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An Efficient Genetic Algorithm for Interval Linear Bilevel Programming Problems

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***Abstract*—This paper deals with a class of interval linear bilevel programming problems, in which some or all of the leader's and follower's objective function coefficients are specified in terms of intervals. The focus of solving this class of problems is on determining the optimal value range when different coefficients of objectives are taken in intervals given. In order to obtain the best and the worst optimal solutions to this class of problems, an efficient genetic algorithm is developed. Firstly, the objective coefficients of the lower level are encoded as individuals using real coding scheme, and the relative intervals are taken as the search space of the genetic algorithm. Secondly, for each encoded individual, a simplified interval linear bilevel program is obtained, in which interval coefficients are simply in the upper level objective function. Finally, the simplified problem is further divided into two linear bilevel programs without interval coefficients and solved by using the optimality theory of linear programming. The optimal values are taken as fitness values, by which the best and the worst optimal solutions can be obtained. In order to illustrate the efficiency of the proposed algorithm, two examples are solved and the results show that the algorithm is feasible and robust.**

***Keywords*-Interval linear bilevel program; genetic algorithm; best optimal solution; worst optimal solution;**

I. INTRODUCTION

Many real-world and theoretical problems may be modeled as traditional mathematical programming problems in which all coefficients involved are exactly known and can be dealt with by using some optimization techniques. However, in practice, it is very common for the coefficient values to be only approximately known because relevant data is nonexistent or scarce, difficult to obtain or estimate, etc. Therefore, mathematical programming models must take explicitly into account the intrinsic uncertainty associated with the model coefficients. Uncertainty can be handled mainly in three manners: interval programming, stochastic programming or fuzzy numbers. Since it is not always easy to specify the membership function or probability distribution in an inexact environment, as a result, interval programming is the most concerned one of the approaches to tackle uncertainty in mathematical programming models. Mathematical programs with interval coefficients in the objective function and/or in the constraints have been addressed in the literature. For this problem, one always needs to compute the optimal value range between the best and the worst optimal objective values. These two extreme values allow the decision maker to better understand the risk involved.

The interval linear program can be viewed as a multi-parametric linear programming with interval domains for parameters, and based on the view point, some efficient approaches have been developed[1]. Lai discussed a class of linear programming problems with interval coefficients in both the objective function and constraints, the noninferior solutions to such problems are defined based on two order relations between intervals. One can obtain the solution by solving a parametric linear programming problem[2]. Molai and Khorram introduced a satisfaction function and defined a satisfactory solution to the problem[3]. Chinneck and Ramadan proposed an approach for a class of linear programming with interval coefficients by analyzing all kinds of variables and constraints[4]. Hladik presented a general approach to the situation the feasible set is described by an arbitrary linear interval system[5].

For nonlinear case, very few results have been obtained. Hladik considered a generalized linear fractional programming problem with interval data and presented an efficient method which reduces the problem to solving from two to four real-valued generalized linear fractional programs[6].

Bilevel programming problem (BLPP) is a hierarchical optimization problem, in which two levels of optimization problems are involved, the leader's problem and the follower's problem[7], [8]. The general bilevel programming problem can be formulated as follows

$$\begin{cases} \min\limits_{x\in X} F(x,y) \\ s.t.\ G(x,y)\leq 0 \\ \min\limits_{y\in Y} f(x,y) \\ s.t.\ g(x,y)\leq 0 \end{cases} \tag{1}$$

In this problem,

$$\begin{cases} \min\limits_{x\in X} F(x,y) \\ s.t.\ G(x,y)\leq 0 \end{cases} \tag{2}$$

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and

$$\begin{cases} \min_{y \in Y} f(x,y) \\ s.t.\ g(x,y) \leq 0 \end{cases} \quad (3)$$

are the leader's and the follower's problems,respectively. Unlike other mathematical programs, any feasible solution to the bilevel program (1) must satisfy the optimality of the follower's problem when the leader's variables are fixed. In the problem, when the coefficients are interval numbers, this problem is known as an interval bilevel program. Calvete addressed a linear bilevel problem whose objective coefficients are assumed to lie between specific bounds and developed two enumerative algorithms to compute optimal value range based on the *k*-th best method[9]. However, one has to execute two different algorithms to compute the upper and lower bounds of optimal objective values, respectively.

In this paper, we concentrate on the linear bilevel programming problems in which the coefficients of the upper and lower level objective functions are all intervals, and presents an efficient genetic algorithm for solving the problem. In the proposed algorithm, the coefficient intervals of the lower level objective are taken as the search space and the optimality conditions of linear program are adopted to evaluate each individual.

This paper is organized as follows. The discussed problem and some notations are presented in Section II, and a real encoding genetic algorithm is given based on the optimality conditions of linear program in Section III. Experimental results are presented in Section IV. We finally conclude our paper in Section V.

II. Discussed Problem

In this paper a linear BLPP with interval coefficients in both the upper and lower level objective functions is considered. Let us denote the problems by

$$\begin{cases} \min_{x \geq 0} F(x,y) = [c_l, c_u]x + [d_l, d_u]y \\ s.t.\ A_1x + B_1y \leq b_1 \\ \min_{y \geq 0} f(x,y) = [e_l, e_u]y \\ s.t.\ A_2x + B_2y \leq b_2 \end{cases} \quad (4)$$

where $x \in R^n, y \in R^m$; A_1 is a $p \times n$-matrix and B_1 is a $p \times m$-matrix; A_2 is a $q \times n$-matrix, and B_2 is a $q \times m$-matrix, and $b_1 \in R^p, b_2 \in R^q$; c_l, c_u, d_l, d_u, e_l and e_u are lower and upper bound vectors of objective coefficients c, d and e, respectively. For simplicity, we denote the bilevel program by $LBIC(c,d,e)$ when the objective coefficients are taken as c, d, e, respectively. Now we introduce some related definitions.

1) Constraint region: $S = \{(x,y) \mid A_ix + B_iy \leq b_i, i = 1,2; x, y \geq 0\}$.

2) For x fixed, the feasible region of follower's problem: $S(x) = \{y \mid A_2x + B_2y \leq b_2, y \geq 0\}$.

3) Projection of S onto the leader's decision space: $S(X) = \{x \mid \exists y, (x,y) \in S\}$.

4) For $LBIC(c,d,e)$, the follower's rational reaction set for each $x \in S(X)$: $M(x) = \{y \mid y \in \operatorname{argmin}\{ev, v \in S(x)\}\}$.

5) For $LBIC(c,d,e)$, inducible region: $IR = \{(x,y) \in S \mid y \in M(x)\}$.

Furthermore, we give:

Definition 1 (Optimal solution): (x,y) is called an optimal solution to (4) if there exist $c \in [c_l, c_u], d \in [d_l, d_u]$ and $e \in [e_l, e_u]$ such that (x,y) is an optimal solution to $LBIC(c,d,e)$.

Definition 2 (Best optimal solution): An optimal solution (x^*, y^*) is called the best optimal solution to (4) if for any optimal solution (x,y), the inequality

$$F(x^*, y^*) \leq F(x,y)$$

holds.

Definition 3 (Worst optimal solution): An optimal solution (x^*, y^*) is called the worst optimal solution to (4) if for any optimal solution (x,y), the inequality

$$F(x^*, y^*) \geq F(x,y)$$

holds.

Since in a linear programming any inequality constraint can be transformed into the equality by adding a slack variable. Hence, we re-written (4) as

$$\begin{cases} \min_{x \geq 0} F(x,y) = [c_l, c_u]x + [d_l, d_u]y \\ s.t.\ A_1x + B_1y \leq b_1 \\ \min_{y \geq 0} f(x,y) = [e_l, e_u]y \\ s.t.\ A_2x + B_2y = b_2 \end{cases} \quad (5)$$

The purpose of solving (4) or (5) is to obtain the best and the worst optimal solutions as well as the corresponding objective coefficient values.

III. Proposed Genetic Algorithm

In the section, we introduce a real coding scheme, and design the fitness function based on a double evaluation technique, which can obtain both the best and the worst optimal solutions in once run of the algorithm.

A. Chromosome Encoding

We begin with the coefficients of the follower objective function, and encode each individual using a real-number-coding scheme in the space $[e_l, e_u]$.

B. Fitness Evaluation

For each individual $e = (e_1, e_2, \cdots, e_m) \in [e_l, e_u]$, the following bilevel program is taken into account

$$\begin{cases} \min\limits_{x\geq 0} F(x,y) = [c_l, c_u]x + [d_l, d_u]y \\ s.t.\ A_1x + B_1y \leq b_1 \\ \min\limits_{y\geq 0} f(x,y) = ey \\ s.t.\ A_2x + B_2y = b_2 \end{cases} \tag{6}$$

The problem can be further divided into two linear bilevel programs as follows

$$\begin{cases} \min\limits_{x\geq 0} F(x,y) = c_lx + d_ly \\ s.t.\ A_1x + B_1y \leq b_1 \\ \min\limits_{y\geq 0} f(x,y) = ey \\ s.t.\ A_2x + B_2y = b_2 \end{cases} \tag{7}$$

and

$$\begin{cases} \min\limits_{x\geq 0} F(x,y) = c_ux + d_uy \\ s.t.\ A_1x + B_1y \leq b_1 \\ \min\limits_{y\geq 0} f(x,y) = ey \\ s.t.\ A_2x + B_2y = b_2 \end{cases} \tag{8}$$

One can evaluate the individual e by solving (7) and (8), in which the objective values provide two evaluation criteria.

C. Solving Linear Bilevel programs

In fact, it is very difficult to solve (7) and (8), especially when the scale is very large. Here, we simply consider the problems with a small number of follower variables. Since both (7) and (8) are linear bilevel programs, considering that (8) can also be solved by the same procedure as done in (7), we only present the algorithmic procedure for (7).

First, all potential bases of the follower linear program are denoted by $B_2^1, \cdots, B_2^k$, which only means $B_2^i, i = 1, \cdots, k$, are nonsingular. Further, for each B_2^i, we have $y_B = (B_2^i)^{-1}(b_2 - A_2x)$, here, y_B is basic variable vector, whereas the nonbasic variables are 0. In addition, we replace y by y_B and the follower's problem by the optimality conditions, as a result, a linear programming problem

$$\begin{cases} \min\limits_{x\geq 0} c_lx + d_{lB}y_B \\ s.t.\ A_1x + B_{1B}y_B \leq b_1, \\ (B_2^i)^{-1}(b_2 - A_2x) \geq 0, \\ e - e_B(B_2^i)^{-1}B_2 \geq 0 \end{cases} \tag{9}$$

is obtained, where d_{lB}, B_{1B} and e_B are basic components of d_l, B_1 and e, respectively. When one consider all $B_2^i, i = 1, \cdots, k$, k linear programs can be gotten. Finally, we solve these linear programs, compare the objective values and take the minimum one as the optimal value of (7).

D. Crossover and Mutation Operators

In the designed algorithm the arithmetical crossover and Gaussian mutation are adopted.

1) Crossover Operator: Let l_1 and l_2 be crossover parents. For $\forall\alpha \in (0,1)$, the offspring can be generated as follows:

$$o_1 = \alpha l_1 + (1-\alpha)l_2$$
$$o_2 = \alpha l_2 + (1-\alpha)l_1$$

2) Mutation Operator: $\hat{l}$ is taken for mutation, then the offspring is given as follows:

$$o_m = \hat{l} + N(0, \sigma^2)$$

here $N(0,\sigma^2)$ is a variable distributed normally with mean 0 and variance σ^2.

E. Description of the Proposed Algorithm

In the subsection we propose a genetic algorithm for interval linear bilevel programs (GA-IBP).

- Step1 (Initial population) Randomly generate N initial points to form the initial population $pop(0)$. Let $g = 0$.
- Step2 (Fitness) Evaluate each point by solving (7) and (8), and record the best solution (x_b, y_b) regarding the fitness using (7) and the worst (the largest one) solution (x_w, y_w) regarding the fitness using (8).
- Step3 (Crossover and mutation) The arithmetic crossover and Gaussian mutation operators are adopted to generate all genetic offspring.
- Step4 Evaluate the fitness values of all offspring, and update (x_b, y_b) and (x_w, y_w).
- Step5 For each fitness the best $[N/2]$ individuals are selected from both $pop(g)$ and offspring set. The total of $2*[N/2]$ individuals are put into the next generation population $pop(g+1)$.
- Step6 If the termination condition is satisfied, then the algorithm is stopped; otherwise, let $g = g+1$, go to Step 3.

Noting that (7) and (8) can be solved exactly and the elitism preservation scheme is applied in the section process, one can obtain the following convergence result:

Theorem 1: The proposed algorithm GA-IBP is convergent with probability one.

IV. Computational Examples

In the section, In order to illustrate the performance of GA-IBP, we execute the proposed algorithm on two examples (Ex.1 and Ex.2) from [9]. Also, for the purpose of comparison we denote the known best and worst optimal values by F_b and F_w, respectively.

$$[Ex.1] \begin{cases} \min\limits_{2\geq x\geq 0} x + y_1 + y_2 \\ \min\limits_{y\geq 0} [-1,2]y_1 + [1,3]y_2, \\ s.t.\ -3x - 3y_1 + 2y_2 \leq 1, \\ x + 2y_1 \leq 4, y_2 \leq 2 \end{cases} \tag{10}$$

Table I
BEST OPTIMAL SOLUTIONS PROVIDED BY GA-IBP.

$No.$	F_b^b	F_b^m	F_b^w	(x_b, y_b)	F_b
$Ex.1$	0	0	0	$(0, 0, 0)$	0
$Ex.2$	0.5	0.5	0.5	$(0, 0, 0.5)$	0.5

Table II
WORST OPTIMAL SOLUTIONS PROVIDED BY GA-IBP.

$No.$	F_w^b	F_w^m	F_w^w	(x_w, y_w)	F_w
$Ex.1$	2	2	2	$(0, 2, 0)$	2
$Ex.2$	7	7	7	$(0, 1, 2)$	7

$$[Ex.2]\begin{cases} \min\limits_{2\geq x\geq 0}[1,4]x + [1,3]y_1 + [1,2]y_2 \\ \min\limits_{y\geq 0}[2,5]y_1 + [-3,-1]y_2, \\ s.t.\ -3x - 3y_1 + 2y_2 \leq 1, \\ x + 2y_1 \leq 4, y_2 \leq 2 \end{cases} \tag{11}$$

The parameters are taken as follows: $N = 5$, the crossover probability $p_c = 0.8$, the mutation probability $p_m = 0.1$, and the maximum number of generations $G_m = 10$. GA-IBP is executed 20 runs on a PC, and record the best optimal solutions (x_b, y_b) as well as the worst optimal solutions(x_w, y_w) . In all 20 runs the best ones (F_b^b, F_w^b), mean values (F_b^m, F_w^m), and worst ones (F_b^w, F_w^w) of two classes of solutions are shown in Tables I-II, and the values of e corresponding to the best and the worst optimal solutions are also shown in Table III, in which e_{best} and e_{worst} mean the values of e corresponding to the best and the worst optimal solutions, respectively.

From Tables I-II, one can see that GA-IBP found the same optimal results as those provided by the literature for these examples. Considering that the results including the best and the worst optimal solutions can be got in one execution of GA-IBP, we can conclude that the proposed algorithm is feasible and efficient.

V. CONCLUSION

In this paper a BLPP with interval coefficients in both the leader's and the follower's objectives is discussed. It should be noted that the change of the follower's objective will cause the different feasible region, which causes the problem more complex than other cases with specified follower's coefficients. We design an GA based on a real coding scheme, which makes the best and the worst optimal solutions can be obtained in one execution of the algorithm.

Table III
VALUES OF e CORRESPONDING TO THE BEST AND THE WORST RESULTS

$No.$	e_{best}	e_{worst}
$Ex.1$	$(1.7882, 2.5514)$	$(1.7882, 2.5514)$
$Ex.2$	$(4.4442, -1.1884)$	$(3.8971, -2.8049)$

Since there exist the solution procedure of linear bilevel programming problems in the fitness evaluation, hence, the proposed algorithm can not be easily extended to nonlinear cases. In the future work, we will try to introduce the hybrid coding scheme to design GA for solving nonlinear bilevel programming problems with interval coefficients.

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