3

Phasors

Complex numbers find one of their best applications in analyzing electronic circuits, because cosine signals can be efficiently represented by complex numbers called phasors. With phasors we can analyze circuits with inductors and capacitors almost as easily as resistor circuits, without worrying about calculating derivatives and integrals. In addition, we can use phasors to calculate average power and stored energy.

3.1 Complex Numbers

Complex numbers are often introduced in a mathematics class by writing $\sqrt{-1}$ as i. Electrical engineers use j instead of i, so that i can be reserved for current. It is a good idea to use j in electrical-engineering problems and i in mathematics and physics problems, because the fields follow different sign conventions. Typically

$$j = -i. \tag{3.1}$$

Using j will let people know that you are following the electrical-engineer's sign convention, and i will tell them that you are following the mathematician's or physicist's convention.

In electrical engineering, it may be best to start by thinking of a complex number as a pair of numbers that we call the *real* and *imaginary* parts. In this sense, a complex number is like a two-dimensional vector, and we can draw it like a vector in a plane (Figure 3.1a). We call this the *complex plane*. The horizontal axis is used for the real part, and the vertical axis for the imaginary part. We will use several different notations for writing a complex number, depending on what we want to emphasize. If we let z be a complex number, we can write

$$z = x + jy, \tag{3.2}$$

where x is the real part and y is the imaginary part. The *complex conjugate* z^* is given by

$$z^* = x - jy. \tag{3.3}$$

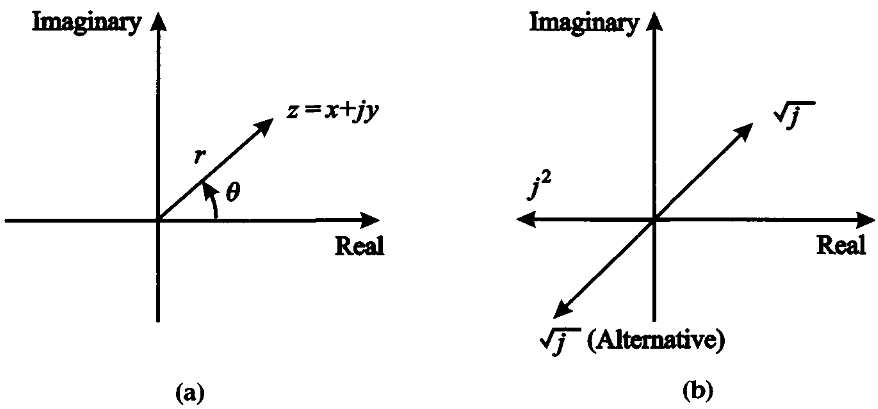


Figure 3.1. Representing a complex number z in a plane by drawing a line from the origin to the point $z = x + jy$ (a). The square and square roots of j (b).

We will also indicate that x and y are the real and imaginary parts of z by writing

$$\mathrm{Re}(z) = x, \tag{3.4}$$

$$\mathrm{Im}(z) = y. \tag{3.5}$$

We add and subtract complex numbers by adding and subtracting the real and imaginary parts separately. This is like vector addition.

We can also represent a complex number in terms of its magnitude and phase. The *magnitude* is the distance from the origin to z in the complex plane. We let the magnitude be given by r and calculate it from the Pythagorean theorem as

$$r = \sqrt{x^2 + y^2}. \tag{3.6}$$

The *phase* is the angle from the real axis, and we write it as θ (the Greek letter *theta*),

$$\theta = \tan^{-1}(y/x). \tag{3.7}$$

We may use either degrees or radians to represent the angle. As a shorthand notation we can write

$$z = r\angle\theta. \tag{3.8}$$

The number j itself can be written as

$$j = 1\angle 90^\circ \tag{3.9}$$

and -1 is given by

$$-1 = 1\angle 180^\circ. \tag{3.10}$$

With trigonometry, we can express the real and imaginary parts in terms of the magnitude and phase as

$$x = r\cos\theta, \tag{3.11}$$

$$y = r\sin\theta. \tag{3.12}$$

We can also indicate that r and θ are the magnitude and phase of z by writing

$$|z| = r, \tag{3.13}$$

$$\angle z = \theta. \tag{3.14}$$

So far complex numbers only seem to be a funny form of vector notation, and if this were all there was to it, we would not need complex numbers. A key difference is in how we multiply and divide. The magnitude and phase are convenient for this calculation. If we have two complex numbers s and t, we can write the magnitude and phase of the product as

$$|st| = |s|\,|t|, \tag{3.15}$$

$$\angle(st) = \angle s + \angle t. \tag{3.16}$$

This means that the magnitude of the product of two complex numbers is the product of the magnitudes, and the phase is the sum of the phases. For example, the product $-1 \cdot z$ is given by

$$|-z| = 1 \cdot |z| = |z|, \tag{3.17}$$

$$\angle(-z) = \angle z + 180^\circ. \tag{3.18}$$

Similarly the quotient s/t is given by

$$\left|\frac{s}{t}\right| = \frac{|s|}{|t|}, \tag{3.19}$$

$$\angle(s/t) = \angle s - \angle t. \tag{3.20}$$

In words, the magnitude of the quotient is given by the quotient of the magnitudes, and the phase is the difference of the phases. As a special case, the quotient $1/s$ is given by

$$|1/s| = 1/|s|, \tag{3.21}$$

$$\angle(1/s) = -\angle s. \tag{3.22}$$

We can deduce the formulas for squares and square roots from the product formulas. We write z^2 as

$$|z^2| = |z|^2, \tag{3.23}$$

$$\angle(z^2) = 2\angle z. \tag{3.24}$$

For example, we can write j^2 as

$$|j^2| = 1, \tag{3.25}$$

$$\angle(j^2) = 180^\circ, \tag{3.26}$$

so that $j^2 = -1$ (Figure 3.1b). If we think of taking the square root as the inverse of squaring, we can write

$$|\sqrt{z}| = \sqrt{|z|}, \tag{3.27}$$

$$\angle(\sqrt{z}) = \frac{\angle z}{2}. \tag{3.28}$$

As in the ordinary square root of a positive number, we have a choice of two roots that differ only in sign. This is shown in Figure 3.1b. The other root can be written as

$$|\sqrt{z}| = \sqrt{|z|}, \tag{3.29}$$

$$\angle(\sqrt{z}) = \frac{\angle z}{2} + 180°. \tag{3.30}$$

For example, consider the square root of j:

$$|\sqrt{j}\,| = 1, \tag{3.31}$$

$$\angle(\sqrt{j}\,) = 45° \quad \text{or} \quad 225°. \tag{3.32}$$

In rectangular coordinates, we would write

$$\sqrt{j} = 1/\sqrt{2} + j/\sqrt{2} \quad \text{or} \quad -1/\sqrt{2} - j/\sqrt{2}. \tag{3.33}$$

3.2 Exponential Function

The exponential function $\exp(x)$ has a deep connection to the cosine and sine functions through complex numbers. We will start with a fundamental definition of the exponential function. There are two parts. First, the exponential function is its own derivative:

$$\frac{d\exp(x)}{dx} = \exp(x). \tag{3.34}$$

To completely determine the function, we must specify its value at some point, because any multiple of the exponential function also satisfies this equation. We set

$$\exp(0) = 1. \tag{3.35}$$

It is interesting to consider the exponential of an imaginary number $j\theta$. We can write the derivative with the chain rule as

$$\frac{d\exp(j\theta)}{d\theta} = j\exp(j\theta). \tag{3.36}$$

This expression indicates that the derivative has the same magnitude as the exponential, but the angle differs by 90°. If we start at $\theta = 0$, where the exponential is just 1, then the function will move up as θ increases. The interesting thing is that the function always moves at right angles to the arrow that represents it.

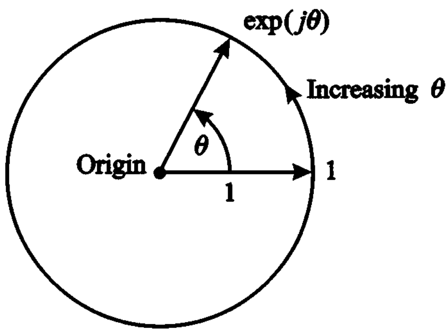


Figure 3.2. Locus of $\exp(j\theta)$ as θ increases from 0.

This causes the function to follow a circular path (Figure 3.2). It is as if the arrow were a rod pinned at one end, so that movement is always at right angles to the rod. More than this, the derivative always has magnitude 1, so that the distance traveled around the circle is equal to θ. Thus θ is the angle in radians as we move around the circle. The path followed by a function as its argument changes is called a *locus*. We would say that the locus of $\exp(j\theta)$ is a unit circle centered at the origin.

From Figure 3.2, we can see by trigonometry that the real and imaginary parts of $\exp(j\theta)$ are $\cos\theta$ and $\sin\theta$. We can write

$$\exp(j\,\theta) = \cos\theta + j\sin\theta. \tag{3.37}$$

This is Euler's formula, and it is one of the most elegant (and surprising) formulas in all of mathematics. We can use Euler's formula to represent the cosine and sine functions in terms of exponentials:

$$\cos(\theta) = \frac{\exp(j\theta) + \exp(-j\theta)}{2}, \tag{3.38}$$

$$\sin(\theta) = \frac{\exp(j\theta) - \exp(-j\theta)}{2j}. \tag{3.39}$$

If you are not familiar with these formulas, it is a good idea to work out the details by substituting Euler's formula into these expressions.

3.3 Phasors

Let us summarize the circuit relations that we have learned for resistors, capacitors, and inductors:

$$V(t) = RI(t), \tag{3.40}$$

$$V(t) = LI'(t), \tag{3.41}$$

$$I(t) = CV'(t). \tag{3.42}$$

The primes denote derivatives. Often it is not very convenient to work with derivatives. However, our radio signals can often be described by cosine functions, and these have simple derivatives, particularly when we consider the relation between the cosine function and an exponential.

We may write a cosine voltage $V(t)$ as

$$V(t) = A\cos(\omega t + \theta), \tag{3.43}$$

where A is the *peak amplitude* in volts, ω (the Greek lower-case *omega*) is the *frequency* in radians per second, and θ is the *phase* in radians. The frequency in radians per second differs from the frequency in cycles per second, or hertz, by a factor of 2π, and so we can write

$$\omega = 2\pi f. \tag{3.44}$$

We will be careful to write the frequency in radians per second as ω and the frequency in hertz as f, so that we can distinguish them. We can write a current $I(t)$ at the same frequency in a similar form:

$$I(t) = B\cos(\omega t + \phi), \tag{3.45}$$

where ϕ (the Greek letter *phi*) is the phase of the current. If the current phase ϕ is different from the voltage phase, then the current can either be ahead of the voltage or behind it (Figure 3.3). If $\phi > \theta$, then we say the current *leads* the voltage, and if $\phi < \theta$, we say the current *lags* the voltage.

If the voltage $V(t) = A\cos(\omega t + \theta)$ is applied to a capacitor C, we can write the current $I(t)$ as

$$I(t) = CV'(t) = -CA\omega\sin(\omega t + \theta) = CA\omega\cos(\omega t + \theta + \pi/2). \tag{3.46}$$

We would say that the current in a capacitor leads the voltage by $\pi/2$, or 90°. In an inductor, the situation is reversed, and the current lags the voltage. An interesting thing happens if we use Euler's formula to express the cosine as the real part of an exponential and repeat this calculation. We write

$$V(t) = \mathrm{Re}[A\exp(j\omega t + j\theta)]. \tag{3.47}$$

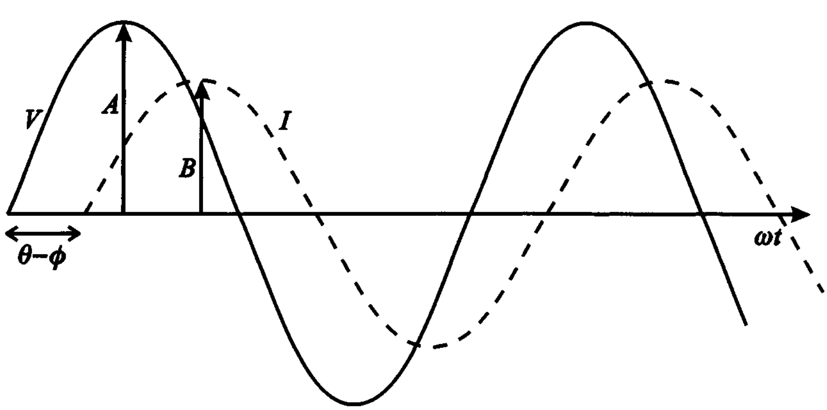


Figure 3.3. Cosine voltage $V(t)$ with a current $I(t)$ lagging it.

We can write the current as

$$I(t) = CV'(t) = \mathrm{Re}\left(C\frac{d}{dt}[A\exp(j\omega t + j\theta)]\right). \tag{3.48}$$

We take the derivative of the exponential by multiplying by $j\omega$. This gives us

$$I(t) = \mathrm{Re}[j\omega CA\exp(j\omega t + j\theta)]. \tag{3.49}$$

This formula is equivalent to Equation 3.46, and you should work out the details to show this. The exponential allowed us to replace a derivative with a multiplication by $j\omega$. We can put this approach on firmer ground if we define complex numbers V and I, given as

$$V = A\exp(j\theta), \tag{3.50}$$

$$I = B\exp(j\phi). \tag{3.51}$$

In terms of the magnitude and phase, we can write

$$|V| = A, \tag{3.52}$$

$$\angle V = \theta, \tag{3.53}$$

$$|I| = B, \tag{3.54}$$

$$\angle I = \phi. \tag{3.55}$$

V and I are called *phasors*. Because they are fixed complex numbers rather than functions of time, t does not appear.

The magnitude of the phasor is equal to the peak amplitude of the original cosine voltage or current, and the phase is the same. To recover the cosine function, we multiply by $\exp(j\omega t)$ and take the real part:

$$V(t) = \mathrm{Re}[V\exp(j\omega t)] = |V|\cos(\omega t + \angle V), \tag{3.56}$$

$$I(t) = \mathrm{Re}[I\exp(j\omega t)] = |I|\cos(\omega t + \angle I). \tag{3.57}$$

Taking the derivative with respect to time is equivalent to multiplying by $j\omega$ for a phasor. For example, for a capacitor we write

$$I = j\omega CV \tag{3.58}$$

as the phasor equivalent of $I(t) = CV'(t)$. We can write a similar relation between current and voltage phasors for an inductor, if we repeat these steps. This gives us

$$V = j\omega LI \tag{3.59}$$

as the equivalent of $V(t) = LI'(t)$. For a resistor we have

$$V = RI \tag{3.60}$$

for phasors, which looks the same as before.

3.4 Impedance

We will be writing voltage and current as phasors most of the time. The ratio of V and I is called the *impedance* and it is written as Z:

$$V = ZI. \tag{3.61}$$

The units of impedance are ohms, like resistance. However, because V and I are complex numbers, the impedance is a complex number with real and imaginary parts. It is traditional to write the real and imaginary parts as

$$Z = R + jX, \tag{3.62}$$

where R is the resistance and X is the *reactance*. We can compare this formula to Equation 3.59, and say that the reactance of an inductor is given by

$$X = \omega L. \tag{3.63}$$

The reactance of an inductor is positive. It is trickier to get the reactance of a capacitor. If we invert Equation 3.58, we get

$$V = \frac{I}{j\omega C}, \tag{3.64}$$

and so we would say that the reactance of a capacitor is given by

$$X = -1/\omega C. \tag{3.65}$$

The minus sign takes the j in the denominator into account. The reactance of a capacitor is negative. Be forewarned: We will often work with the absolute value of the reactance, given by

$$|X| = 1/\omega C. \tag{3.66}$$

People often call this quantity "the reactance," even though it is positive. This is ambiguous but convenient. You have to get the sign from the context.

Impedance is a powerful idea, because it lets us include inductors and capacitors in our analysis without having to take derivatives and integrals. The arithmetic is like that for resistors, except that we use complex numbers, although we have to be careful to remember that impedance is only used for cosine voltages and currents. For example, the impedance Z_s of a series connection of components is the sum of the impedances,

$$Z_s = \sum_i Z_i, \tag{3.67}$$

and the impedance Z_p of a parallel connection is given by the formula

$$\frac{1}{Z_p} = \sum_i \frac{1}{Z_i}. \tag{3.68}$$

You can also find Thevenin and Norton equivalent circuits and voltage and current dividers for impedances in just the same manner that you did for resistances. Even the name impedance suggests the same idea as resistance – a large impedance will *impede* current.

We will often use the inverse of impedance. This is the *admittance*, and the units are siemens (S). We write admittance with a Y:

$$I = YV. \tag{3.69}$$

The real and imaginary parts of the admittance are traditionally written as

$$Y = G + jB, \tag{3.70}$$

where G is the conductance and B is the *susceptance*. We say that the susceptance of a capacitor is ωC and the susceptance of an inductor is $-1/\omega L$. Admittances behave like conductances, so that we write

$$Y_p = \sum_i Y_i \tag{3.71}$$

for components in parallel. Using admittances in parallel circuits is convenient because we can just add the admittances. For components in series we get

$$\frac{1}{Y_s} = \sum_i \frac{1}{Y_i}. \tag{3.72}$$

3.5 RC Filters

We can use phasors to analyze the RC circuits that we build in the lab. These act as low-pass or high-pass filters, selecting either the high or low frequencies in a signal. For example, the circuit in Figure 3.4a allows signals at low frequencies through but blocks higher frequencies. This is a low-pass filter. We find the response of the circuit with phasors and impedances.

We can write the current in terms of the input voltage V_i as

$$I = \frac{V_i}{Z} = \frac{V_i}{R + 1/(j\omega C)}. \tag{3.73}$$

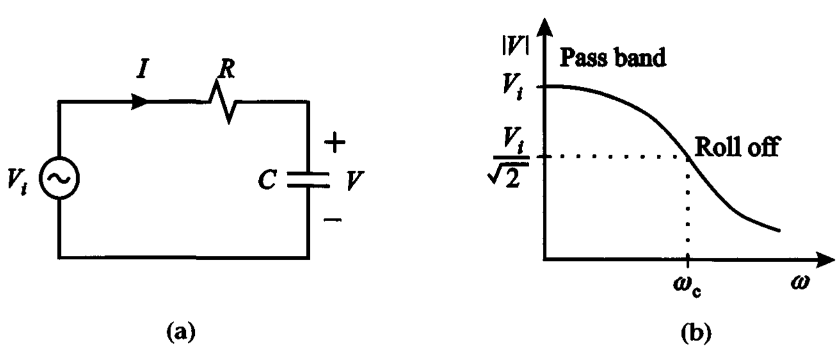


Figure 3.4. RC low-pass filter (a), and the response (b).

The output voltage V is given by

$$V = \frac{I}{j\omega C} = \frac{V_i}{1 + j\omega RC} = \frac{V_i}{1 + j\omega\tau}, \tag{3.74}$$

where $\tau = RC$ is the time constant. Figure 3.4b is a plot of $|V|$. In the pass band, where $\omega\tau \ll 1$, the output voltage is close to the input voltage. When $\omega\tau = 1$, the output voltage is given by

$$|V| = \frac{V_i}{|1 + j|} = \frac{V_i}{\sqrt{2}}. \tag{3.75}$$

This means that the output voltage has dropped by a factor of $\sqrt{2}$. Because power is proportional to the square of the voltage, we can think of this as the half-power frequency, or the 3-dB frequency. This means that we can write the cut-off frequency as

$$\omega_c = 1/\tau. \tag{3.76}$$

In the roll-off region above the cut-off frequency, the response drops as the frequency increases. For $\omega \gg \omega_c$, we can write

$$V \approx \frac{V_i}{j\omega\tau}. \tag{3.77}$$

For phasors, multiplying by $j\omega$ is equivalent to differentiating, and dividing by $j\omega$ is equivalent to integrating. This means that in the roll-off region, this circuit acts as an integrator. One application of this filter would be in an audio system to remove the hiss that you often hear. The hiss comes primarily from frequencies that are higher than the frequency range we use for speaking. A filter with a cut-off frequency of about 3 kHz can remove hiss without hurting speech quality.

The circuit in Figure 3.5a acts as a high-pass filter, letting high frequencies through and blocking low frequencies. We can write the response with the potential-divider formula:

$$V = \frac{V_i R}{R + 1/(j\omega C)} = \frac{V_i}{1 + 1/(j\omega RC)} = \frac{V_i}{1 + 1/(j\omega\tau)}. \tag{3.78}$$

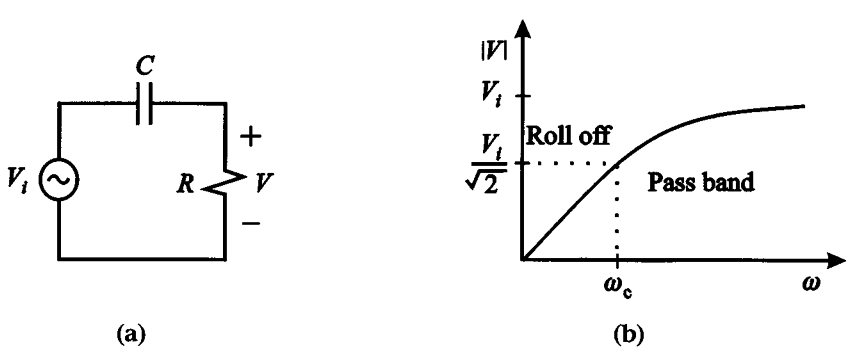


Figure 3.5. RC high-pass filter (a), and the response (b).

This is a high-pass response (Figure 3.5b). The cut-off frequency is the same as for the low-pass filter. This time, however, the pass band is above the cut-off frequency, and the stop band is below it. In the stop band, where $\omega \ll \omega_c$, we can write

$$V \approx j\omega\tau V_i. \tag{3.79}$$

The roll off is proportional to frequency. A circuit like this could be used in an audio system to remove hum. Hum is the low-frequency buzzing that is associated with the AC wall supply.

3.6 Series Resonance

Consider a voltage source with an inductor, capacitor, and load resistor (Figure 3.6a). This is a common circuit for band-pass filters that select signals near a particular frequency. We can use the potential-divider formula to write the output voltage as

$$V = \frac{V_i R}{Z}, \tag{3.80}$$

where Z is the circuit impedance, given by

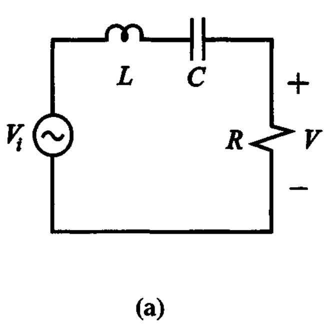
$$Z = R + jX = R + j\omega L + 1/(j\omega C). \tag{3.81}$$

Let us consider the reactance X first, which is the imaginary part:

$$X = \omega L - 1/(\omega C). \tag{3.82}$$

At low frequencies, the capacitive reactance dominates, and the reactance is large and negative. At high frequencies, the inductive reactance dominates, and the reactance is large and positive. The frequency where the reactance is zero is called the *resonant* frequency, and we write it as

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{3.83}$$



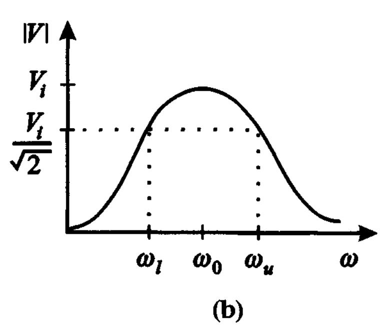


Figure 3.6. Series resonant circuit with a source (a), and the response (b).

At the resonant frequency, the inductive and capacitive reactance cancel, and Equation 3.80 becomes

$$V = V_i \tag{3.84}$$

so that the output voltage is equal to the input voltage.

Away from the resonant frequency, the reactance increases, and the output voltage drops (Figure 3.6b). When the reactance and resistance are equal, the output voltage is given by

$$|V| = \frac{V_i}{|1 \pm j|} = \frac{V_i}{\sqrt{2}}. \tag{3.85}$$

This means that the upper and lower half-power frequencies ω_u and ω_l are where the reactance and resistance are equal. We can find formulas for these frequencies by setting $X = \pm R$:

$$\omega_u L - 1/(\omega_u C) = +R, \tag{3.86}$$

$$\omega_l L - 1/(\omega_l C) = -R. \tag{3.87}$$

Working with these formulas is messy, but we need to go through the details because the results are important. Let us divide through by the resonant inductive reactance $\omega_0 L$ and substitute $1/(\omega_0^2 L)$ for C. We can write

$$\omega_u/\omega_0 - \omega_0/\omega_u = +R/(\omega_0 L), \tag{3.88}$$

$$\omega_l/\omega_0 - \omega_0/\omega_l = -R/(\omega_0 L). \tag{3.89}$$

The ratio of reactance to resistance in a series circuit is called the *quality factor*, or Q for short:

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}. \tag{3.90}$$

We will see later that this corresponds physically to the ratio of the energy stored in reactive elements to the energy lost in the resistor. We can write the quality factor in terms of either the inductive reactance or capacitive reactance, but it is important to realize that in a resonant circuit, it is not the total reactance that we are talking about, but one or the other. The idea of Q is useful because we can relate it to the bandwidth in a simple way. In terms of Q our formula becomes

$$\omega_u/\omega_0 - \omega_0/\omega_u = +1/Q, \tag{3.91}$$

$$\omega_l/\omega_0 - \omega_0/\omega_l = -1/Q. \tag{3.92}$$

If you study these formulas, you can see that ω_u and ω_l must be related by

$$\omega_u/\omega_0 = \omega_0/\omega_l. \tag{3.93}$$

We can rewrite this relation as

$$\sqrt{\omega_l \omega_u} = \omega_0. \tag{3.94}$$

In words, the resonant frequency is the geometric mean of the upper and lower half-power frequencies. Now we substitute Equation 3.93 back in Equation 3.91 to get

$$\omega_u/\omega_0 - \omega_l/\omega_0 = 1/Q. \tag{3.95}$$

I have skipped arithmetic here, but you should fill in the details. It is easier to relate this formula to measurements if we rewrite it in terms of the frequency f by dividing ωs by 2π. We get

$$Q = \frac{\omega_0}{\omega_u - \omega_l} = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}, \tag{3.96}$$

where $\Delta\omega$ is the half-power bandwidth in radians per second and Δf is the half-power bandwidth in hertz. In words, Q is the ratio of the resonant frequency to the bandwidth. If we want a selective filter with a small bandwidth, then we need a large Q. The Q of the resonant circuits that you build with inductors and capacitors is rather low, less than 100. Later on, we will study quartz crystal resonators that have Qs in the range of 50,000 to 100,000; these make extremely selective filters.

Now consider the behavior of the circuit in the stop band, far away from the resonant frequency. At high frequencies, where $\omega L \gg 1/(\omega C)$ and $\omega L \gg R$, the inductive reactance dominates the circuit, and we can write the circuit impedance approximately as

$$Z \approx j\omega L. \tag{3.97}$$

The output voltage becomes

$$V = \left(\frac{R}{j\omega L}\right) V_i = \frac{V_i}{j\omega\tau_l}, \tag{3.98}$$

where $\tau_l = L/R$ is the inductive time constant. This resembles the equation for the roll off in a low-pass filter (Equation 3.77).

At low frequencies, where $1/\omega C \gg \omega L$ and $1/\omega C \gg R$, the capacitive reactance dominates, and we can write the circuit impedance approximately as

$$Z \approx 1/j\omega C. \tag{3.99}$$

The output voltage becomes

$$V = j\omega RC V_i = j\omega\tau_c V_i, \tag{3.100}$$

where $\tau_c = RC$ is the capacitive time constant. This resembles the equation for the roll off in a high-pass filter (Equation 3.79). We can use Equations 3.98 and 3.100 to predict the rejection ratio of filters at different frequencies in the stop band.

3.7 Parallel Resonance

We have learned that if we want a small bandwidth, we need a large Q. For a large Q in a series resonant circuit, the reactance must be large compared to the resistance. This makes it convenient to use series circuits when the resistances are low, like the 50-Ω input of an antenna and a receiver. However, if the resistance is large, it becomes more difficult to make a high-Q series resonant circuit. For example, the input resistance of the mixers in the NorCal 40A is 1,500 Ω, and a high-Q filter would require extremely large reactances. A parallel resonant circuit may be a good choice in this case. This is because a high-Q parallel circuit requires that the reactance be small compared to the resistance. Let us consider a current source with an inductor, capacitor, and load resistor in parallel (Figure 3.7a). Our analysis will be like that of the series resonance, except that we use admittance instead of impedance. We write the output voltage as

$$V = I/Y \tag{3.101}$$

and the load admittance Y as

$$Y = G + jB = G + j\omega C + 1/(j\omega L). \tag{3.102}$$

We start with the susceptance B, which is the imaginary part of the admittance:

$$B = \omega C - 1/(\omega L). \tag{3.103}$$

At low frequencies, the inductive susceptance dominates, and the susceptance is large and negative. At high frequencies, the capacitive susceptance dominates, and the susceptance is large and positive. The susceptance is zero at the resonant frequency ω_0 given by

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{3.104}$$

This is the same formula we found for series resonant circuits. At the resonant frequency, the inductive and capacitive susceptance cancel, and we have the

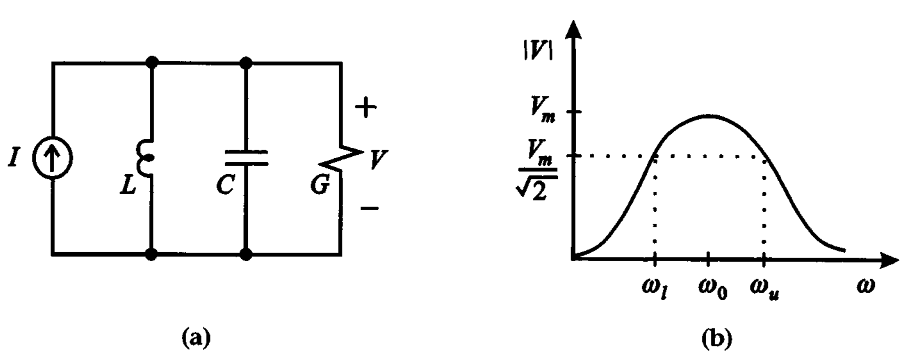


Figure 3.7. Parallel resonant circuit (a), and the response (b).

maximum output voltage. We can write it as

$$V_m = I/G. \tag{3.105}$$

Away from the resonant frequency, the susceptance increases, and the voltage falls (Figure 3.7b). The arithmetic for finding the upper and lower half-power frequencies is similar to that for the series circuit. The half-power frequencies occur when the susceptance equals the conductance. We can write the upper and lower half-power frequencies as

$$\omega_u/\omega_0 - \omega_0/\omega_u = +G/(\omega_0 C), \tag{3.106}$$

$$\omega_l/\omega_0 - \omega_0/\omega_l = -G/(\omega_0 C). \tag{3.107}$$

We define the Q for a parallel circuit as the ratio of susceptance to conductance:

$$Q_p = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 L G}. \tag{3.108}$$

We write this as Q_p, with p standing for parallel. We can rewrite these as

$$Q_p = \frac{R}{\omega_0 L} = \omega_0 RC, \tag{3.109}$$

where $R = 1/G$. This is the inverse of the series formula (Equation 3.90). This means that for high Q in a parallel resonant circuit, we need small reactances. This is different from series resonance. The bandwidth formula, however, is the same as before:

$$Q_p = \frac{f_0}{\Delta f}. \tag{3.110}$$

3.8 Phasor Power

We can write the instantaneous power as

$$P(t) = V(t)I(t). \tag{3.111}$$

Here power is a function of time. This expression includes power that is dissipated as heat in resistors or radiated from antennas and power that goes into inductors and capacitors. In Equation 2.9, we wrote the average power for a resistor with a cosine current as

$$P_a = I_p^2 R/2, \tag{3.112}$$

where I_p is the peak amplitude. We can also write the power in terms of phasors. We define the *complex power* P as

$$P = VI^*/2, \tag{3.113}$$

where * denotes the complex conjugate. We substitute in terms of the circuit impedance $V = ZI$ and get

$$P = ZII^*/2. \tag{3.114}$$

We can write this in terms of the magnitude $|I|$ as

$$P = Z|I|^2/2. \tag{3.115}$$

We can interpret this expression if we write Z in terms of the resistance R and the reactance X:

$$P = R|I|^2/2 + jX|I|^2/2. \tag{3.116}$$

The first term on the right side is real. It is equal to the average power (Equation 3.112), so that we can write

$$P_a = \mathrm{Re}(P) = \mathrm{Re}(VI^*/2). \tag{3.117}$$

The second term on the right side of Equation 3.116 is imaginary. This is the *reactive power*, and it is related to the energy stored in the inductors and capacitors. We can illustrate this for a series combination of an inductor and capacitor. We write the reactive power P_r as

$$P_r = \mathrm{Im}(P) = \frac{\omega L|I|^2}{2} - \frac{|I|^2}{2\omega C} = \omega\left(\frac{L|I|^2}{2} - \frac{C|V_c|^2}{2}\right), \tag{3.118}$$

where V_c is the capacitor voltage. We can rewrite this in terms of the stored energy as

$$P_r = \omega(E_l - E_c), \tag{3.119}$$

where E_l is the peak energy stored in the inductor and E_c is the peak energy stored in the capacitor. This calculation is for a series RLC circuit, but the result also holds for more complicated circuits. The reactive power is proportional to the difference between the peak magnetic energy and the peak electric energy. At resonance, the reactive power is zero, and the peak electric energy equals the peak magnetic energy.

Equation 3.119 allows us to develop a more general formula for Q that includes the series and parallel circuits as special cases. We can rewrite the series Q as

$$Q = \omega\frac{L}{R} = \omega\frac{L|I|^2/2}{R|I|^2/2}, \tag{3.120}$$

or

$$Q = \omega\frac{E_l}{P_a}. \tag{3.121}$$

At resonance the peak inductor energy E_l is equal to the peak capacitor energy E_c, and this energy oscillates back and forth between the inductor and capacitor. When the stored energy in the inductor is at its peak, the stored energy in the capacitor is zero, and this means that E_l is actually the total energy stored in the circuit. We drop the subscript and get

$$Q = \omega\frac{E}{P_a}, \tag{3.122}$$

where E is the total stored energy. This says that Q is proportional to the ratio of the stored energy to the average power. To raise Q, we should increase the stored energy or decrease the loss. You should verify that this general formula is equivalent to the Q_p we defined for parallel resonant circuits. We will also apply the formula to resonant transmission lines in the next chapter.

FURTHER READING

Complex numbers are a fascinating part of mathematics, and students who would like to learn more should read Paul Nahin's *An Imaginary Tale: The Story of* $\sqrt{-1}$, published by Princeton University Press, on the history and application of complex numbers. Nahin has developed geometric interpretations that provide powerful insights into the solution of many physics and engineering problems. The classic textbook on complex numbers is *Theory of Functions of a Complex Variable,* by A. I. Markushevich, published by Chelsea Publishing Company.

PROBLEM 7 – PARALLEL-TO-SERIES CONVERSION

A. It is often useful in discussing circuits to be able to convert a parallel combination of reactance and resistance to an equivalent series combination. Starting with the parallel circuit in Figure 3.8a, find expressions for the components in a series circuit (Figure 3.8b) that give the same impedance. One way to approach this problem is to define a Q for each circuit that is the ratio of the reactance to the resistance. We let

$$Q_s = X_s/R_s \tag{3.123}$$

and

$$Q_p = R_p/X_p. \tag{3.124}$$

First show that if the two circuits are to have the same impedance, the two Qs must be the same. This means that in the rest of the problem, you can drop the subscripts, and just write Q.

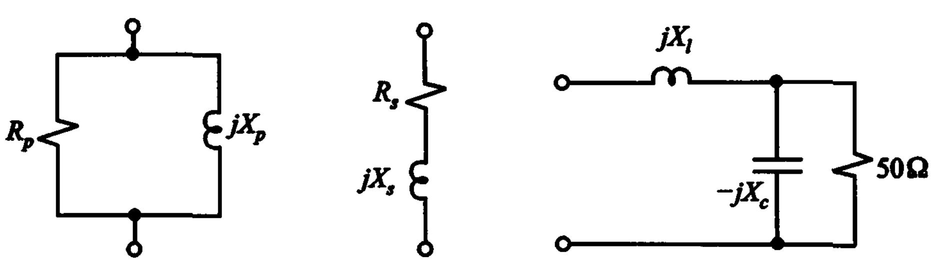


Figure 3.8. Parallel circuit with a resistance R_p and a reactance X_p (a), equivalent series circuit with R_s and X_s (b), and matching circuit with a parallel capacitor and a series inductor (c).

B. Find an approximate formula for X_s when Q is large. Find an approximate formula for R_s when Q is small.

There is one thing you should think about. The Q we define here is not quite the same as the one we use for resonant circuits. This Q involves the total reactance, whereas the resonant circuit Q uses only one of the two reactances. In practice, people use the same letter Q for both situations, and you have to figure out which is intended by the context.

Many transmitters have a low output impedance so that the output power varies inversely with the load resistance. For example, if an amplifier has an output of 1 W with a 50-Ω load, we would hope for 10 W with a 5-Ω load.

C. We will use the network in Figure 3.8c to transform a 50-Ω antenna to 5 Ω. We need our parallel–series conversion formulas. The first step is to find the capacitor reactance X_C. When the capacitor and resistor are converted to a series circuit, the resistance should be 5 Ω. Next choose the inductor reactance X_l to cancel the capacitive reactance. What capacitance (in nF) and inductance (in nH) are required at a frequency of 7 MHz?

PROBLEM 8 - SERIES RESONANCE

In this problem, we solder an inductor and capacitor on the NorCal 40A circuit board and make measurements. It is convenient to mount the board in an electronics vise for soldering. The components mount on the side with the white lettering, and the solder is applied to the other side. Insert the parts that you plan to solder. They should be close to the board, but you may want to leave a millimeter of space so that you can hook up scope probes conveniently. You may need to bend the wires a bit so that the parts do not fall out. Before you solder, check that the parts are in the right holes. They can be unsoldered if you make a mistake, but this is difficult if the part has more than two leads.

Before you start, put some water on a sponge. Turn on a soldering iron, and when it is warm, apply solder to the tip of the iron to tin it. Wipe the tip on the sponge to remove the excess solder. This wiping leaves a shiny surface on the tip that heats up parts much better than a tip without solder. Apply the tip and solder at the same time to the hole and the wire. Be alert when soldering parts with plastic packages, or the plastic will melt. Do not use more solder than you need to flow through the hole and coat the wire, or you run the risk that there will be short circuits to other holes. Clip off the wire ends close to the board after you finish so that they will not touch other wires. Inspect the hole and the wire. The solder should flow completely through the hole and coat the wire. If the wire is not hot enough, the solder will not coat the wire well. This is called a cold solder joint. Cold solder joints often cause open circuits.

If you do make a mistake and put the parts in the wrong holes, be careful when you take them out so that you do not damage the parts or the board. I like to remove solder with wick before I remove the part. Solder wick is a copper braid that absorbs molten solder. Melt the solder with the iron and coax the solder into the wick. Cut off pieces of the wick that get solder on them and throw the pieces away. When you have taken off as much solder as you can, apply the soldering iron at the joints to melt the remaining solder and loosen the part with pliers. You may have to do this repeatedly with each lead

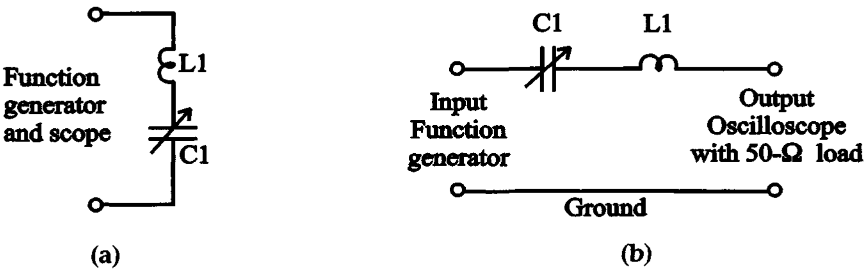


Figure 3.9. Connections for observing a series resonance (a), and band-pass filter connection (b).

until the part comes out. Finally, use the solder wick to remove the solder from the hole before you insert another part.

Solder contains lead, which is dangerous if you eat it; therefore you should wash your hands with soap and water after soldering. It is a good idea to remember to turn off the soldering iron when you have finished working. Leaving the irons on is hard on the tips.

Install C1 and L1 on the NorCal 40A board. C1 is a variable capacitor with a nominal range from 8 to 50 pF, and L1 is an inductor with an inductance of 15 μH. The variable capacitor has meshing metal vanes separated by ceramic insulators. It is best adjusted with a plastic screwdriver. Metal screwdrivers have extra capacitance that shifts the resonant frequency. We will tune the capacitor for resonance at 7 MHz. The inductor and capacitor form part of the RF Filter. Look inside the front cover of this book to see where this circuit goes in the receiver. Appendix D also has data on the components that you use.

A. Make the connections shown in Figure 3.9a, with a tee connection to the scope. Set the function generator for a 7-MHz sine wave with an amplitude of 1 Vpp. When a circuit is resonant, the capacitive reactance cancels the inductive reactance, leaving us with the source resistance of the function generator (50 Ω) and the resistance of the capacitor and the inductor. If we tune the capacitor through resonance, we will see a dip in the input voltage when the circuit is resonant. Adjust the capacitor for minimum input voltage. This sets the resonant frequency to 7 MHz. What is this voltage? Use the voltage to calculate the total resistance of the capacitor and inductor.

B. Now make the connections for the band-pass filter shown in Figure 3.9b, with a 50-Ω scope load. Adjust the capacitor for maximum output voltage. What is this voltage? Calculate the voltage that you expect.

C. Find the half-power bandwidth by measuring the frequencies f_u and f_l where the output voltage has dropped by a factor of $\sqrt{2}$. One way to do this is to use a larger amplitude setting of 1.41 Vpp, and look for the two frequencies that give the same output voltage as 1 Vpp at 7 MHz.

D. Now we will calculate the half-power bandwidth Δf that we expect. Start by finding the resonant inductive reactance. Calculate the Q from the inductive reactance and the total circuit resistance, and then calculate the half-power bandwidth.

E. Return the amplitude setting to 1 Vpp. Measure the output voltage at 1-MHz intervals from 1 MHz to 15 MHz and make a plot. The response changes dramatically between 6 MHz and 8 MHz, and you will need some additional data points to keep them from getting too far apart.

AM radio transmitters in the frequency range from 0.5 to 1.5 MHz are a big problem for receivers because they are usually close and powerful. For example, some broadcasters use 50 kW, and they may only be a few miles away. The NorCal 40A is for 2-W stations that might be a thousand miles away. We will find the AM *voltage rejection factor* R_{am}, given by

$$R_{am} = V_{rf}/V_{am}, \tag{3.125}$$

where V_{rf} is the output at 7 MHz and V_{am} is the output at 1 MHz.

F. Measuring V_{am} is tricky, because the output is small at 1 MHz. One way to approach the problem is to use an amplitude setting of 10 Vpp at 1 MHz. This increases the output voltage by a factor of 10, making it easier to measure. You will need to divide your output voltage by a factor of 10 to take this into account before you compare it with the 7-MHz voltage. It is not a good idea to use an amplitude of 10 Vpp at 7 MHz, because the voltages on the inductor and capacitor get large enough to change their response. What is R_{am}?

G. Use the low-frequency approximation (Equation 3.100) to calculate the value of R_{am} that we would expect.

PROBLEM 9 – PARALLEL RESONANCE

In the NorCal 40A, the transmitter signal is produced by mixing the VFO at 2.1 MHz with the Transmit Oscillator at 4.9 MHz. The transmitter frequency is the sum of these two frequencies, 7.0 MHz. The Transmit Mixer also produces other frequencies that are removed by the Transmit Filter. This filter uses a parallel resonance. A parallel resonance is a good choice for a band-pass filter if the source and load resistances are large, because we can easily make capacitors and inductors with much smaller reactances to give high Q. This filter is made up of C37, C38, C39, and L6. You should study the endpaper to see how this circuit works in the transmitter.

Start by soldering C37 (5-pF disk) and C38 (100-pF disk) on the board. Do not include the variable capacitor C39 yet – we will make some measurements first. L6 is the first inductor that you make yourself by winding wire on a toroidal core. *Toroidal* means donut-shaped. This shape is good for radio inductors because it keeps the magnetic field inside the magnetic material. Compared with the smaller rod inductors we have worked with so far, the toroidal inductors have a better Q and can operate at higher power. L6 uses a T37–2 core. "T" indicates toroidal core, 37 is the outside diameter in hundredths of an inch, and 2 refers to the particular mix of material. Material #2 is an iron powder mix that is useful from 1 to 30 MHz. #2 cores are traditionally painted red to distinguish them from other mixes.

The L6 coil has 28 turns of #28 wire. Cut a 40-cm length and wrap it around the core, being careful with the count (Figure 3.10a). It is easy to be low by one turn. For example, the figure shows a core with 6 turns, not 5. After you finish winding, spread the turns evenly around the core, leaving a gap between the first and last turn so that the

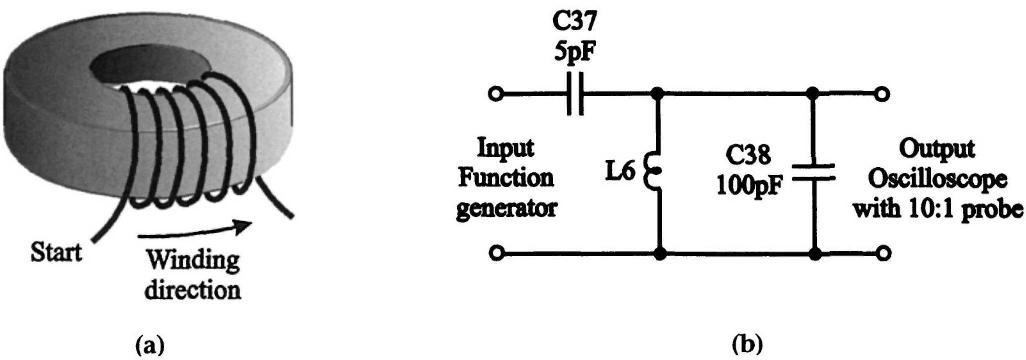


Figure 3.10. (a) Winding a toroid. Six turns are shown. (b) Initial band-pass filter connections.

wire ends line up with the holes in the board. Strip the wire ends using a cigarette lighter to burn the enamel. Sand the ends until the enamel is completely gone. If the enamel is not completely removed, the solder may not stick to the wire. If you are lucky, you get an open circuit. More likely is an intermittent contact that depends on temperature, pressure, and the phase of the moon.

Solder L6 onto the board. Connect the function generator and 10:1 probe as shown in Figure 3.10b. The coil wire is thin, and it is a bad idea to attach probes to it. It will take a little practice to follow the traces on the circuit board so that you can tell where to connect the probes. Most of the traces are on the solder side of the board, but there are also a few on the component side. Moreover, most of the component side is a single connected ground plane. If you see solder pads that do not appear to lead anywhere, they are likely to be ground connections.

A. The function generator should be set for a sine wave, with an amplitude setting of 1 Vpp. Find the resonant frequency f_0 that gives the largest output voltage. From f_0 and the total capacitance (C37, C38, and the probe capacitance), calculate the inductance of the coil that you wound.

B. We will discuss inductance calculations in Chapter 6, but for now you need to know that the inductance is proportional to the square of the number of turns. We write

$$L = A_l N^2, \tag{3.126}$$

where A_l is an inductance constant and N is the number of turns. Core manufacturers provide the inductance constant in their data sheets. For the T37–2 core, A_l is 4.0 nH/turn^2. Calculate the inductance that you expect for L6.

C. Now solder the variable capacitor C39 into the circuit. Set the frequency to 7 MHz and adjust the capacitor carefully for maximum output. Record the output voltage. Measure the half-power bandwidth Δf, and calculate the Q.

D. Calculate the inductor reactance X at 7 MHz. Use this reactance and the Q you measured to find the effective parallel resistance R. This resistance is not a separate component but is associated with the inductor, the capacitors, the function generator, and the scope probe.

E. We can also calculate the output voltage that we expect. One way to start is to find a Norton equivalent circuit for the series combination of the function generator and the 5-pF capacitor. The output voltage can be calculated from the Norton current and the effective parallel resistance R.

F. In addition to the sum frequency at 7 MHz, the mixer produces a strong difference-frequency signal at 2.8 MHz. We do not want to transmit the difference frequency, because it might interfere with other services. Measure the response of the filter at the difference frequency. Express the difference-frequency voltage rejection factor R_- as

$$R_- = V_{rf}/V_-, \tag{3.127}$$

where V_{rf} is the 7-MHz voltage and V_- is the difference voltage. At 2.8 MHz, you should turn up the function generator to 10 Vpp to make the output signal as large as you can, and you should take this into account in calculating the voltage ratio. The output signal will be quite small, and the trace will become fuzzy because of scope noise. You need to be careful to measure at the same place in the noise at the top and bottom of the sine wave.

G. Although dB are units for comparing power levels, we can also write dB expressions in terms of voltage or current if we take into account the fact that the power is proportional to the square of the voltage or current. We write

$$10\log(P_1/P_2) = 20\log(V_1/V_2) = 20\log(I_1/I_2)\,\text{dB}. \tag{3.128}$$

For example, if V_1 is twice V_2, then we would say that the first signal is 6 dB bigger than the second. For these voltage and current formulas to make sense, the resistance associated with each power must be the same, because the power depends on the resistance. This is appropriate for the rejection factor of a filter. Now express the rejection factor as a dB difference, using the formula

$$R_- = 20\log(V_{rf}/V_-)\,\text{dB}. \tag{3.129}$$

H. Calculate what the difference-frequency rejection should be. You will need to consider how the circuit quantities vary with frequency.

I. What would the Q of the filter be if the 5-pF input capacitor (C37) were bypassed and the 50-Ω function generator were connected directly to C38?