APPENDIX B

Fourier Series

We can interpret the waveforms we see in terms of components at different frequencies. For example, we might consider a voltage $V(t)$ that is the sum of three cosine components:

$$V(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + a_3 \cos(\omega_3 t). \tag{B.1}$$

We can include a DC term as a special case if we assume that one of the frequencies is zero. Representing a function in terms of its frequency components is helpful in understanding how the filter in a Class-C or Class-D amplifier operates. We also use the frequency components to define the relationship between DC and AC currents in oscillators and to predict mixer output frequencies.

B.1 FOURIER COEFFICIENTS

If a function is periodic, it has a special representation called a Fourier series, where each frequency component is a harmonic of the fundamental frequency. We will leave the discussion of why a function can be written in a Fourier series to a mathematics text, but we will see how to find the coefficients. We start by writing the function as an infinite sum of cosines and sines:

$$V(t) = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t) + a_2 \cos(2\omega t) + b_2 \sin(2\omega t) + \cdots, \tag{B.2}$$

where a_0, a_1, b_1, a_2, and b_2 are the *Fourier coefficients*. For our functions we can simplify this sum. Notice that if we change the sign of t, the cosine terms do not change. We say that the cosine is an *even* function. In contrast, the sine is an *odd* function, and it changes sign. It turns out that the functions that we are interested in are even, and for this reason we do not need the sine terms. This means that we can write our series as

$$V(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t). \tag{B.3}$$

Now we can find a formula for the Fourier coefficients. To do this we need to use a remarkable property of the integrals of cosine products. Let us consider the integral I_{mn}, where

$$I_{mn} = \int_{-T/2}^{+T/2} \cos(m\omega t) \cos(n\omega t)\, dt, \tag{B.4}$$

where T is the period, given by

$$T = 2\pi/\omega. \tag{B.5}$$

There are several possibilities. If m and n are 0, the integrand is just 1, and the integral is T. If m and n are positive, and $m = n$, the integrand is $\cos^2(m\omega t)$, and the integral is $T/2$. If $m \neq n$, then we rewrite the product as a sum of cosines

$$I_{mn} = \frac{1}{2}\int_{-T/2}^{+T/2} (\cos[(m+n)\omega t] + \cos[(m-n)\omega t])\, dt. \tag{B.6}$$

The integrals of cosines are sine functions. These are periodic, also, so that they have the same value at each limit, and the integrals will vanish. This means that if we integrate the product of two different harmonics over a period, the integral is zero. We say that different harmonics are *orthogonal*. We can summarize these results as follows:

$$I_{mn} = \begin{cases} T & \text{for } m = n = 0, \\ T/2 & \text{for } m = n > 0, \\ 0 & \text{for } m \neq n. \end{cases} \tag{B.7}$$

Now consider the integral V_n of the function $V(t)$, defined by

$$V_n = \int_{-T/2}^{+T/2} V(t)\cos(n\omega t)\, dt. \tag{B.8}$$

We substitute for $V(t)$ from Equation B.3 to obtain

$$V_n = \int_{-T/2}^{+T/2} \left(\sum_m a_m \cos(m\omega t)\right) \cos(n\omega t)\, dt. \tag{B.9}$$

I have used m as the index in the sum rather than n to keep them distinct. Let us bring the cosine factor $\cos(n\omega t)$ inside the sum. This gives us

$$V_n = \int_{-T/2}^{+T/2} \left(\sum_m a_m \cos(m\omega t)\ \cos(n\omega t)\right) dt. \tag{B.10}$$

We can do this integral by taking the integral of each of the terms in the sum separately and adding. We write

$$V_n = \sum_m a_m \left(\int_{-T/2}^{+T/2} \cos(m\omega t)\cos(n\omega t)\, dt\right) = \sum_m a_m I_{mn}. \tag{B.11}$$

The integrals are zero except where $m = n$. This means that we are only left with one term in the sum, given by

$$V_n = \begin{cases} T a_0 & \text{if } n = 0, \\ T a_n/2 & \text{if } n > 0. \end{cases} \tag{B.12}$$

I have replaced a_m with a_n, because $m = n$ for this term. We can invert this to find

$$a_0 = \frac{1}{T}\int_{-T/2}^{+T/2} V(t)\, dt. \tag{B.13}$$

This is the DC component, and we can think if it as just the average value of $V(t)$. For $n > 0$, we have

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} V(t)\cos(n\omega t)\,dt. \tag{B.14}$$

Doing the integrals to find the Fourier coefficients requires practice. You should fill in the details of the following examples.

B.2 SQUARE WAVE

Now we find the Fourier coefficients for a square wave with voltages of +1 and −1 (Figure B.1a). The average value is zero, and so there is no DC component. The AC components are given by

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} V(t)\cos(n\omega t)\,dt. \tag{B.15}$$

If n is even, the integral over the time the square wave is positive is zero, and so is the integral over the time that the square wave is negative. This means a_n is zero if n is even. If n is odd, the integrals over the positive part of the square wave are the same as the integrals over the negative part. You should sketch the cosines and the square wave to convince yourself of this. This means that we can write

$$a_n = \frac{4}{T}\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-T/4}^{T/4} = \frac{2}{\pi}\left[\frac{\sin(n\omega t)}{n}\right]_{-T/4}^{T/4}. \tag{B.16}$$

The sines evaluate to +1 or −1, and we can write the first four coefficients as

$$a_1 = +\frac{4}{\pi}, \tag{B.17}$$

$$a_3 = -\frac{4}{3\pi}, \tag{B.18}$$

$$a_5 = +\frac{4}{5\pi}, \tag{B.19}$$

$$a_7 = -\frac{4}{7\pi}. \tag{B.20}$$

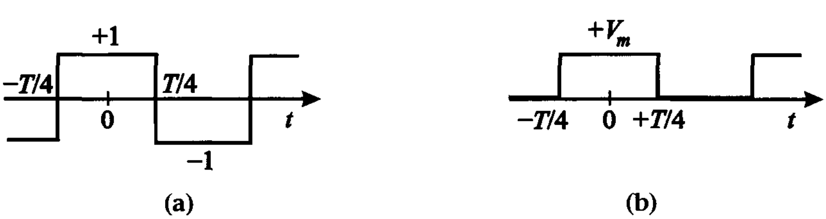

Figure B.1. Square waves for Fourier analysis (a), and pulse train with rectangular pulses with 50% duty cycle (b).

There are only odd harmonics, and the coefficients alternate in sign. The coefficients decrease as $1/n$. We can write the Fourier series for the square wave as

$$V(t) = \frac{4}{\pi}\left(\cos(\omega t) - \frac{\cos(3\omega t)}{3} + \frac{\cos(5\omega t)}{5} - \cdots\right). \tag{B.21}$$

We use these coefficients for studying mixers in Chapter 12.

We can also use this series to deduce the coefficients for rectangular pulses whose width is half the period (Figure B.1b). We say these pulses have a *duty cycle* of 50%. The DC component is half the pulse height V_m. The other components are the same as for the square wave, except that they need to be multiplied by $V_m/2$ to take the pulse height into account. This means that we can write the series for the rectangular pulses with a 50% duty cycle as

$$V(t) = \frac{V_m}{2} + \frac{2V_m}{\pi}\left(\cos(\omega t) - \frac{\cos(3\omega t)}{3} + \frac{\cos(5\omega t)}{5} - \cdots\right). \tag{B.22}$$

We use this series in analyzing a Class-D amplifier in Chapter 10 and in finding the channel spacing for pulsed transmissions in Chapter 12.

B.3 RECTIFIED COSINE

The voltage for the Class-C amplifier that we study in Chapter 10 looks like a rectified cosine (Figure B.2a). We use the first two terms of the Fourier series to find the relationship between the AC and DC components. We can write the DC term as

$$a_0 = \frac{1}{T}\int_{-T/4}^{T/4} V_m \cos(\omega t)\, dt = V_m/\pi \tag{B.23}$$

and the fundamental component as

$$a_1 = \frac{2}{T}\int_{-T/4}^{T/4} V_m \cos^2(\omega t)\, dt = V_m/2. \tag{B.24}$$

The fundamental frequency component is half of the original cosine. This makes sense, because the cosine is only on during half the cycle. The harmonics are

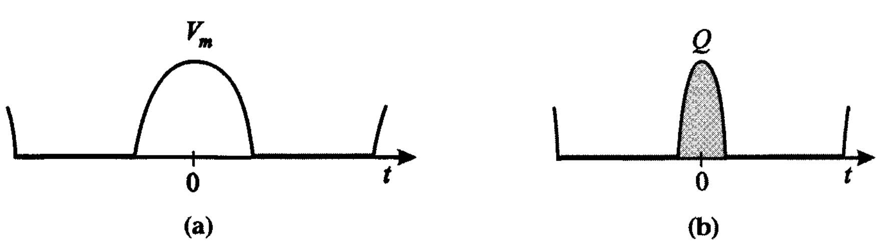

Figure B.2. Rectified cosine for Fourier analysis (a), and narrow pulses (b).

given by

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} V_m \cos(\omega t)\cos(n\omega t)\,dt$$

$$= \frac{V_m}{T}\int_{-T/4}^{T/4} (\cos[(n-1)\omega t] + \cos[(n+1)\omega t])\,dt. \tag{B.25}$$

If n is odd and greater than 1, the integrals are zero. It may help to sketch the cosines to see this. If n is even, we can write

$$a_n = \frac{V_m}{2\pi}\left[\frac{\sin[(n-1)\omega t]}{n-1} + \frac{\sin[(n+1)\omega t]}{n+1}\right]_{-T/4}^{T/4}. \tag{B.26}$$

The sines evaluate to +1 or −1, and we can write the first four harmonics as

$$a_2 = +\frac{2V_m}{3\pi}, \tag{B.27}$$

$$a_4 = -\frac{2V_m}{15\pi}, \tag{B.28}$$

$$a_6 = +\frac{2V_m}{35\pi}, \tag{B.29}$$

$$a_8 = -\frac{2V_m}{63\pi}. \tag{B.30}$$

There are only even harmonics, and the coefficients alternate in sign. The coefficients decrease as $1/(n^2-1)$. We can write the series as

$$V(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos(\omega t) + \frac{2V_m}{\pi}\left(\frac{\cos(2\omega t)}{3} - \frac{\cos(4\omega t)}{15} + \frac{\cos(6\omega t)}{35} - \cdots\right). \tag{B.31}$$

B.4 NARROW PULSES

As a final example, let us find the Fourier coefficients of narrow pulses of current (Figure B.2b). In Chapter 11, we use these coefficients to find the large-signal transconductance of a JFET and the output voltage in bipolar oscillators. We will let the total charge in each pulse be Q. We will assume that the pulses are narrow enough so that, for the harmonics we are interested in, the cosine will be equal to one over the entire pulse. We can write the DC term as

$$a_0 = \frac{1}{T}\int_{-T/2}^{+T/2} I(t)\,dt = Qf \tag{B.32}$$

and the AC terms as

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} I(t)\cos(n\omega t)\,dt = 2Qf. \tag{B.33}$$

We write the series as

$$I(t) = Qf(1 + 2(\cos(\omega t) + \cos(2\omega t) + \cos(3\omega t) + \cdots)). \tag{B.34}$$

All harmonic components are present, and the coefficients are all the same. This is an idealization, and in practice the higher-order harmonics will begin to drop off when the pulse width is no longer narrower than the cosine. In the oscillators in Chapter 11, we compare the DC component a_0 and the fundamental AC component a_1. We can see that

$$a_1/a_0 = 2. \tag{B.35}$$

In words, the peak value of the fundamental component is twice the DC component. In measurements, we would likely use the peak-to-peak value of the fundamental, and this is *four* times the DC component.

FURTHER READING

The idea of representing functions in terms of frequency components is a central theme in electrical engineering, mathematics, and physics. These are called spectral representations, and they may be in the form of a series such as the Fourier series, or an integral such as the Laplace transform, or a combination of the two. *Signals and Systems,* by Oppenheim and Willsky, published by Prentice-Hall, gives a good overview of these series and transforms.