4

Transmission Lines

Cables allow us to transmit electrical signals from one circuit to another. For example, we might attach coaxial cable between a function generator and an oscilloscope (Figure 4.1a) and plastic-coated twin lead between an antenna and a television (Figure 4.1b). Usually, when we analyze the circuit, we assume that the voltage at one end of the cable is the same as the voltage at the other end and that the current at the beginning is the same as the current at the end. This is appropriate if the frequency is low. However, at high frequencies the cable itself begins to have an effect. A fundamental limitation is the speed of light. If the voltage at one end of the cable changes appreciably in less time than it takes light to propagate to the other end, we should expect the voltage to be different at the two ends. Another way of saying this is that we would expect the voltages at the ends to be different when the length of the cable becomes an appreciable fraction of a wavelength.

4.1 Distributed Capacitance and Inductance

However, even when the cable is considerably shorter than a wavelength, it can have a large effect. We found in Problem 3 that a cable has capacitance. This capacitance is associated with the charges that the voltages on the line induce. We can take the capacitance into account in a circuit by adding a capacitance between the wires (Figure 4.1c). Some of the current will return through this capacitance. This means that the current at the end of the cable will not be the same as the current at the beginning. This is apparently a violation of Kirchhoff's current law. In addition, the cable has inductance. The inductance comes from the magnetic field that the currents make. We can include this effect by adding a series inductor (Figure 4.1c). There will be a voltage drop across the inductance, so that the voltage at the end of the cable will not be the same as at the beginning. This is an apparent violation of Kirchhoff's voltage law. Now we have an equivalent circuit for our cable with a series inductance and a parallel capacitance. There is another effect, resistance in the wires, that we will take into account later.

It is not obvious whether the inductance or capacitance is more important. It depends on the load impedance. If the impedance is high, the current is relatively small, and the inductance has little effect. However, the capacitive current will

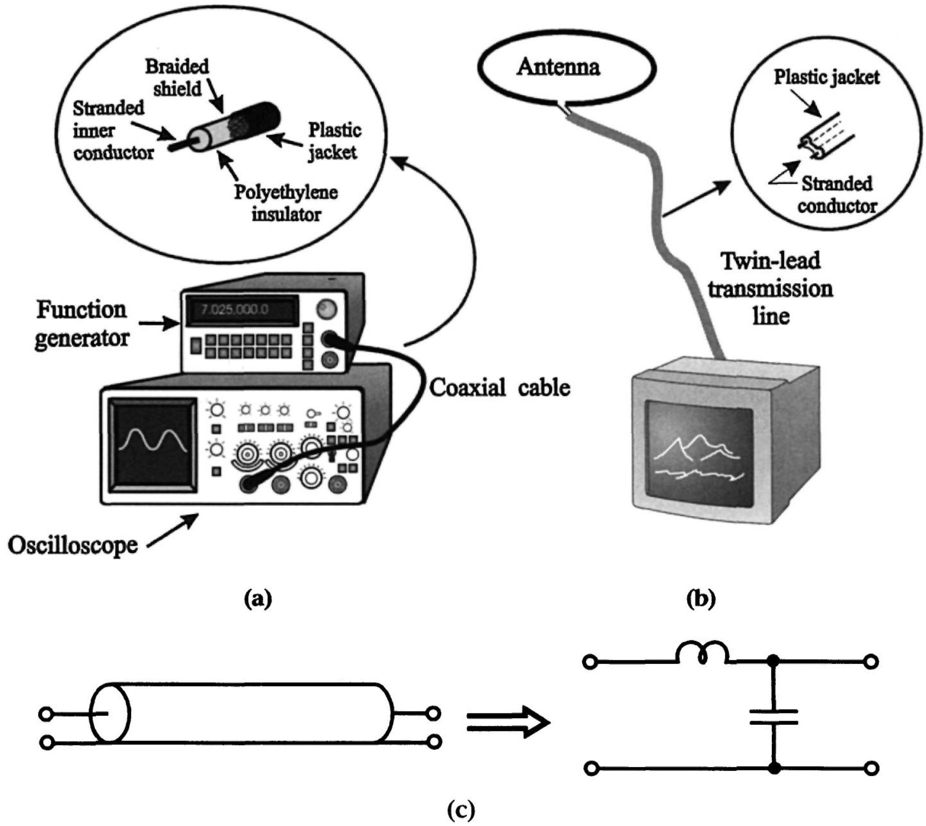

Figure 4.1. (a) Connecting a function generator to an oscilloscope with coaxial cable, and coaxial-cable construction (inset). (b) Connecting an antenna to a television with twin lead, and twin-lead construction (inset). (c) An equivalent circuit for the cable that includes a series inductance and a shunt capacitance.

be relatively important. For example, let us assume that we are making a coaxial-cable connection to an oscilloscope with an input resistance of 1 MΩ and a parallel capacitance of 20 pF. This is a relatively high impedance, and it is usually more important to consider the effect of the cable capacitance than the inductance. The capacitance of a typical coaxial cable is 100 pF/m. A one-meter cable increases the capacitance of the oscilloscope connection from 20 pF to 120 pF, and we would notice delays that are much larger than we would expect without the cable. However, if the load impedance is small, the load current will be large, and the inductance will be more important.

Our circuit model is really a simplification. We cannot really say that the inductor should go before the capacitor, or the other way around, because the capacitance and inductance are spread out along the cable. This capacitance and inductance are called *distributed* elements to distinguish them from ordinary *lumped* capacitors and inductors. There is an elegant approach to calculate the effect of distributed elements, called *transmission-line theory*. We will derive the transmission-line theory by analyzing a network of small inductors and capacitors.

4.2 Telegraphist's Equations

Our transmission line will have two parallel conductors with uniform cross section. We assume that they are long enough that we need not worry about the ends. We do not assume anything about the shape of the conductors – they could be two adjacent wires, or they could be coaxial. They should not touch each other, because then there would be just one conductor. We divide the line into small sections of length l (Figure 4.2a). Each of these sections has an inductor L_l and a capacitor C_l associated with it. We can draw a network that represents our transmission line and define voltages and currents (Figure 4.2b). We can write the inductor voltage as

$$V_{n+1} - V_n = -L_l \frac{dI_{n+1}}{dt} \tag{4.1}$$

and the capacitor current as

$$I_{n+1} - I_n = -C_l \frac{dV_n}{dt}. \tag{4.2}$$

When we draw a model for a transmission line with small inductors and capacitors, we are implicitly assuming that the inductance and capacitance are proportional to the length. This can be shown by electromagnetic theory. If the inductance and capacitance are proportional to the length, then we can let L and C be equal to the proportionality constants and write

$$L = L_l/l, \tag{4.3}$$

$$C = C_l/l. \tag{4.4}$$

Here L and C are called the distributed inductance and capacitance. These are the fundamental quantities that characterize a transmission line. They are determined by the shape of the conductors and the nature of the insulators. The units of distributed inductance are henries per meter, and the units of distributed capacitance

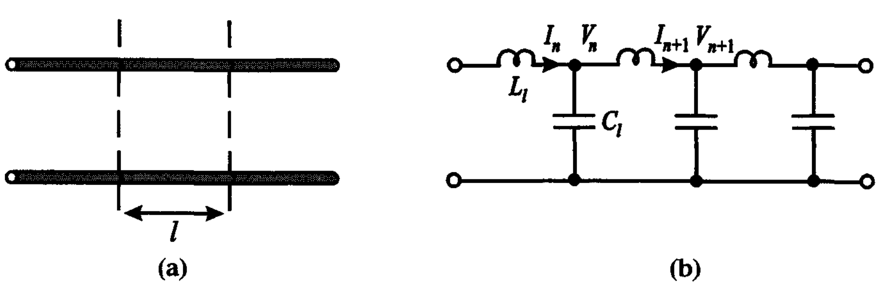

Figure 4.2. Dividing a transmission line into sections of length l (a). Representing the transmission line as a network of inductors and capacitors (b). This kind of network is called a *ladder network*.

are farads per meter. We can rewrite our equations in terms of L and C as

$$\frac{V_{n+1} - V_n}{l} = -L\,\frac{dI_{n+1}}{dt}, \tag{4.5}$$

$$\frac{I_{n+1} - I_n}{l} = -C\,\frac{dV_n}{dt}. \tag{4.6}$$

Notice that if we take the limit as l approaches 0, the quotients on the left become derivatives with respect to distance. To be more precise, these are *partial derivatives*, since the current and voltage are also functions of time. We will let our distance variable be z. In the limit, the equations become

$$\frac{\partial V}{\partial z} = -L\,\frac{\partial I}{\partial t}, \tag{4.7}$$

$$\frac{\partial I}{\partial z} = -C\,\frac{\partial V}{\partial t}. \tag{4.8}$$

Here we use the partial derivative sign, ∂ (which is a funny d), to show that we are taking a derivative with respect to a particular variable. These formulas are known as the *telegraphist's equations* or *transmission-line equations*. They were developed by Oliver Heaviside for telegraph cables more than one hundred years ago. They are extremely important in science and engineering. Similar equations describe radio waves, light, sound, and heat. Consequently, once you understand how to work with the equations, you can solve a wide variety of problems.

The telegraphist's equations predict the propagation of waves. We can derive a wave equation by differentiating the first formula with respect to z and the second with respect to t:

$$\frac{\partial^2 V}{\partial z^2} = -L\,\frac{\partial^2 I}{\partial t \partial z}, \tag{4.9}$$

$$\frac{\partial^2 I}{\partial t \partial z} = -C\,\frac{\partial^2 V}{\partial t^2}. \tag{4.10}$$

We can eliminate I between these two formulas to get the voltage wave equation,

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}. \tag{4.11}$$

4.3 Waves

We can write a voltage wave in the form $V(z-vt)$, where V is a voltage function and v is the velocity. We will consistently use an upper-case V for voltage and a lower-case v for velocity to keep them distinct. We will assume that V is a pulse function centered around $z=0$ at $t=0$ (Figure 4.3a). Some time t_0 later, we sketch the function again (Figure 4.3b). We get the same pulse, displaced to the right by an amount $z=vt_0$. The wave moves in the $+z$ direction, and we call this a *forward* wave. We can write a voltage wave that propagates in the $-z$ direction in the form $V(z+vt)$.

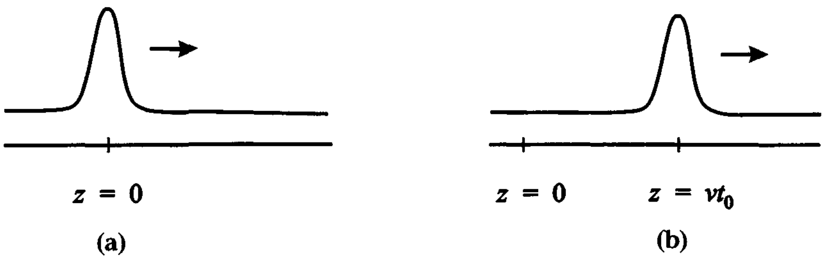

Figure 4.3. A forward wave of the form $V(z-vt)$. At $t=0$, the wave peaks at $z=0$ (a). At time $t=t_0$ later, the wave has moved a distance $z=vt_0$ to the right (b).

This is a *reverse* wave. We can have both a forward and a reverse wave on a transmission line at the same time. The reverse wave is often the reflection from a load.

Now we can show that these wave functions satisfy Equation 4.11. We substitute a forward wave $V(z-vt)$ into the equation, and use the chain rule to write the partial derivatives in terms of the second derivative of V, which we write as V'':

$$\frac{\partial^2 V}{\partial z^2} = V'' = LC\frac{\partial^2 V}{\partial t^2} = LC\,v^2 V''. \tag{4.12}$$

This gives us

$$v = 1/\sqrt{LC}. \tag{4.13}$$

This formula allows us to predict the velocity if we know L and C. For coaxial cable, the velocity is typically 2/3 the speed of light, or 2×10^8 m/s. The twin lead that is commonly used for connecting FM and TV antennas has a velocity of 4/5 the speed of light, or 2.4×10^8 m/s.

Now we can use our transmission-line equations to relate the current to the voltage. The wave equation for current is the same as that for the voltage; thus the solutions are also waves with the same velocity. We can use the chain rule again to rewrite Equation 4.7 as

$$V' = vLI'. \tag{4.14}$$

Notice that both the voltage and the current appear only as derivatives. When we integrate this equation we will have arbitrary constants, which correspond to constant voltages and currents on the line. We will neglect these, because we already know how transmission lines work at DC. We integrate this formula, setting the integration constants to zero, and substitute for v from Equation 4.13 to find the ratio of the voltage to the current:

$$V/I = \sqrt{L/C}. \tag{4.15}$$

This ratio of voltage and current in a forward wave is called the *characteristic impedance*, and it is written as Z_0:

$$Z_0 = \sqrt{L/C}. \tag{4.16}$$

Coaxial cables usually have a characteristic impedance of 50 Ω or 75 Ω, whereas twin lead is typically 300 Ω. If we know Z_0 and v for a transmission line, we can work backwards and calculate L and C. Using Equation 4.13 and Equation 4.16 we can write

$$L = Z_0/v, \tag{4.17}$$

$$C = 1/(Z_0 v). \tag{4.18}$$

We can repeat this analysis for a reverse wave of the form $V(z+vt)$. Equation 4.12 does not change, and so the velocity is the same for a reverse wave as it is for a forward wave. This makes sense, because we have not assumed anything about the line that would make the wave go faster in one direction than the other. We can find the ratio of the voltage and current by substituting into Equation 4.7, and we find

$$V' = -vLI'. \tag{4.19}$$

We integrate to find that

$$V/I = -\sqrt{L/C}. \tag{4.20}$$

This tells us that the ratio of voltage and current in a reverse wave changes sign. We can write formulas that relate the voltage and current as

$$V_+/I_+ = +Z_0, \tag{4.21}$$

$$V_-/I_- = -Z_0, \tag{4.22}$$

where the + subscript is for a forward wave and the − subscript is for a reverse wave. Figure 4.4 shows how these voltages and currents look.

We can understand why the ratio changes sign if we consider the power. Power is positive if it flows to the right and negative if it flows to the left. For a forward wave the power is

$$P_+(t) = V_+(t)I_+(t) = V_+^2(t)/Z_0. \tag{4.23}$$

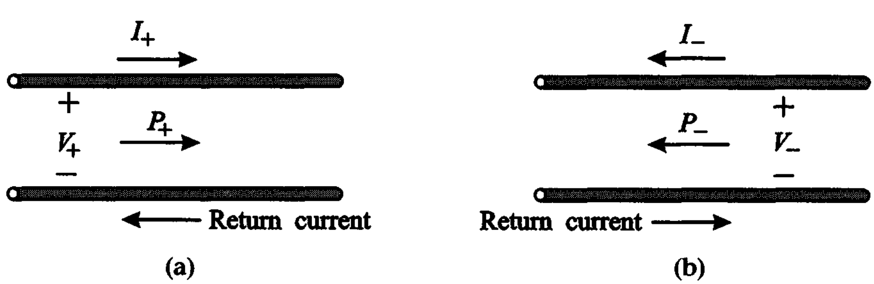

Figure 4.4. Voltages and currents on a transmission line for a forward wave (a) and a reverse wave (b). Both V_+ and V_- are taken to be positive. The current I_+ is positive and it flows to the right in the top conductor. In addition there is a return current in the bottom conductor. The current I_- is negative and it flows to the left in the top conductor.

This is a positive number, indicating that power flows to the right, in the direction of propagation. Notice that the power does not depend on the sign of the voltage. For a reverse wave, the voltage–current ratio changes sign, and we have

$$P_-(t) = V_-(t)I_-(t) = -V_-^2(t)/Z_0. \tag{4.24}$$

This is a negative number, and so power flows to the left, again in the direction of propagation. The sign change reverses the direction of power flow.

4.4 Phasors for Waves

We found that signals that vary in time as cosines can be described in a simple way by phasors, and this allows many circuits to be solved by algebra alone. Waves generated by cosine signals can also be represented by phasors. Let us consider a forward wave of the form

$$V(z - vt) = A\cos(\omega t - \beta z), \tag{4.25}$$

where β (the Greek letter *beta*) is called the *phase constant*, because it determines the phase. The units of β are radians per meter. We can write the corresponding expression for a reverse wave by changing the $-$ signs to $+$ signs. You should work through the details to show that the cosine expression actually has the correct form to be a forward wave. If we compare the right and left sides of the equation, we find that we can write the following expression for v:

$$v = \omega/\beta. \tag{4.26}$$

In addition, the wave is periodic in z, and its wavelength, written as λ, given by

$$\lambda = 2\pi/\beta. \tag{4.27}$$

To convert to phasors, we start by writing the wave as the real part of a complex exponential,

$$V = A\cos(\omega t - \beta z) = \mathrm{Re}[A\exp + j(\omega t - \beta z)]. \tag{4.28}$$

We can rewrite the equation as

$$V = \mathrm{Re}[A\exp(-j\beta z)\exp(j\omega t)]. \tag{4.29}$$

In phasor notation, we consider the complex factor of $\exp(j\omega t)$, given by

$$V = A\exp(-j\beta z). \tag{4.30}$$

It is interesting to plot the locus for wave phasors in the complex plane. For example, consider a forward wave given by

$$V_+ = \exp(-j\beta z). \tag{4.31}$$

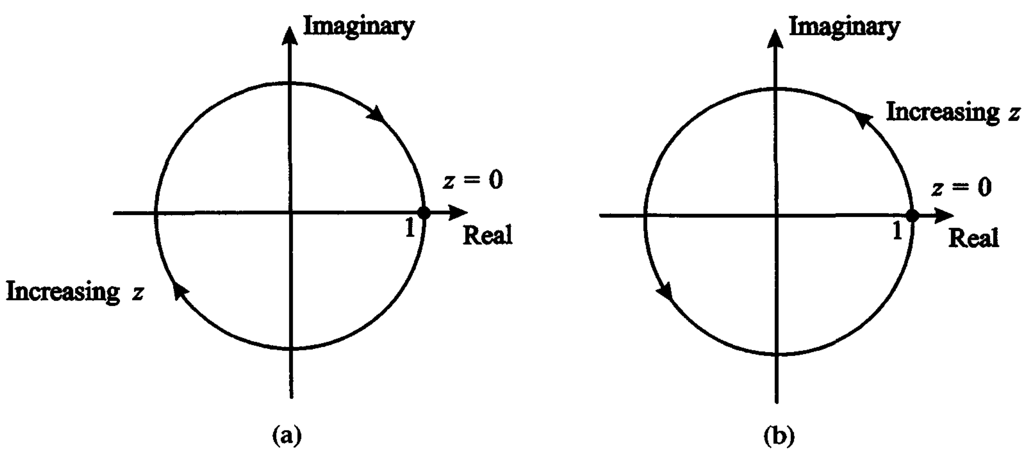

Figure 4.5. Plotting the loci of wave phasors in the complex plane. A forward wave, $V_+ = \exp(-j\beta z)$ (a), and a reverse wave, $V_- = \exp(+j\beta z)$ (b). The phasor for the forward wave rotates clockwise as z increases, and the phasor for the reverse wave rotates counterclockwise.

This path is shown in Figure 4.5a. The phase lags as z increases, and the phasor traces out a clockwise circle. For comparison, in Figure 4.5b we also plot the locus for a reverse wave,

$$V_- = \exp(+j\beta z). \tag{4.32}$$

We see a progressive phase lead as z increases and a counterclockwise circle. Notice that for both waves, the magnitude is constant but the phase varies along the line.

Now we can develop power formulas for phasor waves. We write the complex power P as

$$P = VI^*/2, \tag{4.33}$$

where V and I are phasors. For a forward wave, we have

$$P_+ = \frac{V_+ I_+^*}{2} = \frac{V_+ V_+^*}{2Z_0} = \frac{|V_+|^2}{2Z_0}, \tag{4.34}$$

assuming that Z_0 is real. The power is real and positive. For a reverse wave, the sign of the impedance changes and we get

$$P_- = \frac{V_- I_-^*}{2} = -\frac{V_- V_-^*}{2Z_0} = -\frac{|V_-|^2}{2Z_0}, \tag{4.35}$$

and the average power is negative.

4.5 General Lines

We have seen that phasors allow us to define impedances and admittances for circuit elements. This makes it natural to consider a transmission line with a

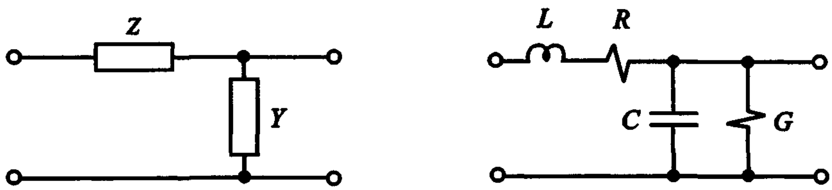

Figure 4.6. General transmission lines. (a) A transmission line with distributed series impedance Z and parallel admittance Y. It is traditional to use a thin rectangle as the symbol for impedances and admittances, because they could be combinations of capacitors, resistors, and inductors. We can consider an LC transmission line as a special case with $Z = j\omega L$ and $Y = j\omega C$. (b) The distributed circuit elements for a transmission line with series resistance R and parallel conductance G.

distributed series impedance and a distributed parallel admittance. We will analyze a transmission line having a distributed impedance Z, with units of ohms per meter, and an admittance Y, with units of siemens per meter (Figure 4.6a). We can follow the same limiting procedure that we used for LC transmission lines to derive more general telegraphist's equations:

$$\frac{dV}{dz} = -ZI, \tag{4.36}$$

$$\frac{dI}{dz} = -YV. \tag{4.37}$$

Now consider a forward wave with a voltage V and current I that vary as $\exp(-jkz)$. We use k here rather than β because we want to allow for the possibility that k will be complex. k is called the *propagation constant*. It is traditional to characterize the real and imaginary parts of k by writing

$$jk = \alpha + j\beta, \tag{4.38}$$

where α is the Greek letter *alpha*. The forward wave phasor is then of the form

$$\exp(-jkz) = \exp(-\alpha z - j\beta z). \tag{4.39}$$

We can see that α determines the loss of the wave as it propagates, and for this reason, it is called the *attenuation constant*. It should be positive, or else the wave will grow instead of decay. The units of α are given their own name, *nepers*/m. The word neper is pronounced "neeper," and it is derived from a Latin version of the name Napier. John Napier was the Scottish mathematician who invented logarithms. Since we often quote losses in decibels, we need to figure out how to convert between nepers and dB. An attenuation of 1 neper corresponds to a voltage reduction by a factor of e. This means we can relate dB and nepers by the formula

$$\alpha_{\mathrm{dB/m}} = \alpha_{\mathrm{nepers/m}} \cdot 20\log_{10}(e) = 8.686 \cdot \alpha_{\mathrm{nepers/m}}. \tag{4.40}$$

Now we return to the general telegraphist's equations, assuming forward waves of the form $\exp(-jkz)$. The equations become

$$jkV = ZI, \tag{4.41}$$

$$jkI = YV. \tag{4.42}$$

In addition, if we let the ratio of V to I be Z_0, we get

$$jkZ_0 = Z, \tag{4.43}$$

$$jk/Z_0 = Y. \tag{4.44}$$

We can write the solutions as

$$jk = \sqrt{ZY}, \tag{4.45}$$

$$Z_0 = \sqrt{Z/Y}. \tag{4.46}$$

In general, all these quantities are complex, and there are two complex roots differing only in sign. It can be difficult to choose the correct sign. Ordinarily, we should choose the sign of jk so that α is positive, to keep the wave from growing as it propagates. In addition, Z_0 should have a positive real part to keep the average power positive.

As an example, let us consider loss in transmission lines. Loss is associated with either the metal or the insulator. We can model the metal loss as a distributed series resistance R, with units of Ω/m (Figure 4.6b). In practice, R is not a constant but usually increases as the square root of the frequency because of an electromagnetic phenomenon called the skin effect. We can write the distributed impedance Z as

$$Z = j\omega L + R. \tag{4.47}$$

We can take the insulator loss into account by a distributed parallel conductance G with units of S/m. The conductance also varies with frequency. In practical transmission lines, G is often small enough that it can be neglected. With conductance, the distributed admittance Y becomes

$$Y = j\omega C + G. \tag{4.48}$$

When we substitute these into our formulas for jk and Z_0, we get

$$jk = \sqrt{(j\omega L + R)(j\omega C + G)}, \tag{4.49}$$

$$Z_0 = \sqrt{(j\omega L + R)/(j\omega C + G)}. \tag{4.50}$$

In both formulas, the correct root is the one with a positive real part.

4.6 Dispersion

The velocity v and the attenuation constant α may vary with frequency. This frequency variation is called *dispersion*, and it is a problem. For example, if v depends on frequency, then different frequency components travel at different velocities and one part of a message interferes with another. If α increases with frequency, then we lose the high-frequency information in our signal. There is in principle a simple solution to this problem, however, that was discovered by Oliver Heaviside. If the transmission-line parameters can be adjusted to satisfy

$$R/L = G/C \tag{4.51}$$

then the attenuation and velocity become constants. Equation 4.49 can be written as

$$jk = j\omega\sqrt{LC}\sqrt{\left(1+\frac{R}{j\omega L}\right)\left(1+\frac{G}{j\omega C}\right)}. \tag{4.52}$$

The terms in parentheses are the same, and so we can rewrite this as

$$jk = j\omega\sqrt{LC}\left(1+\frac{R}{j\omega L}\right), \tag{4.53}$$

or

$$v = \omega/\beta = 1/\sqrt{LC} \tag{4.54}$$

and

$$\alpha = \sqrt{RG}. \tag{4.55}$$

The velocity is the same as that of a lossless line and is independent of frequency. There is loss, but it is independent of frequency, and an amplifier can compensate for it. The impedance is also independent of frequency. Equation 4.50 can be written as

$$Z_0 = \sqrt{L/C}\sqrt{\left(1+\frac{R}{j\omega L}\right)\Big/\left(1+\frac{G}{j\omega C}\right)}. \tag{4.56}$$

Again the terms in parentheses are the same, and so we have

$$Z_0 = \sqrt{L/C}, \tag{4.57}$$

as it is in a line with no loss.

The telephone company uses an approach like this in phone lines. Typically R is considerably larger than ωL and this causes v and α to depend strongly on frequency. In practice, Heaviside's zero-dispersion condition is hard to satisfy, because G is usually close to zero. However, we can come close to zero dispersion by making ωL much larger than R. The phone company does this by adding inductor coils to the lines, usually 88-mH inductors at intervals of one mile. To see how this works, consider a line where $\omega L \gg R$ and $G = 0$. This is a *large-reactance* approximation. We start with the exact formula and derive approximate expressions for Z_0 and jk:

$$Z_0 = \sqrt{(j\omega L + R)/(j\omega C)} \approx \sqrt{L/C} \tag{4.58}$$

and

$$jk = \sqrt{(j\omega L + R)j\omega C} \approx j\omega\sqrt{LC} + (R/2)\sqrt{C/L}, \tag{4.59}$$

where we have used the first-order Taylor-series formula

$$\sqrt{1+z} \approx 1 + z/2, \tag{4.60}$$

which holds when $|z| \ll 1$. From Equation 4.59, we can write α and v as

$$\alpha = R/(2Z_0), \tag{4.61}$$

$$v = \omega/\beta = 1/\sqrt{LC}. \tag{4.62}$$

These are independent of frequency. As an example, in 50-Ω coaxial cable at 5 MHz, the series resistance might be 0.5 Ω/m and the inductance 250 nH/m. The reactance ωL is 7.9 Ω/m, and thus the high-reactance approximation is justified. The loss is given by

$$\alpha = R/(2Z_0) = 0.005 \text{ nepers/m}. \tag{4.63}$$

Now consider a *high-resistance line*, where $R \gg \omega L$. We write

$$jk = \sqrt{(j\omega L + R)j\omega C} \approx \sqrt{j\omega RC}. \tag{4.64}$$

The square root of an imaginary number has an angle of 45°. This means that α and β are equal. We can write

$$\alpha = \sqrt{\omega RC/2}, \tag{4.65}$$

$$v = \sqrt{2\omega/(RC)}. \tag{4.66}$$

Because both α and v vary as $\sqrt{\omega}$, the line is highly dispersive. As an historical example, we can analyze the first transatlantic telegraph cable, laid in 1865. This cable was 3,600 km long and weighed 5,000 tons. The insulator was a vegetable gum called gutta-percha. For this cable L = 460 nH/m, C = 75 pF/m, and R = 7 mΩ/m. At a frequency of 2.4 kHz, $\omega L = R$, and so the high-resistance assumption is well satisfied for frequencies below 100 Hz. At 12 Hz, we can write α and v as

$$\alpha = \sqrt{\omega RC/2} = 4.4 \times 10^{-3} \text{ nepers/km}, \tag{4.67}$$

$$v = \sqrt{2\omega/(RC)} = 17{,}000 \text{ km/s}. \tag{4.68}$$

The loss for the entire line is αl = 140 dB and the delay is l/v = 210 ms. For comparison, at 3 Hz, the loss in dB and the delay change by a factor of 2, to 70 dB and 420 ms. Thus the 12-Hz component attenuates 70 dB more than the 3-Hz component. In addition, the 12-Hz component arrives 210 ms ahead of the 3-Hz component. In order to improve these characteristics, the signalling speed had to be drastically reduced, to about one word per minute, which was twenty times slower than hoped for. You might be interested to know that the renowned physicist, Lord Kelvin, did this analysis, but the project chief ignored it. His name was

Dr. Whitehouse (a medical doctor), and he said, "In electricity, there is seldom any need of any mathematical or other abstractions, ... and the formulas may for all practical purposes be dispensed with." The end was tragic. The cable operators thought they could improve the signalling rate by increasing the voltage, and they drove the line with 2-kV pulses. The insulation was not good enough to take this large voltage, and within two weeks, a short developed, somewhere along the 3,600-km line.

4.7 Reflections

Until now, we have not worried about what happens at the end of a transmission line. However, most of the time we are really more interested in what is going on at the ends than we are in the middle, because our sources and loads are usually at the ends. First we consider a transmission line with a load (Figure 4.7). Let the line impedance be real and be given by Z_0 and the load impedance be Z. Assume a forward wave V_+ is incident on the load. The effect of the load will be to make a reflected wave V_-. We will see that the amplitude of the reflected wave is determined by how different Z is from Z_0.

We call the ratio of V_- to V_+ the *reflection coefficient*. It is given by

$$\rho = V_-/V_+, \tag{4.69}$$

where ρ is the Greek letter *rho*. Notice that ρ is a *voltage* reflection coefficient. Sometimes we need a *current reflection coefficient*, ρ_i, defined in a dual way as

$$\rho_i = I_-/I_+. \tag{4.70}$$

The current reflection coefficient has the same magnitude as the voltage reflection coefficient because the voltages and currents in the waves are proportional. However, since the current in the reverse wave changes sign, we can write

$$\rho_i = -\rho. \tag{4.71}$$

The load voltage V is also proportional to the incident voltage, and this ratio is called the *transmission coefficient*. We write this as τ:

$$\tau = V/V_+. \tag{4.72}$$

Now we can find a simple formula that relates ρ and τ. The load voltage V is the

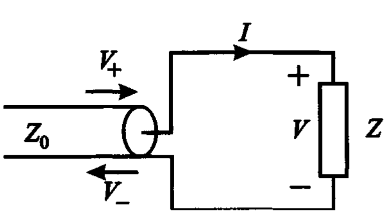

Figure 4.7. Reflection and transmission at a load.

sum of the incident wave V_+ and the reflected wave V_-. We can write

$$V = V_+ + V_-. \tag{4.73}$$

If we divide by V_+, we get

$$\tau = 1 + \rho. \tag{4.74}$$

This is an important expression, because it means that ρ and τ are not independent – we can calculate one if we know the other.

Next we can find formulas that relate ρ to Z. We can write an expression for the load current I as the sum of the current in the incident wave I_+ and the current in the reflected wave I_-:

$$I = I_+ + I_-. \tag{4.75}$$

Now divide this formula into Equation 4.73 to get

$$\frac{V}{I} = \frac{V_+ + V_-}{I_+ + I_-} = \frac{V_+}{I_+}\,\frac{1 + V_-/V_+}{1 + I_-/I_+}. \tag{4.76}$$

We can substitute for all these ratios and rewrite the formula as

$$\frac{Z}{Z_0} = \frac{1+\rho}{1-\rho}. \tag{4.77}$$

This formula lets us calculate Z if we know ρ. Microwave instruments measure reflection coefficient rather than impedance, and we can use this formula to see what Z really is. We can also solve for ρ as

$$\rho = \frac{Z - Z_0}{Z + Z_0}. \tag{4.78}$$

This is one of a family called *bilinear transforms*, and because of this it turns out that the loci of constant resistance, reactance, conductance, and susceptance are circles or straight lines. It is typical to plot ρ in the complex plane (Figure 4.8).

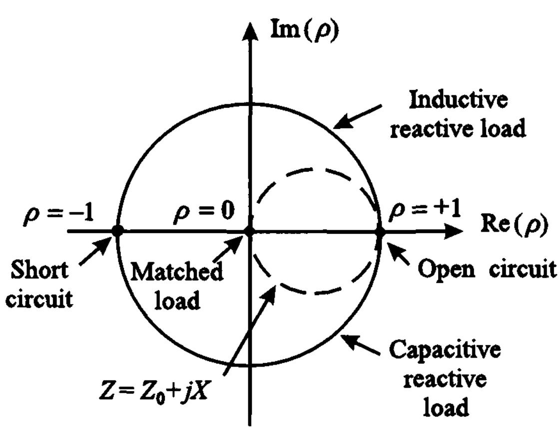

Figure 4.8. Plot of ρ in the complex plane.

These plots are usually called Smith charts, after Philip Smith, the engineer at Bell Labs who thought of this approach.

Let us consider some special cases and see what the reflection coefficients are. When $Z = Z_0$, the load is said to be *matched*. The reflection coefficient is 0; hence there is no reflection. We use this idea in the lab when we work with oscilloscopes and fast signals, where we may see ringing on waveforms from repeated reflections. If we put a matching resistor in parallel with the oscilloscope input we can stop the ringing. When Z is real, then ρ is also real. This means that if the load is a resistor R, then ρ lies along the real axis. If $R > Z_0$, then ρ is positive, and the reflected wave has the same phase as the incident wave. If $R < Z_0$, then ρ is negative, and the reflected wave is 180° out of phase with the incident wave. At the extremes, a short circuit has a reflection coefficient of -1, and an open circuit has a reflection coefficient of $+1$. Now consider a reactive load, with $Z = jX$. We can write

$$\rho = \frac{jX - Z_0}{jX + Z_0}. \tag{4.79}$$

The absolute values of the real and imaginary parts of the numerator and denominator are the same, and so the magnitudes of the numerator and denominator are the same. This means that the reflection coefficient lies on the unit circle. The inductive reactances are along the top half. Let us start at $X = 0$ and consider the locus as X increases. When $X = 0$, the load is a short, and $\rho = -1$. As X increases we move clockwise along the top half of the unit circle. When $X = Z_0$, we are at the top, and as X approaches ∞, we reach $\rho = 1$. The capacitive reactances are along the bottom half. One other interesting case to consider is an impedance of the form $Z = Z_0 + jX$. This locus appears in Figure 4.8 as a dashed circle that passes through $\rho = 0$ and $\rho = +1$.

4.8 Available Power

We can find the transmission coefficient τ by combining Equation 4.74 and Equation 4.78 to get

$$\tau = 1 + \rho = \frac{2Z}{Z + Z_0}. \tag{4.80}$$

Notice that τ can be larger than one. For an open-circuited load, $\tau = 2$, so that the transmitted voltage is twice as large as the incident voltage. We can use this fact to find the Thevenin equivalent circuit for a transmission line. The open-circuit voltage V_o is given by

$$V_o = 2V_+. \tag{4.81}$$

The look-back resistance R_s is just the characteristic impedance of the cable. We get

$$R_s = Z_0. \tag{4.82}$$

This gives us the Thevenin circuit shown in Figure 4.9.

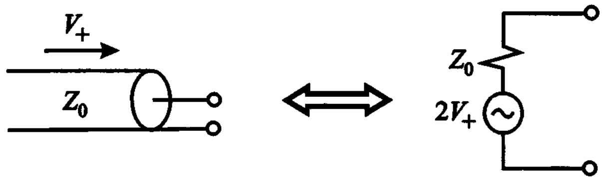

Figure 4.9. Thevenin equivalent circuit for a transmission line with a characteristic impedance Z_0 and an incident wave V_+.

There is more to this result than might appear at first. The Thevenin circuit produces the same voltages and currents in a load as the transmission line. However, another way to see this is to note that the transmission line produces the same results as the Thevenin circuit. We can turn things around, and think of the transmission line as an equivalent source. From this point of view, it is easy to calculate the maximum power from a Thevenin source. The power in the incident wave is given by Equation 4.34 as

$$P_+ = \frac{V_+^2}{2Z_0}, \tag{4.83}$$

where we have taken V_+ to be real. This is the power that is delivered to a matched load, where there is no reflection. It is the maximum power that can be delivered to any load. In terms of the Thevenin parameters, we can rewrite this as

$$P_+ = \frac{V_o^2}{8R_s}. \tag{4.84}$$

We call $V_o^2/(8R_s)$ the *available power* from a Thevenin source. This is the AC version of the DC formula we derived in Problem 1. It is a good idea to learn it, because we will use it repeatedly. The formula is for peak voltages, but in the lab, we use peak-to-peak voltages, which are twice as large. In addition, function generators read half the open-circuit voltage. Usually these factors of two cancel, and this makes it easy to apply the formula.

4.9 Resonance

We found in the last chapter that when we combine inductors and capacitors, we make resonant circuits. Because a cable has both inductance and capacitance, it can also resonate. An open-circuited transmission line turns out to be much more interesting than an ordinary open circuit. It shows the effects of delays and reflections and can even be used as a filter. To start, we connect a function generator to a transmission line with the same impedance (Figure 4.10a). We assume that the line is long enough that we do not have to worry about reflections from the far end. We represent the generator by a Thevenin equivalent circuit, with open-circuit voltage V_o and impedance Z_0. The forward voltage V_+ is given by

$$V_+ = V_o/2. \tag{4.85}$$

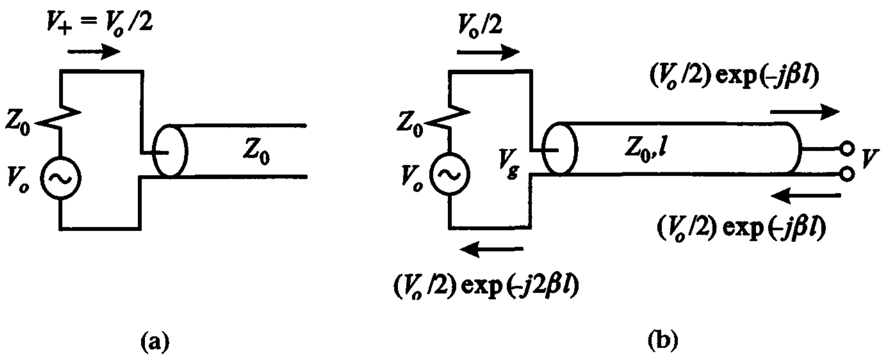

Figure 4.10. Connecting a sine-wave generator to a long transmission line (a) and to an open-circuited line (b).

Now cut the transmission line at some point, leaving the end open-circuited (Figure 4.10b). Starting with the forward voltage at the generator, we can calculate the other voltages on the line by multiplying by phase factors and the reflection coefficient. The forward voltage at the end is given by $(V_o/2)\exp(-j\beta l)$, where l is the length of the line. The reflection coefficient of an open circuit is $+1$; thus the reflected reverse wave is also $(V_o/2)\exp(-j\beta l)$. The total voltage V is

$$V = V_+ + V_- = V_o \exp(-j\beta l). \tag{4.86}$$

This makes sense. It is just the Thevenin voltage with a phase lag due to the transmission line. At low frequencies β approaches zero, and V approaches the Thevenin voltage V_o.

The generator voltage is more surprising. The reflected wave propagates back to the generator where it is absorbed without further reflection. There is an additional phase lag due to the line, so that the reverse wave at the generator is given by

$$V_- = (V_o/2)\exp(-j2\beta l). \tag{4.87}$$

The total voltage at the generator V_g is given by

$$V_g = V_+ + V_- = V_o/2 + (V_o/2)\exp(-j2\beta l). \tag{4.88}$$

We can write this in terms of a cosine by pulling out a factor of $\exp(-j\beta l)$:

$$V_g = V_o \exp(-j\beta l)\cos(\beta l). \tag{4.89}$$

Notice that the phase of V_g is the same as the phase of V (Equation 4.86). In fact, the phase is the same everywhere along the line. We call this a *standing wave.* However, the magnitude of V_g depends on the length of the line. You should notice that V_g is zero when $l = \lambda/4$. In a measurement, we see a resonance at the frequency where the line is a quarter of a wavelength long. This seems mysterious, because the generator output voltage is zero at the same time the voltage at the other end is V_o. However, we do have a current at the generator and we can think

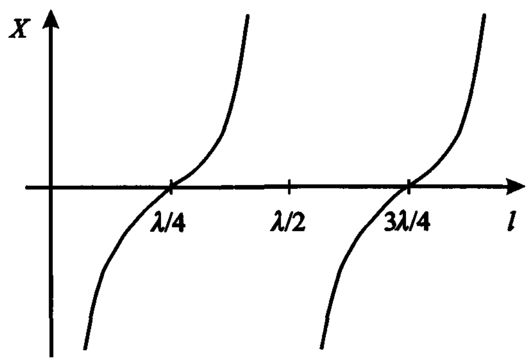

Figure 4.11. Reactance of a section of open-circuited transmission line.

of it as pumping the resonance. We can write the current as

$$I_g = V_+/Z_0 - V_-/Z_0 = jI_s \exp(-j\beta l)\sin(\beta l), \tag{4.90}$$

where $I_s = V_0/Z_0$ is the short-circuit current for the generator. When $l = \lambda/4$, $I_g = I_s$. As far as the generator is concerned, a quarter-wave section of open-circuited line looks like a short. The current is the short-circuit current, and the voltage is zero.

In an open-circuited line, the generator current and voltage are 90° out of phase, and this means that the impedance is reactive. Energy is stored in the waves propagating back and forth. We can find the reactance X by taking the ratio of the generator voltage and current:

$$X = \frac{V_g}{jI_g} = -\frac{Z_0}{\tan(\beta l)}. \tag{4.91}$$

This formula is plotted in Figure 4.11. The curve shows that the line can be used as either an inductor or a capacitor, depending on the length and the frequency.

The figure also shows that an open-circuited line can be used as a resonant circuit. When the reactance is zero, we effectively have a series resonance. When the reactance becomes very large, we have a parallel resonance. When you study the series resonance in the lab, you will see that the input voltage does not really go to zero as the theory predicts, because of loss in the line.

Transmission-line resonators are usually not very practical for filters at frequencies in the MHz range, because the lines turn out to be inconveniently long. For example, if we wanted a series resonance at 5 MHz, we would need a 10-meter cable. However, at the frequencies for microwave radars, in the GHz range, the required length might be a few millimeters, and transmission-line elements are very easy to use. There is a simple transmission line called microstrip, which is just a printed-circuit board with a ground plane on the back (Figure 4.12). To make your circuits on this board, you do not even need capacitors or inductors; you can just etch the copper on the top in the shape that you want. For example, let us suppose that we want to make a filter that can stop signals at a particular frequency. This

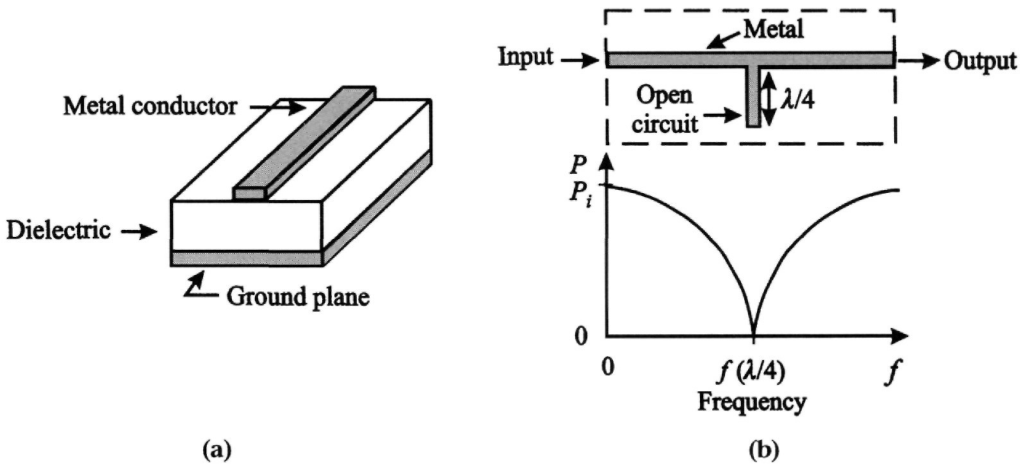

Figure 4.12. Microstrip transmission line (a), and notch filter (b).

is called a *notch* filter. We can add a parallel open-circuited section of transmission line that is a quarter of a wavelength long at the notch frequency. At this frequency, the reactance of the open-circuited line is zero, so that it is effectively a short circuit across the line.

4.10 Quality Factor

We characterize a resonance by a quality factor Q, given in the last chapter as

$$Q = \omega \frac{E}{P_a}, \tag{4.92}$$

where E is the stored energy and P_a is the average power lost. In a transmission line, stored energy is in the form of power propagating down the line. We can write

$$E = P_+(l/v), \tag{4.93}$$

where P_+ is the power in the forward wave and l/v is the delay time for the cable. Next we calculate the dissipated power P_a. As the forward wave travels along the transmission line, the voltage decays by a factor of $\exp(-\alpha l)$. Because the power is proportional to the square of the voltage, it decays as $\exp(-2\alpha l)$. The lost power can thus be written

$$P_a = P_+ - P_+ \exp(-2\alpha l)) \approx 2\alpha l P_+. \tag{4.94}$$

Here we are using the first-order Taylor series approximation

$$\exp(x) \approx 1 + x, \tag{4.95}$$

which is valid when $|x| \ll 1$. When we substitute for E and P_a into Equation 4.92, we get

$$Q = \omega \frac{E}{P_a} = \frac{\beta}{2\alpha}. \tag{4.96}$$

We have calculated Q in terms of the forward wave only. However, the result is the same for a reverse wave. Thus the formula can be applied to a resonator as a whole. A typical Q for a transmission-line resonator is between 10 and 100.

4.11 Lines with Loads

We have found that the impedance of an open-circuited line depends strongly on the length of the line and the frequency. We can also find formulas for lines with loads. Let us consider a section of length l connected to a load with a reflection coefficient $\rho(0)$ (Figure 4.13). We will calculate the reflection coefficient at the other end of the line. To start, let the forward wave at the input be V_+. We can write the forward wave at the load as $V_+ \exp(-j\beta l)$. To find the reverse wave at the load, we multiply the forward wave by the reflection coefficient, $\rho(0)$, to give $\rho(0) V_+ \exp(-j\beta l)$. We can write the reverse wave at the input V_- as

$$V_- = \rho(0) V_+ \exp(-j2\beta l). \tag{4.97}$$

The reflection coefficient at the generator $\rho(l)$ is given by

$$\rho(l) = V_-/V_+ = \exp(-j2\beta l)\rho(0). \tag{4.98}$$

The magnitude of the reflection coefficient does not change, only the phase. Notice that the reflection coefficient at the input lags the reflection coefficient at the load. There are actually two phase lags. One comes from the propagation of the forward wave from the generator to the load, and the other from the propagation of the reflected wave from the load back to the generator. When we plot the reflection coefficient in the complex plane (Figure 4.14), the locus is a clockwise circle.

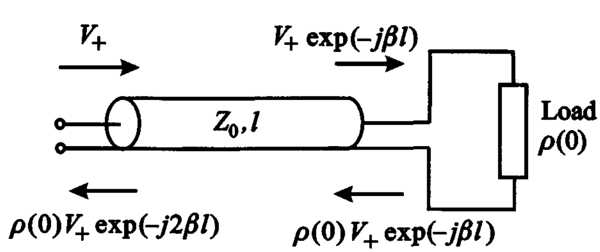

Figure 4.13. Reflection coefficient calculation for a lossless line with a load.

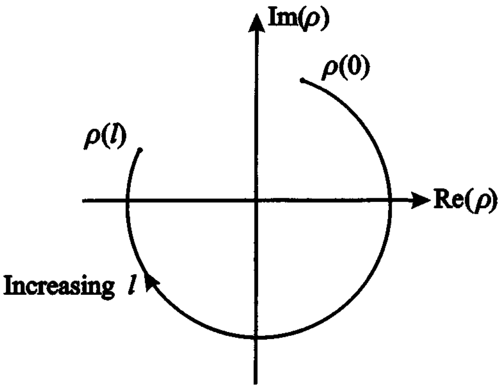

Figure 4.14. Locus of the reflection coefficient as the length increases.

If the transmission line is a half wavelength long, the reflection coefficient is the same as the load reflection coefficient,

$$\rho(\lambda/2) = \rho(0). \tag{4.99}$$

This means that at this frequency, the transmission line has no effect except for a propagation delay. This idea is used to make protective covers for radars on airplanes. The cover is called a *radome* or *half-wave window*. The problem with mounting a radar antenna on the front of an airplane is that it will blow away unless it is protected. If the covering is made a half wavelength thick, the waves go right through it.

The other interesting length to consider is a quarter wavelength. When the line is a quarter wavelength long, the reflection coefficient changes sign. We can write

$$\rho(\lambda/4) = -\rho(0). \tag{4.100}$$

Changing the sign of the reflection coefficient transforms the impedance. We can write the impedance at the generator end, $Z(\lambda/4)$, with Equation 4.77 as

$$\frac{Z(\lambda/4)}{Z_0} = \frac{1+\rho(\lambda/4)}{1-\rho(\lambda/4)} = \frac{1-\rho(0)}{1+\rho(0)} = \frac{Z_0}{Z(0)}, \tag{4.101}$$

where $Z(0)$ is the load impedance. We can rewrite this expression as

$$\frac{Z(\lambda/4)}{Z_0} = \frac{Z_0}{Z(0)}. \tag{4.102}$$

One way to understand this formula is to define a *normalized impedance*, which is the impedance scaled to Z_0. We will use lower-case letters for normalized impedance and write

$$z = Z/Z_0. \tag{4.103}$$

The *normalized admittance* is given by

$$y = 1/z = YZ_0. \tag{4.104}$$

In terms of normalized impedances, Equation 4.102 becomes

$$z(\lambda/4) = 1/z(0). \tag{4.105}$$

This means that we can think of the quarter-wave transmission line as an impedance inverter. We will study other impedance inverters in the next chapter. We will see that we can make excellent band-pass filters by combining impedance inverters and resonators. Another application of quarter-wave sections is in eliminating reflections. We know that if the load resistance is different from the characteristic impedance of a cable, there will be a reflection. We can use a quarter-wave section to transform the impedance of a load to match the cable. To see this, rewrite Equation 4.102 as

$$Z_0 = \sqrt{Z(\lambda/4)Z(0)}. \tag{4.106}$$

In words, the characteristic impedance of the transmission line is the geometric mean of the load impedance $Z(0)$ and the transformed impedance $Z(\lambda/4)$. This means that if we choose Z_0 to be the geometric mean of the load resistance R_l and the source resistance R_s, all of the available power from the source will be delivered to the load. We write the matching condition as

$$Z_0 = \sqrt{R_s R_l}. \tag{4.107}$$

This idea is also used in optics. Lenses are coated with matching layers that are a quarter-wavelength thick to eliminate reflections from the surface of the lens. Typically several layers are used so that the reflections can be reduced for the full range of wavelengths we can see. These are called *antireflection*, or AR, coatings.

FURTHER READING

The classic textbook is *Fields and Waves in Communication Electronics* by Simon Ramo, John Whinnery, and Theodore Van Duzer, published by Wiley. It is comprehensive, covering this material and much more. There is an excellent discussion of distributed inductance and capacitance and the skin effect. Paul Nahin has also written a terrific biography, *Oliver Heaviside: Sage in Solitude*, published by the IEEE Press, that touches directly on many of these topics. Heaviside was an English engineer who helped develop transmission-line theory, Laplace transforms, and the notation that we use for vector calculus and for Maxwell's equations. Nahin also tells the sad story of the failure of the first transatlantic cable.

PROBLEM 10 – COAXIAL CABLE

Coaxial cable has many advantages for transmitting electrical signals. It can be used from DC to very high frequencies (cables are available that operate as high as 100 GHz). A common laboratory cable is RG58/U, costing about a dollar per meter. The shield is a weave of fine tinned-copper wires around an insulating polyethylene tube. This cable

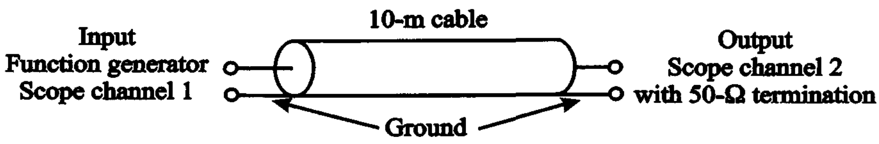

Figure 4.15. Measuring the velocity v.

typically has twist-lock BNC connectors. (A short note is needed on abbreviations. BNC stands for "Bayonet Neill Concelman," after the Bell Laboratories engineers Paul Neill and Carl Concelman, who developed the connector. RG/U is "radio-guide/universal," and different varieties come with different identifying numbers.) RG58 coax and BNC connectors are commonly used up to a frequency of 1 GHz. In this problem, you will measure the velocity and characteristic impedance for the cable and use these to calculate the distributed inductance and capacitance.

A. First measure the velocity on a 10-m cable with the connections in Figure 4.15. Set the function generator for 5-V pulses with a width of 50 ns that repeat at a frequency of 20 kHz. To measure the delay accurately, we need a fast time scale on the scope. A convenient scale is 10 ns per division. You should be able to see an incident pulse on channel 1 and a delayed pulse on channel 2. Measure the delay, and calculate the velocity v. Express v as a fraction of the speed of light c, where $c = 3.00 \times 10^8$ m/s.

B. Disconnect the 10-meter cable and plug an antenna cable into the channel-1 tee. Now your pulses are sent up to the antenna. At the antenna, the pulses are reflected and come back down the cable. Use the delay to deduce the length of the antenna cable, assuming that the velocity is the same as before.

C. Next we find the characteristic impedance Z_0 with the circuit shown in Figure 4.16. The voltage is measured by a tee connection to channel 1 of the oscilloscope. The current is measured through a 1:1 transformer. We will study transformers in Chapter 6, but for now you should know that the 1-Ω resistor effectively appears in series for our signals. This means that the voltage on channel 2 is numerically equal to the current in amperes. The transformer is needed to avoid a short to the scope ground. Measure the voltage and current in the middle of the pulse and calculate Z_0.

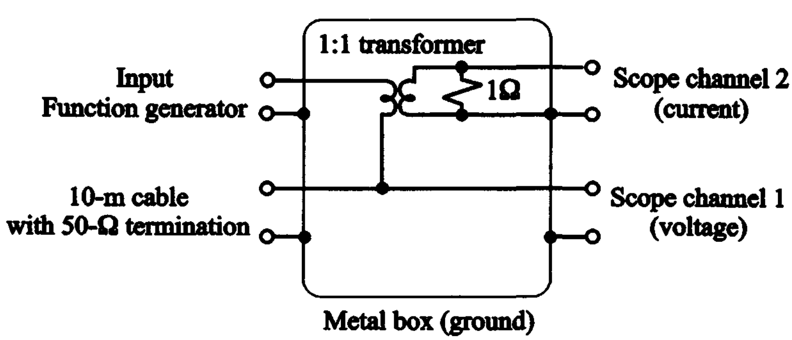

Figure 4.16. Circuit for measuring the characteristic impedance Z_0.

D. Now remove the 50-Ω load from the end of the cable so that the cable is open-circuited. Sketch and interpret the voltage and current waveforms.

E. Use your measurements of v and Z_0 to calculate L and C.

PROBLEM 11 – WAVES

A. We saw that for either a forward wave or a reverse wave alone, the magnitude remains the same at different positions, but the phase changes. We call this a *traveling wave.* However, if both a forward wave and a reverse wave are present at the same time, we need to add the two phasors, and the locus changes dramatically. Sketch the locus as z varies when both a forward voltage wave $\exp(-j\beta z)$ and a reverse wave $\exp(+j\beta z)$ are present. How does the locus change if the reverse wave becomes $-\exp(+j\beta z)$? Sketch the new locus. How does the locus change if the reverse wave becomes $\rho\exp(+j\beta z)$, where $|\rho| \leq 1$? The *standing wave ratio* (SWR) is defined as the ratio of the maximum magnitude to the minimum magnitude. Find a formula for the SWR in terms of $|\rho|$. The SWR is often used to characterize connectors, filters, and antennas.

B. We analyzed a transmission line by breaking it up into short sections and letting the length of the sections go to zero. We found a formula for the characteristic impedance, which is the ratio of the voltage and current in a forward wave. This would be the impedance we would measure at the input of a transmission line that is sufficiently long that we do not have to consider the effect of the far end. It is interesting to consider the impedance of a ladder network of discrete cascaded components (Figure 4.17). We let the impedance of the series element be Z and the admittance of the parallel element be Y. We will assume that the number of elements is large enough that we do not need to consider the effect of the far end. Find the input impedance Z_0 of the discrete line in terms of Z and Y. One way to approach this problem is to consider that adding another Z and Y section at the beginning should not change the input impedance. You can use this fact to find Z_0.

C. Suppose we want to transmit voice signals over 100 km of cable with $L = 250$ nH/m and $C = 100$ pF/m. The distributed resistance at voice frequencies is 50 mΩ/m. The distributed conductance may be neglected. Using the high-resistance approximation, calculate the total loss in dB and the delay in ms at 500 Hz, 1 kHz, and 2 kHz.

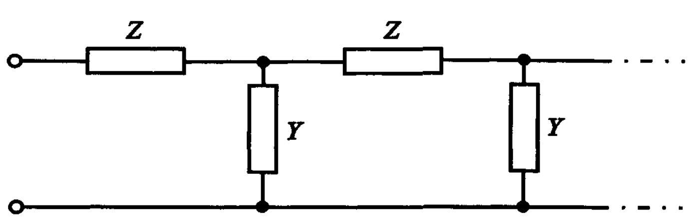

Figure 4.17. A discrete transmission line with lumped series impedance Z and parallel admittance Y.

D. Now add 100-mH series inductor coils at 1-km intervals. You may assume that the added inductance is effectively distributed uniformly along the line, and you may neglect the resistance of the coils. Using Equations 4.49 and 4.50, calculate the total loss in dB and the delay in ms at 500 Hz, 1 kHz, and 2 kHz. For comparison, calculate the total loss and delay using the high-reactance approximation.

PROBLEM 12 – RESONANCE

A. We will consider an open-circuited section of transmission line connected to a generator (Figure 4.18). Let the attenuation constant of the line be α and the phase constant be β. Derive an expression for the ratio $|V_g/V|$ at the first series resonant frequency. Find a first-order approximation for the ratio, assuming that α is small.

B. Now we find α. Make the connections shown in Figure 4.19, with the end of the cable connected to channel 2 of the oscilloscope. Do not use a 50-Ω load. Use an amplitude setting of 1 Vpp. Adjust the frequency to find the first series resonance where $|V_g|$ is a minimum. Use the ratio $|V_g/V|$ to calculate α.

C. Next we use the resonant frequency to find the velocity. Because the scope capacitance shifts the resonance, you should disconnect the cable from channel 2 for this part. Readjust the frequency for resonance. Use the frequency and the length to calculate the cable velocity v. How large was the frequency shift caused by the scope capacitance? Calculate the frequency shift that you would expect, using the scope and cable capacitance.

D. Next we consider the bandwidth. For a series resonance, we defined the half-power frequencies f_l and f_u where R and X are equal and the load voltage changed by a factor of $\sqrt{2}$. Here the load is effectively the distributed resistance of the cable and we do not have access to the resistance by itself. However, at resonance, the cable resistance is only a few ohms, and the function generator current is very close to the short-circuit current $I_s = V_o/Z_0$. The voltage $|V_g|$ will be a minimum at the resonant

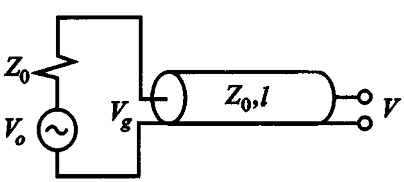

Figure 4.18. Open-circuited section of transmission line connected to a generator.

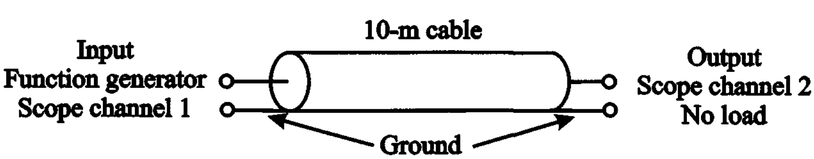

Figure 4.19. Measuring the attenuation constant α.

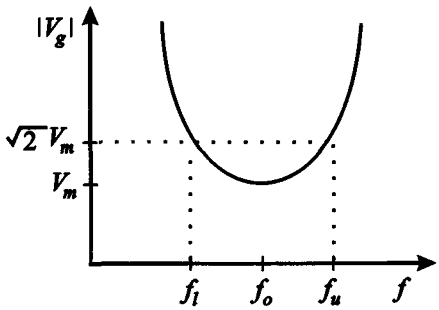

Figure 4.20. Variation of $|V_g|$ near the resonant frequency. V_m is the voltage minimum at resonance.

frequency and will increase as we move away from it. At f_u and f_l, where the input resistance and reactance are equal, $|V_g|$ will rise by a factor of $\sqrt{2}$. One way to make the measurement is to first measure $|V_g|$ at resonance and then reduce the amplitude setting by $\sqrt{2}$. Now measure the frequencies f_l and f_u that give the same value of $|V_g|$ that you measured before. What Q does this bandwidth indicate?

E. Now calculate the Q that you expect from the energy formula

$$Q = \frac{\beta}{2\alpha}. \tag{4.108}$$