\mu) = \partial(\varepsilon E)/\partial t$ gives us $B_x = -(\omega\mu\varepsilon/k)E_y$. Equating these two expressions for B_x gives $k^2 = \omega^2\mu\varepsilon$.

characteristic impedance is simply given by V_0/I_0. (Note that the current in the outer conductor is just the negative of the current in the inner conductor.)

Consider an incremental segment of the inner conductor from z to $z + \delta z$. The rate at which charge accumulates on this element is $\partial/\partial t\ (\rho_L)\delta z$, where ρ_L is the charge per unit length. But the rate at which charge accumulates in δz is nothing more than the difference between the current flowing into δz and the current flowing out of δz. Therefore, we can write

$$\frac{\partial \rho_L}{\partial t} = \frac{-\partial I}{\partial z} = -k\, I_0 \sin(\omega t - kz). \tag{10.9}$$

Integrating $d\rho_L/dt$ with respect to time gives us the linear charge density (coulombs/meter):

$$\rho_l(z,t) = \frac{1}{\omega}(kI_0 \cos(\omega t - kz)) = \frac{kI_0}{\omega}\cos(\omega t - kz). \tag{10.10}$$

Now that we know the charge density, we can find the electric field. The field is radial with field lines like spokes of a wheel. Imagining a Gaussian "pillbox" of radius r and height δz around the center conductor, we use Gauss's law: the integral of the E field over the sidewall surface must be equal to the enclosed charge divided by ε:

$$E(r,z,t)(2\pi r\delta z) = \frac{1}{\varepsilon}\rho_l(z,t)\delta z. \tag{10.11}$$

Substituting for λ and solving for E, we have

$$E(r,z,t) = \frac{kI_0 \cos(\omega t - kz)}{2\pi r \varepsilon \omega}. \tag{10.12}$$

Integrating this electric field from $r = a$ to $r = b$ gives us the voltage between the inner and outer conductors:

$$V(z,t) = \int_a^b E(r,z,t)dr = \frac{kI_0 \cos(\omega t - kz)\ln(b/a)}{2\pi\varepsilon\omega}. \tag{10.13}$$

Finally, we divide $V(z,t)$ by $I(z,t)$ to get the characteristic impedance:

$$Z_0 = \frac{V(z,t)}{I(z,t)} = \frac{k\ln(b/a)}{2\pi\varepsilon\omega} = \frac{\ln(b/a)}{2\pi\varepsilon v} = \frac{1}{2\pi}\left(\frac{\mu}{\varepsilon}\right)^{1/2}\ln(b/a), \tag{10.14}$$

which is the same as the result we obtained using the δL δC ladder network equivalent circuit.

This derivation (as well as the LC derivation) for Z_0 is for TEM waves, where both $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to z. For TEM solutions to exist, the line must be uniformly filled with homogenous dielectric material or vacuum. The dielectric can be lossy, but the metal conductors must, strictly speaking, have no resistance. In practice, these conditions are usually not satisfied perfectly, and the waves will be slightly different from the TEM waves corresponding to ideal conditions. In particular, the waves will have a small E_z or H_z field, or both. Microstrip lines are a case of nonuniform dielectric; some of the E-field lines arch through the air above the conductor, before plunging through the dielectric to the ground plane. The wave must have a unique phase velocity, but $(\mu\varepsilon)^{-1/2}$ has one value in the air and another value in the dielectric. The waves, therefore, cannot be TEM. They turn out to have both E_z and H_z components. Known as quasi-TEM waves, they show some frequency dependence in both Z_0 and v_{phase}, which can be important at millimeter-wave frequencies. Closed form expressions have not been derived for a microstrip; designers find Z_0 and v_{phase} vs. frequency by using graphs or approximate formulas based on numerical solutions of Maxwell's equations.

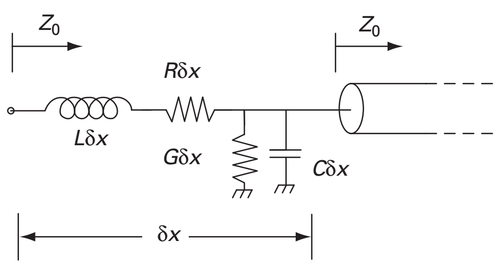
Problems

Problem 10.1. A common 50-ohm coaxial cable, RG214, has a shunt capacitance of 30.8 pF/ft. Calculate the series inductance per ft and the propagation velocity.

Problem 10.2. (a) Use the "tan tan" formula to show that a short length, δz, of transmission line, open-circuited at the far end, behaves as a capacitor, i.e., that it has a positive susceptance, directly proportional to frequency. Express the value of this capacitor in terms of the cable's capacitance per unit length. (Hint: $\tan(\theta) \approx \theta$ for small θ.)

(b) Show that a short length, δz, of transmission line, short-circuited at the far end, acts as an inductor, i.e., that it has a negative susceptance inversely proportional to frequency. Express the value of this inductor in terms of the cable's inductance/unit length.

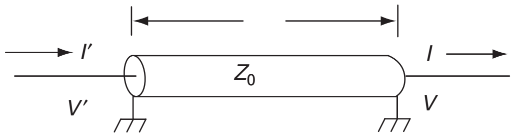
Problem 10.3. (a) Find a formula for the characteristic impedance of a lossy cable where the loss can be due to a series resistance per unit length, R, as well as a parallel conductance per unit length, G. R represents the ohmic loss of the metal conductors while G represents dielectric loss.



Hint: You can generalize the result for the lossless cable by simply replacing L by $L+R/(j\omega)$ and C by $C+G/(j\omega)$.

(b) Find the formula for the propagation constant k of this lossy cable. Hint: apply the substitutions given above to the formula $k = \omega\sqrt{LC}$. What distance (in wavelengths) is required to reduce by 1/e the power of a signal at frequency ω_1 if $R/(\omega_1) = 0.01L$?

Problem 10.4. If the (sinusoidal) voltage, V, and current, I, at the right-hand end of a transmission line are given, find the corresponding voltage, V', and current, I', at the left-hand end.



Hint: assume the (complex) voltage on the line is given by $V(\phi) = V_F e^{-j\phi} + V_R e^{j\phi}$. The corresponding current is given by $Z_0 I(\phi) = V_F e^{-j\phi} - V_R e^{j\phi}$. Let $\phi = 0$ at the right-hand end. Show that $V_F = (V{+}IZ_0)/2$ and $V_R = (V - IZ_0)/2$. Then show that, at the left-hand end, where $\phi = -\theta$, that $V' = V\cos\theta + IZ_0\,\mathrm{j}\sin\theta$ and $I' = I\cos\theta + \mathrm{j}\sin\theta\; V/Z_0$.

Problem 10.5. Use the results of Problem 10.4 to upgrade your ladder network analysis program (Problem 1.3) to handle another type of element, a series lossless transmission line. Three parameters are necessary to specify the line. These could be the characteristic impedance, the physical length, and the velocity of propagation. For convenience in later problems, however, let the three parameters be the characteristic impedance (Z_0), the electrical length (θ_0) in degrees for a particular frequency, and that frequency (f_0). A 50-ohm cable that has an electrical length of 80° at 10 MHZ would appear in the circuit file as "TL, 50, 80, 10E6." For any frequency, f, the electrical length is then $\theta = \theta_0 f/f_0$.

Example answer: For the MATLAB program shown in Problem 1.3, insert the following lines of code in "elseif chain":

```
elseif strcmp(component,'TL')==1
ckt_index=ckt_index+1; Z0=ckt{ckt_index}; %characteristic impedance
ckt_index=ckt_index+1; refdegrees=ckt{ckt_index};%electrical length
ckt_index=ckt_index+1; reffreq=ckt{ckt_index};   %at ref. frequency
eleclength=pi/180*f(i)*(refdegrees/reffreq);
Iold=I; I=I*(cos(eleclength))+V*(1j/Z0*sin(eleclength));
V=V*(cos(eleclength))+1j*Z0*Iold*(sin(eleclength));
```

Problem 10.6. Use your program to analyze the circuit of Figure 10.13. Assume a design frequency, say 1 MHZ, in order to determine the value of the capacitor. Run the analysis from 0 to 2 MHz. Then make the transmission line 360° longer and repeat the analysis. What form will the response take if the transmission line is made very long?

Problem 10.7. A 50-ohm transmission line is connected in parallel with an equal length transmission line of 75 ohms, i.e., at each end the inner conductors are connected and the outer conductors are connected. The cables have equal phase velocities. Show that the characteristic impedance of this composite transmission line is given by (50·75)/(50+75), i.e., the characteristic impedances add like parallel resistors.

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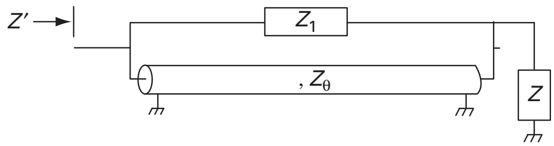
Problem 10.8. In the circuit shown below, the impedance, Z, is modified by a transmission line in parallel with a lumped impedance, Z_1, which could be an R, C, or L or a network.



Show that the admittance looking in from the left, $Y' = 1/Z'$, is given by

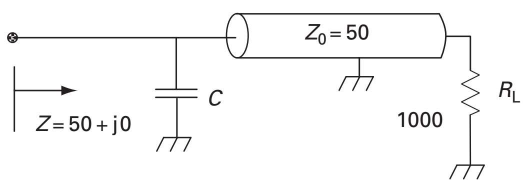
$$\frac{1}{Z'} = Y' = Y_0 \frac{Y + \mathrm{j}Y_0 \tan\theta + \left(2Y_1 - \dfrac{2Y_1}{\cos\theta} + \mathrm{j}\dfrac{YY_1}{Y_0}\tan\theta\right)}{Y_0 + \mathrm{j}Y\tan\theta + (\mathrm{j}Y_1\tan\theta)}.$$

Hint: extend the argument used in the text to find Z' for a cable without a bridging lumped element. Assume a forward and reverse wave in the cable with amplitudes 1 and Γ. The voltage on the cable is then $V(z) = \mathrm{e}^{\mathrm{j}\omega t}\,(\mathrm{e}^{-\mathrm{j}kz} + \Gamma \mathrm{e}^{\mathrm{j}kz})$ and the current is $I(z) = Z_0^{-1}\mathrm{e}^{\mathrm{j}\omega t}\,(\mathrm{e}^{-\mathrm{j}kz} - \Gamma \mathrm{e}^{\mathrm{j}kz})$. The current into Z is the sum of the current from the cable and the current from Z_1 while the current into the circuit is the sum of the current into the cable and the current into Z_1.

Problem 10.9. Using a 50-ohm network analyzer, it is found that a certain device, when tested at 1 GHz, has a (complex) reflection coefficient of 0.6 at an angle of −22° (standard polar coordinates: the positive x-axis is at 0° and angles increase in the counterclockwise direction).

(a) Calculate the impedance, R+jX.
(b) Find the component values for both the equivalent series R_sC_s circuit and the equivalent parallel R_pC_p circuit that, at 1 GHz, represent the device.

Problem 10.10. The circuit below matches a 1000-ohm load to a 50-ohm source at a frequency of 10 MHz. The characteristic impedance of the cable is 50 ohms.



(a) Make a Smith chart sketch that shows the derivation of this circuit.
(b) Find the length of the (shortest) cable and the value of the capacitor. Specify the length in degrees and the capacitance in picofarads. Calculate these values rather than reading them from an accurately drawn Smith chart.
(c) Use your ladder network analysis program (Problems 1.3 and 10.5) to find the transmission from 9 MHz to 11 MHz in steps of 0.1 MHz.

Problem 10.11. Find a transmission line element to replace the capacitor in the circuit of Problem 10.9.

Problem 10.12. Suppose that a transmission line has small shunt susceptance (capacitive or inductive) at a point z. By itself, this will cause a small reflection. If an identical shunt reactance is placed one quarter-wave from the first, its reflection will compensate the first and the cable will have essentially perfect transmission. Show that this is the case (a) analytically, using the "tan tan" formula for Z' and B', and (b) graphically, using the Smith chart (the area around the center of the chart).

Problem 10.13. Find the size and position of the constant resistance circles on the normalized Smith chart. Use the following procedure:

We have $z(x) = r + jx$ where x is a variable and r is a constant. This vertical line in the z-plane maps into the ρ-plane via the equation $\rho(x) = [z(x) - 1]/[z(x) + 1]$. We want to show that the locus of points in the ρ-plane is a circle with radius $1/(r+1)$ centered at $[r/(r+1), 0]$.

Assume that the locus will be a circle centered on the real axis at $[a,0]$. Write the equation $|\rho(x) - a| = radius$. This equation has the form

$$|[N_{\mathrm{Re}}(x) + jN_{\mathrm{Im}}(x)]/[D_{\mathrm{Re}}(x) + j\,D_{\mathrm{Im}}(x)]| = radius, \tag{1}$$

where $N_{\mathrm{Re}}(x)$ and $N_{\mathrm{Im}}(x)$ are the real and imaginary parts of the numerator and $D_{\mathrm{Re}}(x)$ and $D_{\mathrm{Im}}(x)$ are the real and imaginary parts of the denominator. If every point on the circle is to have the same value of r, the radius of the circle must be independent of x.

$$\begin{aligned} |\rho(x) - a|^2 &= [(N_{\mathrm{Re}}(x))^2 + (N_{\mathrm{Im}}(x))^2]/[(D_{\mathrm{Re}}(x))^2 + (D_{\mathrm{Im}}(x))^2] = radius^2 \\ &= \text{function only of } r. \end{aligned} \tag{2}$$

In this case, the way to satisfy Equation (2) is to set $N_{\mathrm{Re}}(x)/D_{\mathrm{Re}}(x) = -N_{\mathrm{Im}}(x)\,/\,D_{\mathrm{Im}}(x)$.

This will let us find a and $radius$. Other ways to make the radius constant will produce circles on which both r and x vary.