CHAPTER

4 Basic filters

Bandpass filters are key elements in radio circuits, for example, in radio receivers, to select the desired station. Here we will discuss lumped-element filters made of inductors and capacitors. We will first look at lowpass filters, and then see how they serve as prototypes for conversion to bandpass filters. We begin with the well-established lowpass filter prototypes – Butterworth, Chebyshev, Bessel, etc. These lowpass prototypes are simple *LC* ladder networks with series inductors and shunt capacitors, as shown in Figure 4.1.

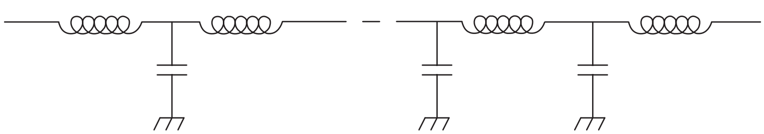

Figure 4.1. Lowpass ladder network.

An *n*-section lowpass filter has *n* components (capacitors plus inductors). The end components can be either series inductors, as shown above, or shunt capacitors, or one of each. Since they contain no (intentional) resistance, these filters are *reflective filters*; outside the passband, it is mismatch that keeps power from reaching the load. The ladder network can be redrawn as a cascade of voltage dividers as in Figure 4.2.

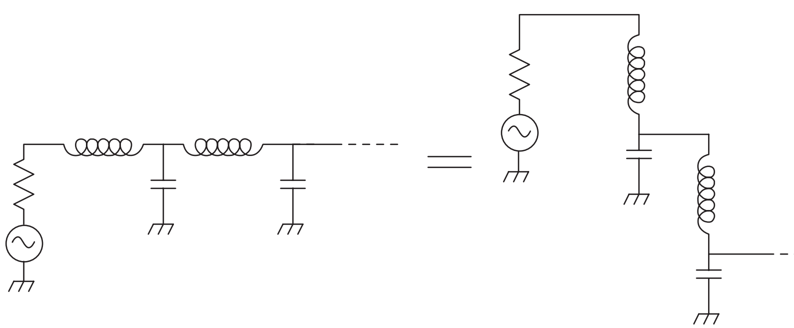

Figure 4.2. Ladder network as a cascade of voltage dividers.

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At high frequencies the division ratio increases so the load is increasingly isolated from the source. For frequencies well above cutoff, each circuit element contributes 6 dB of attenuation per octave (20 dB per decade). Within the passband, an ideal lowpass filter provides a perfect match between the load and the source. Filters with many sections approach this ideal. When the source and load impedances have no reactance (either built-in or parasitic) it is theoretically possible to have a perfect match across a wide band.

4.1 Prototype lowpass filter designs

The Butterworth filter is maximally flat, that is, it is designed so that at zero frequency the first $2n-1$ derivatives with respect to frequency of the power transfer function are zero. The final condition (needed to determine the values of n elements) is the specification of the cutoff frequency, f_0, often specified as the 3-dB or half-power frequency. The frequency response of the Butterworth filter turns out to be

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|^2 = \frac{1}{1+(f/f_0)^{2n}}. \tag{4.1}$$

While it is the flattest filter, the Butterworth filter does not have skirts as sharp as those of the Chebyshev filter. The trade-off is that the Chebyshev filters have some passband ripple. The design criterion for the Chebyshev filter is that these ripples all have equal depth. The response is given by

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|^2 = \frac{1}{1+(V_{\text{r}}^{-2}-1)\cosh^2\left(n\cosh^{-1}(f/f_0)\right)}, \tag{4.2}$$

where V_{r} is the height above zero of the ripple valley (in voltage) relative to the height of the peaks.

You will find tables of filter element values in many handbooks and textbooks. Two tables from Matthaei, Young and Jones [2] are given in Appendix 4.1 at the end of this chapter. These tables are for normalized filters, i.e., the cutoff frequency[1] is 1 radian/sec ($1/2\pi$ Hz). The value of the n-th component is g_n farads or henrys, depending on whether the filter begins with a capacitor or with an inductor. The proper source impedance is $1+\text{j}0$ ohms. This is also the proper load impedance except for the even-order Chebyshev filters, where it is $1/g_{n+1}+\text{j}0$ ohms. Figure 4.3 shows plotted power responses of a Butterworth filter and several Chebyshev filters.

[1] The cutoff frequency for the Butterworth filters is the half-power (3 db) point. For an n-dB Chebyshev filter it is the highest frequency for which the response is down by n dB (see Figure 4.3).

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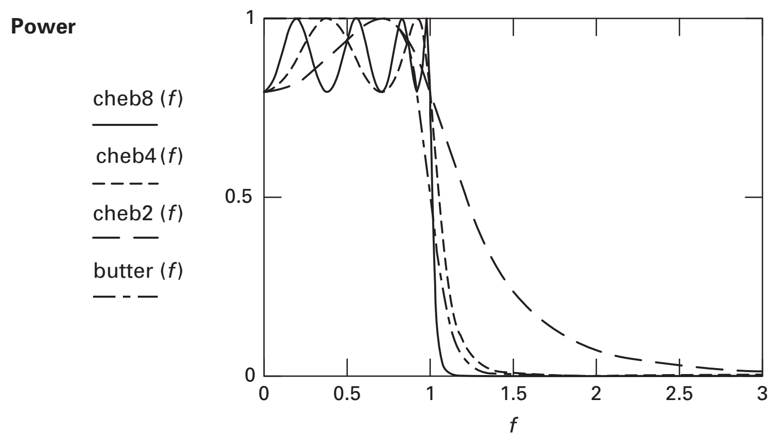
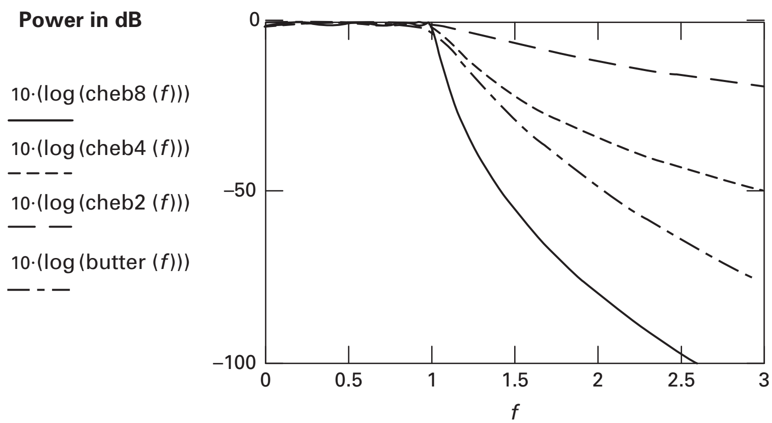
Figure 4.3. Butterworth and Chebyshev responses.





4.2 A lowpass filter example

As an example, we will look at the three-section Butterworth lowpass filter. From the table, the filter has values of 1 H, 2 F, and 1 H (Figure 4.4a) or 1F, 2H, and 1F (Figure 4.4b). The (identical) responses for these two filters are given in Table 4.1 and plotted in Figure 4.5. Note that they work as advertised; the 3-dB point is at 0.159 Hz.

Suppose we need a three-section Butterworth that is 5 kHz wide and works between a 50-ohm generator and a 50-ohm load. We can easily find the element values by scaling the prototype. The values of the inductors are just multiplied by 50 (we need 50 times the reactance) and divided by $2\pi \cdot 5000$ (we need to reach that reactance at 5 kHz, not 1 radian/sec). Similarly, the capacitor values are divided by 50 and divided by $2\pi \cdot 5000$. Figure 4.6 shows the circuit resulting from scaling the values of Figure 4.4b.

The response of the scaled filter is shown below in Table 4.2 and Figure 4.7.

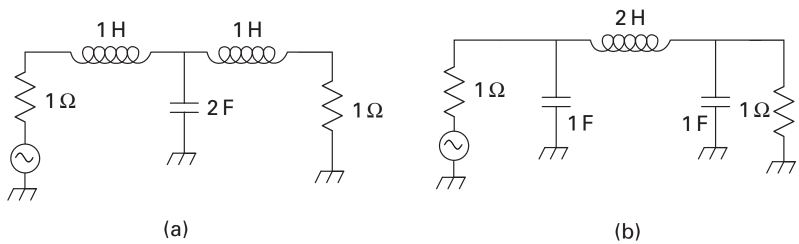


Figure 4.4. Equivalent three-section Butterworth lowpass filters.

Table 4.1 Frequency response for filters of Figure 4.4.

Frequency (Hz)	Power	Response (dB)
0.00	1.000	−0.0
0.0321	0.000	−0.0
0.0640	0.996	−0.02
0.095	0.955	−0.20
0.1270	0.792	−1.01
0.1590	0.500	−3.01
0.1910	0.251	−6.00
0.2230	0.117	−9.31
0.2540	0.056	−12.5
0.2860	0.029	−15.4
0.3180	0.015	−18.1

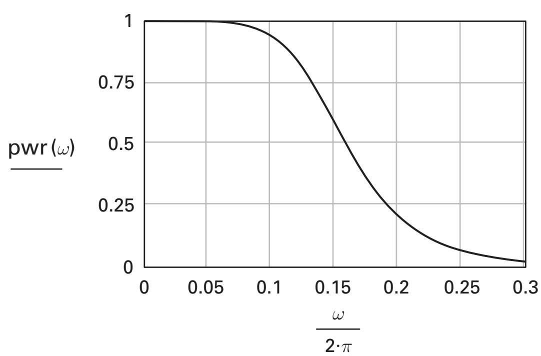


Figure 4.5. Plotted response of filters of Figure 4.4.

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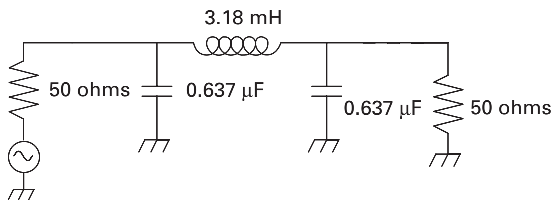


Figure 4.6. Filter of Figure 4.4(b), after conversion to 50 ohms and 5 kHz cutoff frequency.

Table 4.2 Response of the scaled lowpass filter of Figure 4.6.

Frequency (Hz)	Power	Response (dB)
0	1.000	−0.0
1000	1.000	−0.0
2000	0.996	−0.02
3000	0.956	−0.20
4000	0.793	−1.01
5000	0.500	−3.01
8000	0.056	−12.5
6000	0.251	−6.00
7000	0.117	−9.31
9000	0.029	−15.4
10000	0.015	−18.1

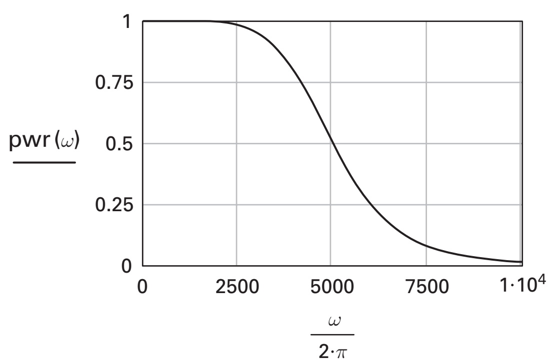


Figure 4.7. Plotted response of the scaled lowpass filter of Figure 4.6.

4.3 Lowpass-to-bandpass conversion

Here we will see how to convert lowpass filters into bandpass filters. Remember how the lowpass filters work: as frequency increases, the series arms (inductors), which are short circuits at dc, begin to pick up reactance. Likewise, the shunt arms (capacitors), which are open circuits at dc, begin to pick up susceptance. Both effects impede the signal transmission, as we have seen. To convert these lowpass filters in the most direct way to bandpass filters, we can replace the inductors by series LC combinations and the capacitors by parallel LC combinations. The series combinations are made to resonate (have zero impedance) at the center frequency of the desired bandpass filter, just as the inductors had zero impedance at dc, the "center frequency" of the prototype lowpass filter. It is important to note that as we move away from resonance, a series LC arm picks up reactance at twice the rate of the inductor alone. This is easy to see: The reactance of the series arm is given by

$$X_{\text{series}} = \omega L - \frac{1}{\omega C}. \tag{4.3}$$

Differentiating with respect to ω, we find

$$\frac{\mathrm{d}X}{\mathrm{d}\omega} = L + \frac{1}{\omega^2 C}. \tag{4.4}$$

At $\omega = \omega_0$,

$$\frac{\mathrm{d}X}{\mathrm{d}\omega} = L + \frac{1}{\omega_0{}^2 C} = 2L. \tag{4.5}$$

As we move off resonance, the inductor and the capacitor provide equal contributions to the reactance. Likewise, the parallel LC circuits, which replace the capacitors in the prototype lowpass filter, pick up susceptance at twice the rate of their capacitors. With this in mind, let us convert our 5-kHz lowpass filter into a bandpass filter. Suppose we want the center frequency to be 500 kHz and the bandwidth to be 10 kHz. As we move up from the center frequency, the series arms must pick up a reactance at the same rate the inductors picked up a reactance in the prototype lowpass filter. Similarly, the shunt arms must pick up susceptance at the same rate the capacitors picked up susceptance in the prototype. This will cause the bandpass filter to have the same shape above the center frequency as the prototype had above dc. If the 3-dB point of the prototype filter was 5 kHz, the upper 3-dB point of the bandpass filter will be at 5 kHz above the center frequency. The bandpass filter, however, will have a mirror-image response as we go below the center frequency. (Below center frequency the reactances and susceptances change sign but the response remains the same.)

Let us calculate the component values. As we leave center frequency, the series circuits will get equal amounts of reactance from the L and the C, as explained above. Therefore the series inductor values should be exactly half what they were in the low pass prototype. Note: no matter how high we make the center frequency, the values of the inductors are reduced only by a factor of 2 from the those of the scaled lowpass filter. The series capacitors are chosen to resonate at the center frequency with the new (half-value) series inductors. The values of the parallel arms are determined similarly; the parallel capacitors must have half the value they had in the prototype lowpass filter. Finally, the parallel inductors are chosen to resonate with the new (half-value) parallel capacitors. These simple conversions yield the bandpass filter shown in Figure 4.8.

The response of this bandpass filter is given below in Table 4.3 and Figure 4.9.

While this theoretical filter works perfectly (since its components are lossless), the component values are impractical; typical real components with these values would be too lossy to achieve the calculated filter shape. When a bandpass filter is

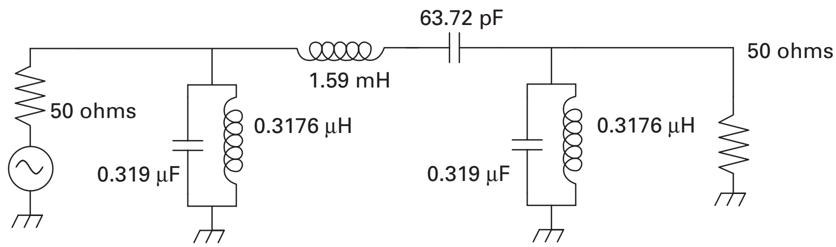


Figure 4.8. Bandpass filter.

Table 4.3 Response of the bandpass filter of Figure 4.8.

Frequency (kHz)	Power	Response (dB)
490	0.014	−18.1
492	0.053	−12.8
494	0.241	−6.19
496	0.785	−1.05
498	0.996	−0.18
500	1.000	−0.00
502	0.996	−1.16
504	0.801	−0.966
506	0.260	−5.84
508	0.059	−12.9
510	0.016	−17.9

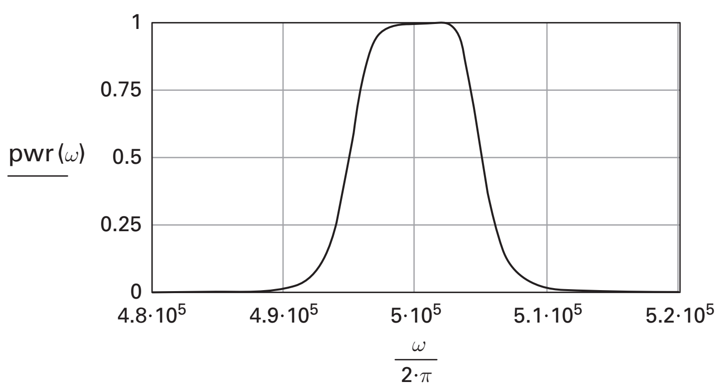


Figure 4.9. Plotted response for Table 4.3.

to have a large fractional bandwidth (bandwidth divided by center frequency) this direct conversion from lowpass to bandpass can be altogether satisfactory. It is when the fractional bandwidth is small, as in this example, that the direct conversion gets into trouble.[2] We will see later that the problem is solved by

[2] The component problem with the straightforward lowpass-to-bandpass conversion is that the values of the series inductors are very different from the values of the parallel inductors. (The same is true of the capacitors, but high-Q capacitors can usually be found.) In the above example, the inductors differ by a factor of about 5000 and it is normally impossible to find high-Q components over this range. (Low-Q inductors, of course, make the filter lossy and, if not accounted for, distort the bandpass

transforming the prototype lowpass filters into somewhat more complicated bandpass circuits known as *coupled resonator filters*. Those filters retain the desired shape (Butterworth, Chebyshev, etc.) and can serve, in turn, as prototypes for filters made from quartz or ceramic resonators and for filters made with resonant irises (thin aperture plates that partially block a waveguide).

Appendix 4.1. Component values for normalized lowpass filters[3]

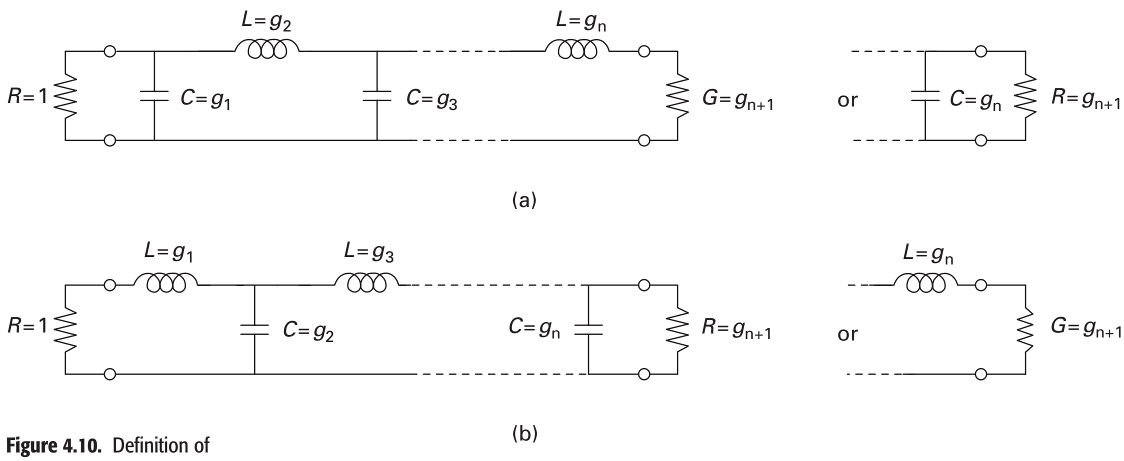


Figure 4.10. Definition of prototype filter parameters, g_1, g_2, …, g_n, g_{n+1}. The prototype circuit (a) and its dual (b) give the same response.

Table A4.1 Element values for Butterworth (maximally flat) lowpass filters (the 3-dB point is at $\omega = 1$ radian/sec).

Value of n	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.000	1.000									
2	1.414	1.414	1.000								
3	1.000	2.000	1.000	1.000							
4	0.7654	1.848	1.848	0.7654	1.000						
5	0.6180	1.618	2.000	1.618	0.6180	1.000					
6	0.5176	1.414	1.932	1.932	1.414	0.5176	1.000				
7	0.4450	1.247	1.802	2.000	1.802	1.247	0.4450	1.000			
8	0.3902	1.111	1.663	1.962	1.962	1.663	1.111	0.3902	1.000		
9	0.3473	1.000	1.532	1.879	2.000	1.879	1.532	1.000	0.3473	1.000	
10	0.3129	0.9080	1.414	1.782	1.975	1.975	1.782	1.414	0.9080	0.3129	1.000

shape.) The inductors in coupled-resonator filters are all of about the same value. If a high-Q inductor can be found, the coupled resonator filter is designed for whatever impedance calls for that value of inductor and then transformers or matching sections are used at each end to convert to the desired impedance.

[3] From Matthaei, Young, and Jones [2].

Table A4.2 Element values for Chebyshev lowpass filters (for a filter with *N*-dB ripple, the last *N*-dB point is at $\omega = 1$ radian/sec).

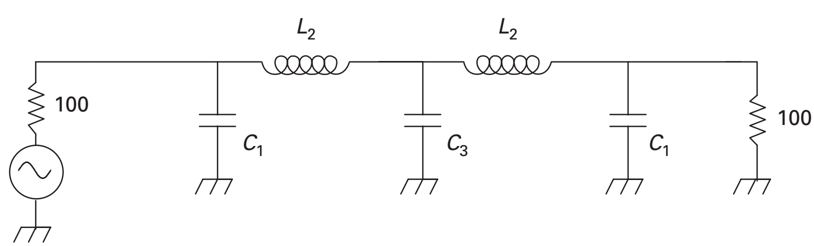
Value of n	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
0.01-dB ripple											
1	0.0960	1.0000									
2	0.4488	0.4077	1.1007								
3	0.6291	0.9702	0.6291	1.0000							
4	0.7128	1.2003	1.3212	0.6476	1.1007						
5	0.7563	1.3049	1.5773	1.3049	0.7563	1.0000					
6	0.7813	1.3600	1.6896	1.5350	1.4970	0.7098	1.1007				
7	0.7969	1.3924	1.7481	1.6331	1.7481	1.3924	0.7969	1.0000			
8	0.8072	1.4130	1.7824	1.6833	1.8529	1.6193	1.5554	0.7333	1.1007		
9	0.8144	1.4270	1.8043	1.7125	1.9057	1.7125	1.8043	1.4270	0.8144	1.0000	
10	0.8196	1.4369	1.8192	1.7311	1.9362	1.7590	1.9055	1.6527	1.5817	0.7446	1.1007
0.1-dB ripple											
1	0.3052	1.0000									
2	0.8430	0.6220	1.3554								
3	1.0315	1.1474	1.0315	1.0000							
4	1.1088	1.3061	1.7703	0.8180	1.3554						
5	1.1468	1.3712	1.9750	1.3712	1.1468	1.0000					
6	1.1681	1.4039	2.0562	1.5170	1.9029	0.8618	1.3554				
7	1.1811	1.4228	2.0966	1.5733	2.0966	1.4228	1.1811	1.0000			
8	1.1897	1.4346	2.1199	1.6010	2.1699	1.5640	1.9444	0.8778	1.3554		
9	1.1956	1.4425	2.1345	1.6167	2.2053	1.6167	2.1345	1.4425	1.1956	1.0000	
10	1.1999	1.4481	2.1444	1.6265	2.2253	1.6418	2.2046	1.5821	1.9628	0.8853	1.3554
0.2-dB ripple											
1	0.4342	1.0000									
2	1.0378	0.6745	1.5386								
3	1.2275	1.1525	1.2275	1.0000							
4	1.3028	1.2844	1.9761	0.8468	1.5386						
5	1.3394	1.3370	2.1660	1.3370	1.3394	1.0000					
6	1.3598	1.3632	2.2394	1.4555	2.0974	0.8838	1.5386				
7	1.3722	1.3781	2.2756	1.5001	2.2756	1.3781	1.3722	1.0000			
8	1.3804	1.3875	2.2963	1.5217	2.3413	1.4925	2.1349	0.8972	1.5386		
9	1.3860	1.3938	2.3093	1.5340	2.3728	1.5340	2.3093	1.3938	1.3860	1.0000	
10	1.3901	1.3983	2.3181	1.5417	2.3904	1.5536	2.3720	1.5066	2.1514	0.9034	1.5386
0.5-dB ripple											
1	0.6986	1.0000									
2	1.4029	0.7071	1.9841								
3	1.5963	1.0967	1.5963	1.0000							
4	1.6703	1.1926	2.3661	0.8419	1.9841						
5	1.7058	1.2296	2.5408	1.2296	1.7058	1.0000					
6	1.7254	1.2479	2.6064	1.3137	2.4758	0.8696	1.9841				
7	1.7372	1.2583	2.6381	1.3444	2.6381	1.2583	1.7372	1.0000			
8	1.7451	1.2647	2.6564	1.3590	2.6964	1.3389	2.5093	0.8796	1.9841		
9	1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504	1.0000	
10	1.7543	1.2721	2.6754	1.3725	2.7392	1.3806	2.7231	1.3485	2.5239	0.8842	1.9841
1.0-dB ripple											
1	1.0177	1.0000									
2	1.8219	0.6850	2.6599								
3	2.0236	0.9941	2.0236	1.0000							

Table A4.2 (cont.)

Value of n	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
4	2.0991	1.0644	2.8311	0.7892	2.6599						
5	2.1349	1.0911	3.0009	1.0911	2.1349	1.0000					
6	2.1546	1.1041	3.0634	1.1518	2.9367	0.8101	2.6599				
7	2.1664	1.1116	3.0934	1.1736	3.0934	1.1116	2.1664	1.0000			
8	2.1744	1.1161	3.1107	1.1839	3.1488	1.1696	2.9685	0.8175	2.6599		
9	2.1797	1.1192	3.1215	1.1897	3.1747	1.1897	3.1215	1.1192	2.1797	1.0000	
10	2.1836	1.1213	3.1286	1.1933	3.1890	1.1990	3.1738	1.1763	2.9824	0.8210	2.6599
2.0-dB ripple											
1	1.5296	1.0000									
2	2.4881	0.6075	4.0957								
3	2.7107	0.8327	2.7107	1.0000							
4	2.7925	0.8806	3.6063	0.6819	4.0957						
5	2.8310	0.8985	3.7827	0.8985	2.8310	1.0000					
6	2.8521	0.9071	3.8467	0.9393	3.7151	0.6964	4.0957				
7	2.8655	0.9119	3.8780	0.9535	3.8780	0.9119	2.8655	1.0000			
8	2.8733	0.9151	3.8948	0.9605	3.9335	0.9510	3.7477	0.7016	4.0957		
9	2.8790	0.9171	3.9056	0.9643	3.9598	0.9643	3.9056	0.9171	2.8790	1.0000	
10	2.8831	0.9186	3.9128	0.9667	3.9743	0.9704	3.9589	0.9554	3.7619	0.7040	4.0957
3.0-dB ripple											
1	1.9953	1.0000									
2	3.1013	0.5339	5.8095								
3	3.3487	0.7117	3.3487	1.0000							
4	3.4389	0.7483	4.3471	0.5920	5.8095						
5	3.4817	0.7618	4.5381	0.7618	3.4817	1.0000					
6	3.5045	0.7685	4.6061	0.7929	4.4641	0.6033	5.8095				
7	3.5182	0.7723	4.6386	0.8039	4.6386	0.7723	3.5182	1.0000			
8	3.5277	0.7745	4.6575	0.8089	4.6990	0.8018	4.4990	0.6073	5.8095		
9	3.5340	0.7760	4.6692	0.8118	4.7272	0.8118	4.6692	0.7760	3.5340	1.0000	
10	3.5384	0.7771	4.6768	0.8136	4.7425	0.8164	4.7260	0.8051	4.5142	0.6091	5.8095

Problems

Problem 4.1. Design a five-element lowpass filter with a Chebyshev 0.5-dB ripple shape. Let the input and output impedances be 100 ohms. Use parallel capacitors at the ends. The bandwidth (from dc to the last 0.5-dB point) is to be 100 kHz. Use Table A4.2 to find the values of the prototype 1 ohm, 1 rad/sec filter and then alter these values for 100 ohms and 100 kHz.



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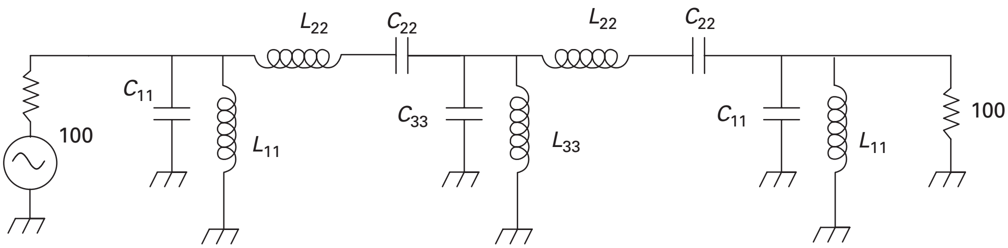
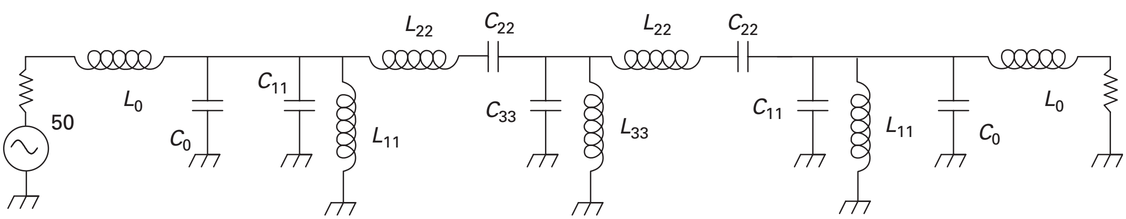
Problem 4.2. Use the results of Problem 4.1 to design a five-element bandpass filter with a Chebyshev 0.5-dB ripple shape. Let the input and output impedances remain at 100 ohms. The center frequency is to be 5 MHz and the total bandwidth (between outside 0.5-dB points) is to be 200 kHz.



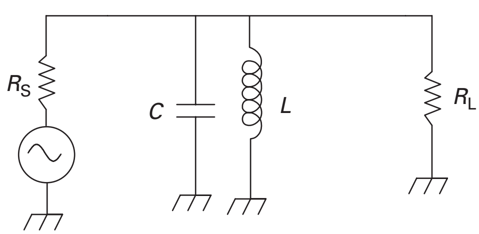
Problem 4.3. Convert the filter of Problem 4.2 to operate at 50 ohms by adding an L-section matching network at each end. Test the filter design using your ladder network analysis program, sweeping from 4.5 to 5.5 MHz in steps of 20 KHz.



Problem 4.4. The one-section bandpass filter shown below uses a single parallel resonator. In its prototype lowpass filter, the resonator is a single shunt capacitor. Show that the frequency response of this filter is given by

$$\frac{P}{P_{\mathrm{max}}} = \frac{1}{1 + Q^2(f/f_0 - f_0/f)^2}$$

where f_0 is the resonant frequency of the LC combination and Q is defined as $R/(\omega_0 L)$, where R is the parallel combination of R_S and R_L.



Problem 4.5. Highpass filters are derived from lowpass filters by changing inductors to capacitors and vice versa and replacing the component values in the prototype lowpass

tables by their reciprocals. (A 2-F capacitor, for example, would become a 0.5-H inductor.) The prototype highpass response at ω will be equal to the prototype lowpass response at 1/ω. Convert the lowpass filter of Figure 4.4(b) into a highpass filter. (Answer: 1 H, 0.5 F, 1 H.) Next, scale it to have a cutoff frequency of 5 kHz and to operate at 50 ohms. Finally, convert the scaled filter into a bandstop filter with a stopband 10 kHz wide, centered at 500 kHz.

Problem 4.6. Enhance your ladder network analysis program (Problem 1.3) to display not just the amplitude response of a network, but also the phase response (phase angle of the output voltage minus phase angle of the input voltage). Calculate the phase response of the Butterworth filter in Figure 4.4(a). Note: ladder networks belong to a class of networks ("minimum phase networks") for which the amplitude response uniquely determines the phase response and vice versa. In Chapter 12 we will encounter "allpass" filters which are not in this class; phase varies with frequency while amplitude remains constant.

Example answer: For the MATLAB program listing in Problem 1.3, simply insert the following two lines ahead of the last two lines in the original program.

```
figure(3);plot(-180/pi*angle(Vgen));
grid;xlabel('Frequency');ylabel('degrees');title('Phase response');
```

References

[1] Fink, D. G., *Electronic Engineers' Handbook*, New York: McGraw-Hill, 1975. See Section 12, Filters, Coupling Networks, and Attenuators by M. Dishal. Contains an extensive list references.

[2] Matthaei, G., Young, L. and Jones, E. M. T., *Microwave Filters, Impedance-Matching Networks, and Coupling Structures* New York: McGraw Hill, 1964, reprinted in 1980 by Artech House, Inc. Contains fully developed designs, comparing measured results with theory (spectacular fits, even at microwave frequencies) and has an excellent introduction and review of the theory.